Using an SPH-based continuum representation of granular terrain to simulate the mobility of an eight-wheel rover

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December 16, 2020
Abstract

We use the SPH method to determine the dynamics of granular material in its interaction with the eight wheels of a rover. The rover wheel geometry is defined through a mesh. The granular material is modeled as an elasto-plastic continuum that dynamically interacts with the rigid wheels of the rover in a Chrono [1] co-simulation setup. Several parametric studies are carried out to assess rover simulation robustness for flat terrain operation scenarios.
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1 Introduction

The interaction of granular material with moving solid bodies is encountered in many engineering applications, e.g. terramechanics, farming, astrophysics, pharma, etc., see, for instance, [2, 3, 4, 5, 6, 7, 8, 9]. DEM is considered to be one of the most accurate methods for the numerical simulation of granular dynamics. However, DEM solves the dynamic equations of motion for each grain, individually. Hence, a fully resolved simulation of a practical granular material flow problem in conjunction with DEM will typically lead to very large degrees of freedom (DOF) counts, which poses both computational and storage challenges. Furthermore, the non-homogeneity of the grains in real-world problems poses another difficulty in the contact model and the contact-detection stage of the DEM, see, for instance, [10].

Continuum models of granular material flows have gained popularity over the last two decades since they proved capable to address scale limitations in the DEM approach. These continuum models express the macro-scale behavior of the material by relating the stress to the strain and strain-rate fields, leading to smaller DOF counts compared to the DEM model. Against this backdrop, we use a two-way coupling algorithm discussed in [11], in which both large deformation of the granular material flows and large overall motion of the solid bodies in three-dimensional space can be captured. In [11], the granular flow is modeled as a SPH-resolved continuum problem. The interaction between the granular material and immersed rigid bodies is posed and solved as an FSI problem using so-called boundary conditions enforcing (BCE) SPH particles [12, 13], which are attached to the boundary of the implements. In previous work, this coupling algorithm was successfully applied to capture the interaction of fluids and rigid/flexible multi-body systems [14, 15, 13, 16]. To represent the dynamics of dense granular material and update the stress field, we employ the rheology proposed in [17].

This technical report is organized as follows. Section 2 describes the two-way coupling method for simulating granular material flows and their interaction with immersed solid bodies based on the SPH method. In Section 3, we model an eight-wheel rover and simulate its mobility on granular terrain. The rover is simulated in different scenarios, and parametric studies are carried out to assess the robustness of the simulation setup. Concluding remarks and directions for future work are discussed in Section 4.

2 Numerical method

2.1 Dynamic equation of the deformable terrain

For a granular material represented as a continuum, the velocity $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ enter the continuity and momentum balance equations as [18]

$$
\begin{align*}
\frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u} \\
\frac{d\mathbf{u}}{dt} &= \frac{\nabla \boldsymbol{\sigma}}{\rho} + \mathbf{f}_b \\
&\text{for } \mathbf{x} \in \Omega_f,
\end{align*}
$$

(1)
where \( \rho \) is the density of the terrain, and \( \mathbf{f}_b \) denotes the external force per unit mass, e.g., the gravity. The stress tensor can be expressed as [18]

\[
\sigma = -pI + \tau,
\]

(2)

where \( p \) is the isotropic pressure and \( \tau \) is the deviatoric component of the stress tensor. The isotropic pressure is defined as the trace of the stress tensor, i.e., \( p = -\frac{1}{3} \text{tr}(\sigma) = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \). According to Hooke’s law, a linear elastic relation between the Jaumann stress rate tensor and elastic strain tensors [19, 20, 21, 17] can be used to obtain the stress rate tensor as

\[
\frac{d\sigma}{dt} = \dot{\phi} \cdot \sigma - \sigma \cdot \dot{\phi} + \Delta \sigma,
\]

(3)

where the rotation rate tensor is defined as \( \dot{\phi} = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^\top) \), the Jaumann rate of the stress tensor is expressed as

\[
\Delta \sigma = 2G(\dot{\varepsilon} - \frac{1}{3} \text{tr}(\dot{\varepsilon})I) + \frac{1}{3} K \text{tr}(\dot{\varepsilon})I,
\]

(4)

and the elastic strain rate tensor is defined as \( \dot{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) \) in the absence of plastic flow. Herein, \( K \) denotes the bulk modulus of the material and satisfies \( K = \frac{2(1+\nu)}{3(1-2\nu)}G \), where \( G \) and \( \nu \) are the shear modulus and Poisson’s ratio, respectively. Once the granular material starts to flow, the elastic strain rate tensor is defined as

\[
\dot{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) - \frac{1}{\sqrt{2}} \frac{\lambda}{\tilde{\tau}} \bar{\tau},
\]

in which the second term of the right-hand side comes from the contribution of the plastic flow of the continuum representation of the granular material. Therein, \( \lambda \) and \( \tilde{\tau} \) are the plastic strain rate and equivalent shear stress, respectively, which will be defined in Section 2.3. For the SPH discretization of the dynamic equations of the granular material terrain, the reader is referred to the work in [11].

