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Autonomous Vehicle Path Tracking via Model Predictive Control with PID Speed Control for Optimal Travel Time

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Abstract

This technical contribution represents a step up in relation to work reported in [1]. Therein, the trajectory tracked was an 8-shaped curve with constant curvature and the optimization step in the model predictive control (MPC) algorithm was carried out without bound constraints. In this report, we consider the trajectory to be an arbitrary curve of continuously varying curvature. The MPC-based path tracking is implemented considering constraints on the control input and state outputs. For the lateral control, the MPC controller generates the optimal wheel steering angle. For the longitudinal control, the PID controller embedded in the solution generates the total accelerating or braking wheel torque. The speed-over-time control was carried out to maximize the speed at which the vehicle moves along the given curve. The speed profile generation was done in a moving prediction window, which was also used for path tracking and speed control. We present simulation results and compare them with reference objectives. We report small tracking errors between the plant and the objectives for the heading angle ‘$\psi$’, the lateral position ‘Y’ and the longitudinal position ‘X’ of vehicle center of mass (C.M.). The work presented here draws on MPC concepts introduced earlier [2], [3] and goes one step beyond the studies reported in [4], [5]. To that end, it uses three vehicle models discussed in [6].

Keywords: Model Predictive Control; Vehicle Dynamics; Path Tracking; Speed Control; Arbitrary Path/Course
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1 Introduction

The last decade has seen significant progress of autonomous vehicles, mainly fueled by advances in software and hardware on sensing, computing and control. The main benefit of autonomous driving is to improve the traffic safety and facilitate the productive use of transit time for daily commuters [7]. In an autonomous vehicle, it can be useful to divide motion control into path planning and path tracking. In this hierarchical structure, the path planner uses data from perception systems to generate a desired path (desired positions and orientations) the vehicle should follow. The path tracker then calculates and applies steering action to guide the vehicle along the path [8].

The aim of path tracking is to control the position and heading angle of an autonomous vehicle through steering. To this end, various controllers have been developed via classical control theory, modern control theory, robust control theory, etc., such as PID control, optimal control, the backstepping and sliding mode control [9]. Almost always, these control methods did not consider the actuator saturation limit and physical constraints. Model predictive control, which combines prediction models, a receding horizon optimization and a feedback correction, has been established as an attractive method to handle these additional challenges due to its consideration of input constraints and states admissible [10]. Recent work has shown that MPC can be used to control the dynamics of multiple vehicles and their safety constraints. Moreover, the stability of these algorithms is also well studied [11]. A path following MPC-based yaw and lateral stabilization scheme for obstacle avoidance maneuver using combined steering and braking was proposed in [12], in which a full tenth-order vehicle model and a simplified bicycle model were used respectively for the nonlinear model predictive control (NMPC) formulation. In [13], an NMPC approach is adopted in order to follow a given path by controlling front steering, braking and traction, while fulfilling various physical and design constraints. In order to reduce the computational burden, the NMPC is converted to a linear time-varying (LTV) MPC based on successive online linearizations.

In this technical report, assuming that the path has been planned and thus points along the reference path are given, we approximate the path curvature and generate a refined reference trajectory for which we design a minimum-time velocity profile that factors in the curvature of the reference trajectory. To explore the proposed MPC path tracking and PID speed controllers in a more general form, we consider two trajectories and take into account state constraints in the MPC formulation. In order to track the desired path as fast as possible, a longitudinal velocity profile is designed with the aim of minimizing the travel time along each of the two trajectories. The reference trajectories are shown in Fig. 1 and Fig. 2. For the lateral control, we adopted MPC and used the 8-DOF vehicle as the prediction model, and the 14-DOF model as the plant. For the longitudinal control, we used PID control to follow the desired speed. The tire model is simplified as linear tire model with the assumption of small tire slip angle and longitudinal slip ratio.
The technical report is organized as follows: In Section 2, we described the tire model, 8-DOF vehicle model (both time-dependent model and arc-length dependent model) and 14-DOF vehicle model, which were used in the MPC controller design and the plant, respectively. In Section 3, we calculated the approximated curvature based on the given points along the reference path and set up the refined reference trajectory for path tracking. In Section 4, we
adopted a forward and backward integration scheme proposed in [14] to design a minimum-time travel speed profile given a fixed path. In Section 3 we specify the MPC algorithm for path tracking and the PID controller for speed tracking. We implemented the proposed control strategy with two different trajectories and provide results in Section 6. Section 7 summarizes the highlights of this study and outlines directions of future work.

2 Vehicle Modeling

2.1 Tire Model

 Except for aerodynamics forces and gravity, all of the forces which affect vehicle behavior are provided by the tires. Because tire forces produce primary external influence and they have highly nonlinear performance, it is essential to use a realistic tire model, especially when investigating large control inputs that results in response near the limits of the linear character scale of the tire. The tire lateral and longitudinal forces are assumed to depend on normal force, slip angle, surface friction, and slip ratio. However, when the slip ratio and slip angle are limited within small values, the tire model can be simplified and generate linearized lateral force and longitudinal force [15].

So, under this assumption, the tire lateral force can be given as \( F_c = C_\alpha(\mu, F_z)\alpha \), where \( C_\alpha \) is tire cornering stiffness related to tire normal force and road-tire friction coefficient; The tire longitudinal force can be given as \( F_l = C_x(\mu, F_z)s_{f,r} \), where \( s_{f,r} \) is the slip ratio of front tire or rear tire, \( C_x \) is the tire longitudinal stiffness which also related to the tire normal force and road-tire friction coefficient.

2.2 8-DOF Vehicle Model

2.2.1 Time-dependent Vehicle Model

An 8-DOF full vehicle model is often used as a simplified lower order model for studying vehicle handling in scenarios which do not involve significant longitudinal accelerations. In this section, a formulation for the 8-DOF model, adapted from various references [16] [17], that can match the 14-DOF model reasonably accurately is presented [16].
The schematic of the 8-DOF full vehicle model is shown in Figure 3. The model has four degrees of freedom for the chassis and one degree of freedom at each of the four wheels representing the wheel spin dynamics. The chassis includes the longitudinal velocity, \( u \), the lateral velocity, \( v \), the roll angular velocity, \( \omega_x \), and the yaw angular velocity, \( \omega_z \). The pitch and heave motions are not modeled and the front and rear suspensions are represented simply by their respective equivalent roll stiffness (\( k_{\psi f}/k_{\psi r} \)) and roll damping coefficients (\( b_{\psi f}/b_{\psi r} \)) [16].

\[
m_t (\dot{u} - \omega_z v) = \sum F_{xgij} + (m_{u_f} a - m_{u_r} b)\omega_z^2 - 2h_{rc} m\omega_z \omega_x \tag{1}
\]

\[
m_t (\dot{v} + \omega_z u) = \sum F_{ygij} + (m_{ur} b - m_{uf} a)\omega_z + h_{rc} m\dot{\omega}_x \tag{2}
\]

\[
J_z \dot{\omega}_z + J_{xz} \dot{\omega}_x = (F_{yglf} + F_{ygrf}) a - (F_{yglt} + F_{ygrr}) b + \frac{(F_{xgrf} - F_{xglf}) c_f}{2} + \frac{(F_{xgrr} - F_{xglt}) c_r}{2} + (m_{ur} b - m_{uf} a) (\dot{v} + \omega_z u) \tag{3}
\]

\[
(J_x + mh_{rc}^2) \dot{\omega}_x + J_{xz} \dot{\omega}_z = mg h_{rc} \phi - (k_{\phi f} + k_{\phi r}) \phi - (b_{\phi f} + b_{\phi r}) \dot{\phi} + h_{rc} m (\dot{v} + \omega_z u), \tag{4}
\]
where

\[ h_{rc} = \frac{h_{rcf} b + h_{rcr} a}{a + b}. \]  

(5)

In these equations, the forces \( F_{xij} \) and \( F_{yij} \) are the longitudinal and lateral forces at the four tire contact patches and the subscript ‘ij’ denotes \( lf, rf, lr, \) and \( rr \). As before, \( h_{rcf} \) and \( h_{rcr} \) are the vertical distances of the front and rear roll centers below the sprung mass C.M., and thus \( h_{rc} \) is the vertical distance from the sprung mass C.M. to the vehicle roll center. It should be noted that as Eq. (4) for the roll degree of freedom is written by considering moments acting about the vehicle roll center rather than the sprung mass C.M., the roll inertia of the sprung mass about the vehicle roll center \((J_x + mh_{rc}^2)\) is considered in Eq. (4) [16].

The 8-DOF model cannot simulate vehicle behavior beyond wheel lift-off. Nevertheless, the model is valid for applications which do not involve wheel lift-off such as active steering and active throttle/brake systems for yaw control [16].

### 2.2.2 Arc Length dependent Vehicle Model

The transformation from time-dependent model to spatial-dependent model allows the position of the vehicle to be known explicitly at each sampling instant of an optimization routine, yet still retains the freedom of the solver to vary the velocity of the vehicle. Furthermore, a spatial horizon allows one to formulate obstacle constraints as simple bounds on \( e_y \) and include them in the state constraints. At each prediction step, the vehicle position along the path is known to be \((s + k \cdot ds)\). According to the position and width of the obstacle, one can determine the bounds on \( e_y \) [18].