### 2.2 Dynamic equation of the solid bodies

We follow the formulation described in [22] to define the configuration of the system of bodies by a set of generalized coordinates for the position and orientation of a rigid body in the 3D Euclidean space as \( \mathbf{r}_A \in \mathbb{R}^3 \) and \( \mathbf{e}_A \in \mathbb{R}^4 \); i.e., the absolute position of the center of mass, and the Euler parameters associated with orientation of body \( A \). The Euler parameters satisfy the normalization constraint \( \mathbf{e}_A^\top \mathbf{e}_A = 1 \). Combining the set of generalized coordinates of different bodies for a system of \( n_b \) bodies, one can write the set of generalized coordinates describing the system at position level as \( \mathbf{q} = [\mathbf{r}_1^\top, \mathbf{e}_1^\top, \ldots, \mathbf{r}_{n_b}^\top, \mathbf{e}_{n_b}^\top]^\top \in \mathbb{R}^{7n_b} \), and at velocity level as \( \dot{\mathbf{q}} = [\dot{\mathbf{r}}_1^\top, \dot{\mathbf{e}}_1^\top, \ldots, \dot{\mathbf{r}}_{n_b}^\top, \dot{\mathbf{e}}_{n_b}^\top]^\top \in \mathbb{R}^{7n_b} \). Instead of using the time derivative of the Euler parameters, one may choose to use angular velocities to describe the state of the system at the velocity level by \( \mathbf{u} = [\dot{\mathbf{r}}_1^\top, \dot{\omega}_1^\top, \ldots, \dot{\mathbf{r}}_{n_b}^\top, \dot{\omega}_{n_b}^\top]^\top \in \mathbb{R}^{6n_b} \), which reduces the problem size. The transformation from the derivatives of Euler parameters, \( \dot{\mathbf{e}}_A \), to angular velocities represented in the body-fixed frame, \( \dot{\omega}_A \), for each body is governed by \( \dot{\mathbf{e}}_A = \frac{1}{2} Q^\top(e_A) \dot{\omega}_A \), where matrix \( Q \in \mathbb{R}^{3\times 4} \) depends linearly on the Euler parameters \( e_A \). Therefore, a block diagonal matrix
Figure 1: Contact between two solid bodies.

\[ L(q) \equiv \text{diag} \left[ I_{3\times3}, \frac{1}{2} Q^T(e_1), \ldots, I_{3\times3}, \frac{1}{2} Q^T(e_{nb}) \right] \in \mathbb{R}^{7nb \times 6nb} \] is used to express via \( \dot{q} = L(q)u \), the relationship between \( \dot{q} \) and \( u \), where \( I_{3\times3} \) is the identity matrix [22].

The constrained Newton-Euler equations of motion that describe the motion of a system of bodies interacting through friction, contact, and bilateral constraints, assume the following form of a differential variational inequality (DVI) problem, see, for instance, [23, 24]:

\[ \dot{q} = L(q)u \]  
\[ M \dot{u} = f(t, q, u) + \sum_{k \in A(q, \delta)} (\gamma_{k,n} D_{k,n} + \gamma_{k,v} D_{k,v} + \gamma_{k,w} D_{k,w}) \]  
\[ k \in A(q, \delta) : 0 \leq \gamma_{k,n} \perp \Phi_k(q) \geq 0 \]  
\[ (\gamma_{k,v}, \gamma_{k,w}) = \arg\min \sqrt{(\gamma_{k,v})^2 + (\gamma_{k,w})^2} \leq \mu \gamma_{k,n} \] ,

where \( f(t, q, u) \) are the external forces; \( M \) is the constant system mass matrix; and, \( A(q, \delta) \) is the set of active and potential unilateral constraints based on the bodies that are mutually less than a gap \( \delta \) apart. For contact \( k \), the tangent space generator \( D_k \equiv [D_{k,n}, D_{k,v}, D_{k,w}] \in \mathbb{R}^{6nb \times 3} \) is defined as [24]