![Figure 4: The curvilinear coordinate system][18]
Figure 4 shows the curvilinear coordinate system used in the spatial bicycle model as well as the states of the model. The dynamics are derived about a curve defining a track. The coordinate $s$ defines the arc-length along the track. The states of the spatial vehicle model are defined as $\xi^s = [\dot{y}, \dot{x}, \dot{\psi}, e_\psi, e_y]^T$. Where $e_\psi$ and $e_y$ are the error of heading angle and lateral position with respect to the arc length. The following kinematic equations can be derived from Figure 4:

$$v_s = (\rho - e_y) \cdot \dot{s}$$  \hspace{1cm} (6)  

$$v_s = \dot{x} \cdot \cos(e_\psi) - \dot{y} \cdot \sin(e_\psi).$$  \hspace{1cm} (7)  

where $v_s$ is the projected vehicle speed along direction of the path, $\rho$ and $\psi_s$ are the radius of curvature and the heading of the lane center line. $\dot{\psi}_s$ is the time derivative of $\psi_s$.

The vehicle’s velocity along the path $\dot{s} = \frac{ds}{dt}$ is then given by:

$$\dot{s} = \rho \cdot \dot{\psi}_s = \frac{\rho}{\rho - e_y} \cdot (\dot{x} \cdot \cos(e_\psi) - \dot{y} \cdot \sin(e_\psi)).$$  \hspace{1cm} (8)  

where $s$ is the projected vehicle position along the arc length of the path.

Using simple relationships in the new curvilinear coordinate system and the fact that $\frac{d\xi^s}{ds} = \frac{d\xi^s}{dt} \cdot \frac{dt}{ds}$ we can calculate the derivative of $\xi^s$ with respect to $s$ as follows: (\cdot)' represents the derivative with respect to $s$):

$$\dot{y}' = \frac{\ddot{y}}{\ddot{s}}; \quad \dot{x}' = \frac{\ddot{x}}{\ddot{s}}; \quad \dot{\psi}' = \frac{\ddot{\psi}}{\ddot{s}};$$  \hspace{1cm} (9)  

$$e_\psi' = (\psi - \psi_s)' = \frac{\ddot{\psi}}{\ddot{s}} - \dot{\psi}_s';$$  \hspace{1cm} (10)  

$$e_y' = \dot{e}_y = \frac{\ddot{e}_y}{\ddot{s}} = (\dot{x} \cdot \sin(e_\psi) + \dot{y} \cdot \cos(e_\psi)) / \dot{s},$$  \hspace{1cm} (11)  

where $\ddot{y}$, $\ddot{x}$ and $\ddot{\psi}$ are computed from a bicycle model, a simplified version of the four wheel model. The road information $\dot{\psi}_s$ is assumed to be known.

The spatial bicycle vehicle dynamics then can be formulated as:

$$\xi^s'(s) = f^s (\xi^s(s), u^s(s)).$$  \hspace{1cm} (12)  

We apply this transformation method to the 8-DOF model, define the states of the spatial vehicle model as $\xi^s = [\dot{y}, \dot{x}, \dot{\phi}, \dot{\psi}, \dot{\psi}, X, Y, e_\psi, e_y]$, and denote $\kappa(s)$ as the curvature along the arc length of the path. Then the vehicle’s velocity along the path is then given as:

$$\dot{s} = \frac{\dot{x} \cdot \cos(e_\psi) - \dot{y} \cdot \sin(e_\psi)}{1 - \kappa(s)e_y}.$$  \hspace{1cm} (13)  

Using simple relationships in the new curvilinear coordinate system and the fact that $\frac{d\xi^s}{ds} = \frac{d\xi^s}{dt} \cdot \frac{dt}{ds}$, we can also calculate the derivative of $\xi^s$ with respect to $s$ as follows:

$$\dot{x}' = \frac{\ddot{x}}{\ddot{s}}; \quad \dot{y}' = \frac{\ddot{y}}{\ddot{s}}; \quad \phi' = \frac{\ddot{\phi}}{\ddot{s}}; \quad \psi' = \frac{\ddot{\psi}}{\ddot{s}};$$  \hspace{1cm} (14a)  

$$\dot{\phi}' = \frac{\ddot{\phi}}{\ddot{s}}; \quad \dot{\psi}' = \frac{\ddot{\psi}}{\ddot{s}}; \quad X' = \frac{\ddot{X}}{\ddot{s}}; \quad Y' = \frac{\ddot{Y}}{\ddot{s}};$$  \hspace{1cm} (14b)  

$$e_\psi' = \frac{\ddot{e}_\psi}{\ddot{s}}; \quad e_y' = \frac{\ddot{e}_y}{\ddot{s}} = (\dot{x} \cdot \sin(e_\psi) + \dot{y} \cdot \cos(e_\psi)) / \dot{s},$$  \hspace{1cm} (14c)
where $\ddot{y}$, $\ddot{x}$, $\dot{\phi}$, $\dot{\psi}$, $\ddot{\phi}$, $\ddot{\psi}$, $\dot{X}$ and $\dot{Y}$ are computed from the 8-DOF vehicle model. The road information $(\dot{\psi}_s)'$ is assumed to be known.

The 8-DOF vehicle model then can be formulated as

$$\xi^s'(s) = f^s(\xi^s(s), u^s(s)).$$ (15)

Note as function of arc length $s$, the time can be retrieved by integrating $t' = 1/\dot{s}$ [18].

### 2.3 14-DOF Vehicle Model

In order to better represent the vehicle lateral and yaw dynamics as well as coupling of yaw-roll motion due to the transient lateral load transfer during extreme maneuvers, higher order model such as 8-DOF and 14-DOF are also used in rollover studies. A 14-DOF vehicle model, which considers the suspension at each corner, has the same benefits of an 8-DOF vehicle model, with the additional capabilities of predicting vehicle pitch and heave motions. It also offers the flexibility of modeling nonlinear springs and dampers and can simulate the vehicle responses to normal force inputs in the case of an active suspension system. Moreover, the 14-DOF model, unlike the 8-DOF model, can predict vehicle behavior even after wheel lift-off and thus can be used in rollover prediction/prevention strategies [16].

Figure 5 shows the schematic of the two axle, 14-DOF vehicle model used to investigate vehicle roll response to steering and torque inputs. This schematic includes 6 DOF at the vehicle lumped mass center of mass and 2 DOF at each of the four wheels, including vertical suspension travel and wheel spin. The body is modeled as being rigid, with body-fixed coordinates, $(xyz)$, attached at the center of mass and aligned in principal directions (coordinate frame 1). $u, v, w$ indicate forward, lateral, and vertical velocities, respectively, of the sprung mass [16].

Figure 5: Schematic of 14-DOF vehicle model with one-dimensional suspension and coordinate frames [16]
The equation of motion for the 6 DOF of the sprung mass model can now be derived from the direct application of Newton’s laws for the system as \[16\]

\[
m(\ddot{u} + \omega_y w - \omega_z v) = \sum (F_{xij}) + mgsin\theta
\]

\[
m(\dot{v} + \omega_z u - \omega_x w) = \sum (F_{yij}) - mgsin\phi\cos\theta
\]

\[
m(\dot{w} + \omega_x v - \omega_y u) = \sum (F_{zij} + F_{dzij}) - mc_s\cos\phi\cos\theta
\]

\[
J_x\dot{\omega}_x = \sum (M_{xij}) + \frac{(F_{zslf} - F_{zsr})c_f + (F_{zsl} - F_{zsr})c_r}{2}
\]

\[
J_y\dot{\omega}_y = \sum (M_{yij}) + (F_{zsl} + F_{zsr})b - (F_{zslf} + F_{zsr})a
\]

\[
J_z\dot{\omega}_z = \sum (M_{zij}) + (F_{yslf} + F_{ysr})a - (F_{ysl} + F_{ysr})b + \frac{(-F_{zsl} + F_{zsr})c_f + (-F_{zslf} + F_{zsr})c_r}{2},
\]

where \(m\) is the sprung mass and the subscript ‘\(ij\)’ denotes left front (lf), right front (rf), left rear (lr), and right rear (rr).

The cardan angles \(\theta, \psi, \phi\) needed in the aforementioned equations are obtained by performing the integration of the following equations \[16\],

\[
\dot{\theta} = \omega_y\cos\phi - \omega_z\sin\phi
\]

\[
\dot{\psi} = \frac{\omega_y\sin\phi}{\cos\theta} + \frac{\omega_z\cos\phi}{\cos\theta}
\]

\[
\dot{\phi} = \omega_x + \omega_y\sin\phi\tan\theta + \omega_z\cos\phi\tan\theta.
\]

More details about the development on 14-DOF vehicle model is specified in techReport \[6\].

### 2.4 Wheel Rotational Dynamics

In the case that the vehicle is front wheel driven, the rotational dynamics for the front right and rear left wheel can be given as \[17\][19][20],

\[
J_w\dot{\omega}_{rf} = T_{arf} - T_{brf} - F_{xtrf}R
\]

\[
J_w\dot{\omega}_{rr} = -T_{brr} - F_{xtrr}R,
\]

where \(T_{arf}, T_{brf}\) is the acceleration torque and braking torque applied to the front wheel. \(T_{brr}\) is the braking torque applied to the rear wheel.