\[ D_k = [0_{3\times3}, \ldots, -A_k^T, A_k^T \bar{s}_{k,A}, 0_{3\times3}, \ldots, 0_{3\times3}, A_k^T, -A_k^T A_B \bar{s}_{k,B}, \ldots, 0_{3\times3}]^T, \]

where \( A_k = [n_k, v_k, w_k] \in \mathbb{R}^{3\times3} \) is the orientation matrix associated with contact \( k \), \( A_A = A(e_A) \) and \( A_B = A(e_B) \) are the rotation matrices, and \( e_A \) and \( e_B \in \mathbb{R}^4 \) are the Euler parameters associated with orientation of body \( A \) and \( B \) respectively; the vectors \( \bar{s}_{k,A} \) and \( \bar{s}_{k,B} \in \mathbb{R}^3 \) represent the contact point positions in body-relative coordinates as shown in Fig. 1. Above, the operator “tilde” applied to a three dimensional vector \( \mathbf{a} \) produces a matrix \( \tilde{\mathbf{a}} \in \mathbb{R}^{3\times3} \) such that \( \mathbf{a} \times \mathbf{b} = \tilde{\mathbf{a}} \mathbf{b} \) for all \( \mathbf{b} \in \mathbb{R}^3 \).

Equation (5c) captures a complementarity condition between \( \Phi \), the gap (distance) between bodies \( A \) and \( B \) at the contact point, and \( \gamma_{k,n} \), the Lagrange multiplier for the normal/contact force associated with the contact \( k \). The complementarity condition states that
of $\gamma_{k,n}$ and $\Phi$, at least one is zero and the other one is nonnegative. Indeed, when the gap function is zero (contact is present), the normal contact force is nonnegative; and, conversely, when the normal contact force is zero the gap function is nonnegative. The forces associated with contact $k$ can be expressed as $f_{k,N} = \gamma_{k,n} n_k$, and $f_{k,T} = \gamma_{k,v} v_k + \gamma_{k,w} w_k$, which are the contact and friction forces, respectively; and, $\gamma_{k,w}$ and $\gamma_{k,v}$ are the components of the friction force in the tangent plane. Finally, $\gamma_{k,w}$ and $\gamma_{k,v}$ are the solution of an optimization problem that is posed in Eq. (5d) to maximize the dissipation energy, see [25]. The DVI problem stated in Eq. (5) can be solved with a variety of techniques, see [26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

### 2.3 Stress update strategy of the granular material terrain

To represent the dynamics of the granular material, we employ a rheology proposed in conjunction with the material point method (MPM) [17]. The stress tensor can be updated explicitly from $t_n$ to $t_{n+1}$ in terms of $\rho^n, u^n$ and $\sigma^n$. The update strategy is implemented in four steps:

**STEP 1**: Update the stress tensor to an intermediate value $\sigma^*$ using the time derivative calculated using Eq. (3)

$$\sigma^* = \sigma^n + \frac{d\sigma^n}{dt} \Delta t,$$

and use Eq. (2) to compute $\tau^*$ and $p^*$. If $p^* < 0$, then simply set $\sigma^{n+1} = 0$ and advance the simulation time by one time step, i.e., start a new integration step all over again.

**STEP 2**: Calculate the candidate equivalent shear stress $\bar{\tau}^*$ through the double inner product of the intermediate deviatoric component of the stress tensor as

$$\bar{\tau}^* = \sqrt{\frac{1}{2} (\tau^*)_{\alpha\beta} : (\tau^*)_{\alpha\beta}},$$

**STEP 3**: Calculate $S_0$ based on the static friction coefficient $\mu_s$

$$S_0 = \mu_s p^*.$$  

**STEP 4**: If $\bar{\tau}^* < S_0$, no plastic flow occurs; use $\tau^*$ as the deviatoric component of the stress tensor at the end of this time step

$$\tau^{n+1} = \tau^*, p^{n+1} = p^*, \sigma^{n+1} = -p^{n+1} I + \tau^{n+1}.$$  

Else, if $\bar{\tau}^* \geq S_0$, plastic flow occurs; the Drucker-Prager yield criterion is used to scale the deviatoric component of stress tensor back to the yield surface

$$\tau^{n+1} = \mu \frac{p^*}{\bar{\tau}^*} \tau^*, \quad p^{n+1} = p^*, \quad \sigma^{n+1} = -p^{n+1} I + \tau^{n+1}.$$  

Above, the friction coefficient $\mu = \mu_s + \frac{\mu_2 \mu_s}{\rho_0 / I + 1}$, $I_0$ is a material constant and was chosen between 0.03 and 0.05 in this work; $I = \lambda d \sqrt{\frac{\mu_0}{p_0}}$ is the inertial number; $\mu_2$ is the limiting value of $\mu$ when $I \rightarrow \infty$; $d$ is the average diameter of the granular particles; and $\lambda = \frac{\tau^* - \bar{\tau}^*}{G \Delta t}$.
is the plastic strain rate. To update the stress tensor, we first use Eq. (3) without considering the contribution from the plastic flow. By adopting the Drucker-Prager yield criterion and the four-step update strategy above, the stress tensor can be corrected at the end of each integration step. More details are provided in [11].