Then, we reconsider the longitudinal vehicle dynamics equation in the plant, 14-DOF vehicle model. The longitudinal model considered here is based on one wheel vehicle model. The sum of the longitudinal forces acting on the vehicle C.M. is given by \[19\][20]:

\[
m(\ddot{u} + \omega_y w - \omega_z v) = F_p - F_r + mgsin\theta,
\]
where $v$ is the vehicle speed, $F_p$ is the propelling force and $F_r$ is the sum of resisting forces. The propelling force $F_p$ is the controlled input resulting from brake and throttle actions. The sum of the resisting force $F_r$ is given by \[ F_r = F_a + F_g + F_{rr}, \] (28)

where $F_a$ is the aerodynamic force, $F_g$ is the gravitational force and $F_{rr}$ is the rolling resistance force. The form of the wheel dynamics has been slightly modified to distinguish the total brake torque $T_b$ and the total traction torque $T_a$ as follows \[ I_w \dot{\omega} = -F_l R + T_a - T_b. \] (29)

For longitudinal controller synthesis, the following simplifying assumptions are considered: for a non-slip rolling then the following relationship hold \[ u = R \omega \]

\begin{equation}
F_p = F_l = \frac{1}{R} (T_a - T_b - I_w \dot{\omega}) = \frac{T_a - T_b}{R} - \frac{I_w}{R^2} \ddot{u}.
\end{equation}

(31)

For simplicity, the sum of the resisting force in the longitudinal dynamics is ignored, i.e., $F_r = 0$, and the acceleration torque is divided equally and applied to the front left and right wheels, the braking torque is divided equally and applied to the front and rear wheels, (i.e., $T_{afl} = T_{arf} = 0.5T_a$, $T_{blf} = T_{brf} = T_{blr} = T_{brr} = 0.25T_b$) so the vehicle longitudinal dynamics equation (27) for the plant becomes:

\[ \dot{u} = \frac{1}{m + \frac{J_w}{r_{lf}^2} + \frac{J_w}{r_{rf}^2} + \frac{J_w}{r_{lr}^2} + \frac{J_w}{r_{rr}^2}} \left( \frac{0.5T_a - 0.25T_b}{r_{lf}} + \frac{0.5T_a - 0.25T_b}{r_{rf}} + \frac{-0.25T_b}{r_{lr}} \right) + \frac{-0.25T_b}{r_{rr}} - m\omega_y w + m\omega_z v + mgsin\theta \].

(32)

3 Reference Path Generation from Given Points

The nominal vehicle path can be as simple as road center line or be a very precise sequence of desired positions from a higher level path planner using its understanding of the vehicle’s behavior and environment. This path is then passed to the controller as a reference to track. Nominal paths here are defined in terms of curvature $K$ and arc lengths. $E(s)$ and $N(s)$ are defined to be the East and North measures of the position of the path at $s$ from a local datum. The path’s heading $\psi_r(s)$ is defined to be the angle from North to a vector parallel to the path at $s$. Curvature $K$ is related to path’s heading $\psi_r$ and position $(E, N)$ through
the following differential equations [8]:

\[
\frac{d\psi_r}{ds} = K(s) \tag{33}
\]

\[
\frac{dE}{ds} = \cos(\psi_r) \tag{34}
\]

\[
\frac{dN}{ds} = \sin(\psi_r) \tag{35}
\]

The curvature \( K \), path’s heading \( \psi_r \) and path’s position in the global inertial coordinate system are provided in terms of arc length \( s \). The equations listed above can be also referred in [21], which shows that the vehicle position state is determined by the curvature. Thus for the vehicle trajectory planner curvature can be the only controlled variable. In other words, the local trajectory can be carried out by determining the curvature [21]. A common controller approach is to split up path tracking and velocity tracking. After comparing geometric, kinematic and dynamic path trackers, it concluded that they all require at least curvature-continuous path reference. So, choosing the polynomial curvature spiral as the path primitive type, the trajectory generation can be formulated as follows [22]:

\[
x(s) = \int_0^s \cos(\psi(s)) \cdot ds \tag{36}
\]

\[
y(s) = \int_0^s \sin(\psi(s)) \cdot ds \tag{37}
\]

\[
\psi(s) = \int_0^s \kappa(s) \cdot ds \tag{38}
\]

where, \((x, y), \psi\) and \(\kappa\) specify position, heading and curvature (implies its steering angle) respectively.

Since the vehicle position state is determined by the curvature, the curvature function of a curve is required and defined as: [23]:

\[
s \text{ be the arc length of the track, } x(s) \text{ and } y(s) \text{ the coordinates on the map of the points } x' = \frac{dx}{ds} \text{ and } y' = \frac{dy}{ds} \text{ be the spatial derivatives at point } s. \text{ Then, the curvature of the path at point } s \text{ is given by}
\]

\[
\kappa = \frac{(x' y'' - y' x'')}{{\sqrt{x'^2 + y'^2}}}. \tag{39}
\]

The curvature can be approximated using finite differences of the sample waypoints as [24].

\[
\kappa_i \approx \frac{x'_i y''_i - y'_i x''_i}{{\sqrt{x'^2_i + y'^2_i}}} \tag{40}
\]
where

\[
x_i' \approx \frac{x_{i+1} - x_i}{t} \quad (41)
\]

\[
x_i'' \approx \frac{x_{i+2} - 2x_{i+1} + x_i}{t^2} \quad (42)
\]

\[
y_i' \approx \frac{y_{i+1} - y_i}{t} \quad (43)
\]

\[
y_i'' \approx \frac{y_{i+2} - 2y_{i+1} + y_i}{t^2} \quad (44)
\]

Likewise, the equations can be expressed with respect to arc length \(s\):

\[
x(s)_{i}' \approx \frac{x(s)_{i+1} - x(s)_i}{\Delta s} \quad (45)
\]

\[
x(s)_{i}'' \approx \frac{x(s)_{i+2} - 2x(s)_{i+1} + x(s)_i}{\Delta s^2} \quad (46)
\]

\[
y(s)_{i}' \approx \frac{y(s)_{i+1} - y(s)_i}{\Delta s} \quad (47)
\]

\[
y(s)_{i}'' \approx \frac{y(s)_{i+2} - 2y(s)_{i+1} + y(s)_i}{\Delta s^2} \quad (48)
\]

Hence, the reference trajectory and heading angle can be generated by integration from the curvature profile with given conditions.

## 4 Speed Profile Generation Given Fixed Reference Path

The speed profile design is to plan a longitudinal velocity by seeking a minimum-time motion and exploring the dynamic capabilities of a vehicle to travel on a given path. Many methods have been introduced in the literature to solve the minimum-time travel problem. A velocity planning scheme that achieves a true global minimum-time along with smooth motion and arbitrary boundary conditions was proposed in [25]. This planning can be easily executed in real-time because the devised Minimum-Time Velocity Planning (MTVP) algorithm relies on the solution of an algebraic quartic equation whose roots can be straightforwardly computed by well-known closed-form expressions. The algebraic solution shows that the addressed MTVP problem can be reduced to the problem of determining the positive real roots of a quartic equation whose coefficients depend on the problem data, i.e. the path length, the jerk bound, and the velocities and accelerations at the endpoints of the planned time interval [25].

While the task of minimum-time speed profile design has been recently formulated as a convex optimization problem, given a fixed reference path, a forward-backward integration scheme to determine the minimum-time longitudinal speed profile, subject to tire friction constraints was proposed in [14] instead. Firstly, given the reference path described by the arc length \(s\) and the curvature \(K\), the minimum-time speed profile the vehicle can achieve should not exceed the available tire-road friction. Given the lumped front and rear tires,
the available longitudinal force $F_x$ and lateral force $F_y$ at each wheel are constrained by the friction circle \[14\],

\[
F_{xf}^2 + F_{yf}^2 \leq (\mu F_z f)^2
\]

\[
F_{xr}^2 + F_{yr}^2 \leq (\mu F_z r)^2,
\]

where $\mu$ is the tire-road friction coefficient and $F_z$ is the available normal force. Given zero longitudinal force, the maximum permissible steady-state vehicle speed is given by \[14\]

\[
U_x(s) = \sqrt{\frac{\mu g}{|K(s)|}},
\]

where the weight transfer and topography effects are neglected for simplified case. The results are obtained by setting $F_y f = \frac{mb}{(a + b)} U_x^2 K$ and $F_z f = mg/(a + b)$. Furthermore, the vehicle speed is determined by the road environment, and the most important factor is the curvature of the road. We assumed that the tire model was linear and the vehicle lateral acceleration was small. According to the curvature of the road, the reference speed of the vehicle is calculated by \[19\]

\[
U_x(s) = \sqrt{0.4 \mu g |K(s)|}.
\]

Secondly, a forward integration step occurs, where the velocity of a given point is determined by the velocity of the previous point and the available longitudinal force $F_{x,max}$ for acceleration \[14\],

\[
U_x(s + \Delta s) = \sqrt{U_x^2(s) + 2 \frac{F_{x,accel,max}}{m} \Delta s}.
\]

A key point of the forward integration step is that at every point, the value of $U_x(s)$ is compared to the corresponding value from Eq.(51), and the minimum value is taken \[14\].

Thirdly, a backward integration step occurs, where the available longitudinal force for deceleration is again constrained by the lateral force demand on all tires \[14\],

\[
U_x(s - \Delta s) = \sqrt{U_x^2(s) - 2 \frac{F_{x,decel,max}}{m} \Delta s}.
\]

The value of $U_x(s)$ is then compared to the corresponding value from Eq.(53) for each point along the path, and the minimum value is chosen, resulting in the desired velocity profile \[14\].