3 Numerical experiments

![Figure 2: Schematic of the eight wheel rover.](image)

The rover simulation considered herein is complementary to the numerical experiments discussed in [11], where the approach was used to simulate several experiments, e.g., angle of repose, ball drop, cone penetration, plowing, and landslide. In this previous tests, all solid bodies were individual and were not organized as a proper mechanical system. In the eight-wheel rover system discussed herein, all eight wheels have been connected to the rover body through a revolute joint at the center of the wheel. The rover was driven by prescribing a constant angular velocity on each of the wheels. The rover body and the wheels have nontrivial 3D shapes (see Fig. 2), defined in both cases by triangular meshes. The granular material-like soil was modeled using SPH particles while the eight-wheel rover system were modeled as rigid bodies with constrains. To handle the interaction between wheels and the terrain, BCE particles are attached onto the wheels using the underlying mesh that defines the rigid body geometry. To generate the BCE particles from a nontrivial 3D geometry defined with mesh, we used the mesh to point cloud tool which discussed in [36]. Figure 3 shows the original mesh representation of the wheel and the particle presentation generated by the spherical decomposition tool. The OBJ files and the particle files generated are provided in the supporting material. The solid-to-solid interaction with friction and contact was handled via a penalty approach [37], and called for mesh-to-mesh collision detection.

The granular material is stored in a container of sizes $(L \times W \times H) 10m \times 4m \times 0.2m$. The container was filled with granular material-like soil. The eight-wheel rover was placed right above the soil, the initial position of the rover body is $[3, 0, 0.3]$. Once started, the rover moves with a constant angular velocity at the wheel ($\pi/2$ rad/s). The Young’s modulus,
density, and Poisson’s ratio of the granular material were set to $1 \times 10^6$ Pa, $1700$ Kg/m$^3$, and 0.3, respectively. The static frictional coefficient for the interaction between granular material and the wheel surface was set to $\mu_s = 0.7$. The mass of the rover was set as 20 Kg. The average diameter of the soil particles used in this simulation was 0.001m while the size of the SPH particles was set as 0.025m. The total number of SPH particles used to model this problem is 1.7 million. For a 20-second simulation, the total computational cost is about 2 days on one Nvidia V100 GPU card. The integration stepsize was 5e-4s. Figure (4) shows the distribution of the SPH particles and the position of the rover at different moments. The source code for this test is available in the public domain as open-source, along with a movie of the simulation [38].

A parametric test was run in order to gauge the robustness of the solver relative to changing the values of several model parameters. To this end, we performed a group of simulations, in which the mass of the rover was set as 30, 40, 50 and 60 Kg, while all other parameters were fixed as before. Figure (5) shows the rover position with different mass after a specific time. The results suggest the following: with the same angular velocity driving on the rover wheels, the rover with a larger mass had a deeper track left on the granular material terrain; the rover with a smaller mass traveled further away within the given amount of time ($T_{end}=18$s). In Fig. 5 (d), the rover of mass 60 Kg got stuck in the granular material terrain.

4 Conclusions and directions of future work

This technical report outlines a methodology that employs the SPH method to simulate the dynamics of granular material and its interaction with a rover. The need for a continuum solution is motivated by the observation that in many practical applications, a fully resolved
Figure 4: SPH particle distribution and the motion of the eight wheel rover at different moments.
(a) mass = 30 Kg  
(b) mass = 40 Kg  
(c) mass = 50 Kg  
(d) mass = 60 Kg

Figure 5: Rover position and particle distribution after 18 s.
dynamic simulation with DEM is prohibitively long [39]. By comparison, the SPH-based continuum solution discussed reduces both the degree of freedom count and computational cost. As future work, it remains to understand how to handle scenarios in which the granular material departs from the mono-disperse assumption thus opening the door to more complex grain geometries.

Acknowledgments

Support was provided by National Science Foundation grant CISE1835674 and US Army Research Office, under grants W911NF1910431 and W911NF1810476.

References