The method determines the normal and lateral tire forces $F_z$ and $F_y$ at each point along the path by accounting for weight transfer, road bank, and grade. The lap time $t$ for a given racing line is provided by the following equation \[14\]:

\[
t = \int_0^L \frac{ds}{U_x(s)}.
\]
Eq. (55) implies that minimizing the vehicle lap time requires simultaneously minimizing the total path length $L$ while maximizing the vehicle’s longitudinal velocity $U_x$. 

In this report, the length $L$ is determined and known since the desired path is fixed and given. Hence, to minimize the travel time along the curved path, we need to find out the maximum longitudinal velocity at each point along the path while taking into account the constraint of tire-road friction. Additionally, we assume the vehicle will arrive at the initially reference speed before MPC path tracking and PID speed control implementation.

5 Controller Design

Model predictive control has its roots in optimal control. The basic concept of MPC is to use a dynamic model to forecast system behavior, and optimize the control move at the current time to produce the best performance in the future [26].

The scheme of MPC-based path tracking with PID speed control is adopted for the controller design. The control objective is to track a given reference path and speed by a combined use of steering, accelerating and braking.

To simplify the controller design, in every simulation step, the combined lateral/longitudinal control problem is replaced by two sequential subproblems where the optimal steering control input is computed by MPC controller assuming given a constant speed, and then the wheel accelerating or braking torque is computed by PID controller through speed comparison between the plant and the reference, and then the calculated optimal front wheel steering angle and total wheel torque are passed to the plant simultaneously to calculate the plant state and implement the path tracking and speed control.

5.1 Lateral Dynamics Control: MPC Controller

The design concept of the MPC controller is to convert an actual control problem into an objective function with constraints. Optimizing the objective function, we can obtain the control sequences in a future horizon section. The first element of them is applied to the plant. The MPC controller is designed based on the linear error model. The objective function is linearized and converted into a quadratic program (QP), which can be easier to calculate [10].

5.1.1 Linearization of the Vehicle Model

We set the control input $u = \delta$ and the state variable $\chi = [\dot{x}, \dot{y}, \phi, \dot{\phi}, \psi, \dot{\psi}, Y, X]^T$, then the general form of the system can be given as

$$\dot{\chi} = f(\chi, u).$$

The general form around the operating point is

$$\dot{\chi}_o = f(\chi_o, u_o).$$
Using the Taylor series expansion at the operating point and ignoring higher order terms, we can obtain
\[
\dot{\chi} = f(\chi_o, u_o) + \frac{\partial f(\chi, u)}{\partial \chi} \bigg|_{\chi = \chi_o, u = u_o} (\chi - \chi_o) + \frac{\partial f(\chi, u)}{\partial u} \bigg|_{\chi = \chi_o, u = u_o} (u - u_o).
\]
(58)

subtracting Eq. (57) from Eq. (58) results in
\[
\dot{\tilde{\chi}} = A \tilde{\chi} + B \tilde{u},
\]
(59)

where
\[
A = \frac{\partial f(\chi, u)}{\partial \chi} \bigg|_{\chi = \chi_o, u = u_o}, \quad B = \frac{\partial f(\chi, u)}{\partial u} \bigg|_{\chi = \chi_o, u = u_o}, \quad \dot{\tilde{\chi}} = \dot{\chi} - \dot{\chi}_o, \quad \tilde{\chi} = \chi - \chi_o, \quad \text{and} \quad \tilde{u} = u - u_o.
\]

Eq. (59) is the linear error model [10].

In order to apply the model to the design of the MPC controller, the linear error model is discretized in the form of state-space representation [10] [27]
\[
\tilde{\chi}(k+1) = A_d \tilde{\chi}(k) + B_d \tilde{u}(k),
\]
(60)

where
\[
A_d = I + TA, \quad B_d = TB \quad \text{and} \quad T \text{ is the sampling time.}
\]

The Eq. (60) can be also given in the following form for the MPC controller design [10]
\[
\chi(k+1) = A_d \chi(k) + B_d u(k) + d_k(k),
\]
(61)

where
\[
d_k(k) = f(\chi_o(k), u_o(k)) - (A_d \chi_o(k) + B_d u_o(k)).
\]

5.1.2 State Prediction

We define \( \xi(k) = \begin{bmatrix} \chi(k) \\ u(k-1) \end{bmatrix} \) as the new state variable, \( \eta(k) \) as the output state variable, and \( \Delta u(k) = u(k) - u(k-1) \) as the control input increment. Then, the discrete state-space controller model can be translated into a new form as follows [10] [27]
\[
\xi(k+1) = \tilde{A}_d \xi(k) + \tilde{B}_d \Delta u(k) + \tilde{d}_k(k)
\]
(62a)
\[
\eta(k) = \tilde{C}_d \xi(k),
\]
(62b)

where \( \tilde{A}_d = \begin{bmatrix} A_d & B_d \\ 0_{m\times n} & I_m \end{bmatrix}, \quad \tilde{B}_d = \begin{bmatrix} B_d \\ I_m \end{bmatrix}, \quad \tilde{d}_k(k) = \begin{bmatrix} d_k(k) \\ 0_m \end{bmatrix}, \quad \tilde{C}_d = \begin{bmatrix} C_d & 0_{p\times m} \end{bmatrix} \) (\( m \) denotes the dimension of control input, \( n \) denotes the dimension of state variable, and \( p \) denotes the dimension of output) [10].

We denote the sequence increment of future control input computed at time \( k \) as \( \Delta U_m \), that is \( \Delta U_m = [\Delta u(k), \ldots, \Delta u(k+m), \ldots, \Delta u(k+N_c-1)]^T \). The control input varies for \( N_c \) time steps (i.e., the control horizon) and then is held constant up to the preview horizon. We define the predicted output for the predictive state-space model as \( \eta_m(k) = [\eta(k+1), \ldots, \eta(k+N_p)]^T \). In this situation, it is straightforward to derive the prediction
model of performance outputs over the prediction horizon $N_p$ in a compact matrix form as

$$\eta_m(k) = \Theta_m \xi(k) + \Gamma_m \Delta U_m + \Psi_m D_k,$$

where

$$\Theta_m = \begin{bmatrix} \tilde{C}_d \tilde{A}_d & \tilde{C}_d \tilde{A}_d^2 & \ldots & \tilde{C}_d \tilde{A}_d^{N_c} & \ldots & \tilde{C}_d \tilde{A}_d^{N_p} \end{bmatrix}^T$$  

$$\eta_m(k) = [\eta(k+1) \ldots \eta(k+N_p)]^T$$  

$$\Delta U_m = [\Delta u(k) \ldots \Delta u(k+m) \ldots \Delta u(k+N_c - 1)]^T$$  

$$D_k = [\tilde{d}_k(k) \tilde{d}_k(k+1) \ldots \tilde{d}_k(k+N_p - 1)]^T$$  

5.1.3 Cost Function Definition

The objective function of the path tracking controller can be given as

$$J(k) = \sum_{j=1}^{N_p} \tilde{\chi}^T(k + j) Q \tilde{\chi}(k + j) + \tilde{u}^T(k + j - 1) R \tilde{u}(k + j - 1),$$

where $Q$ and $R$ represent weight matrices, where the $\tilde{\chi} = \chi - \chi_o$ and $\tilde{u} = u - u_o$. The first term in Eq.(70) reflects the capability of tracking performance, while the second reflects the constraint on the change of the control output.

Considering the soft constraints concept, we can get an alternative form of the objective function as follows:

$$J(k) = \sum_{i=1}^{N_p} \|\eta(k + i|t) - \eta_{ref}(k + i|t)\|^2_Q + \sum_{i=1}^{N_c} \|\Delta U(k + i|t)\|^2_R + \rho \varepsilon^2.$$
Considering Eq. (63), the objective function can be given as

\[ J(k) = \left( \Theta_m \xi(k) + \Gamma_m \Delta U_m + \Psi_m \tilde{d}_k - \eta_{ref} \right)^T Q \left( \Theta_m \xi(k) + \Gamma_m \Delta U_m + \Psi_m \tilde{d}_k - \eta_{ref} \right) + \Delta U_m^T R \Delta U_m + \rho \varepsilon^2. \]  

(72)

To solve the following optimization problem, the objective function is converted into a standard quadratic form [10].

\[ J(\xi(t), u(t - 1), \Delta U(t)) = [\Delta U(t)^T, \varepsilon]^T H_t [\Delta U(t)^T, \varepsilon] + G_t [\Delta U(t)^T, \varepsilon], \]  

(73)

where \( H_t = \begin{bmatrix} \Gamma_m^T Q \Gamma_m + R & 0 \\ 0 & \rho \end{bmatrix} \), \( G_t = \begin{bmatrix} 2e_t^T Q \Gamma_m & 0 \end{bmatrix} \), \( e_t = (\Theta_m \xi(k) + \Psi_m \tilde{d}_k - \eta_{ref}) \) and \( e_t \) is the tracking error in the predictive horizon [10].

After obtaining the solution to Eq. (73) in each control cycle, a series of control input increments in the control horizon can be calculated as [10]

\[ \Delta U^*_t = [\Delta u^*_t, \Delta u^*_{t+1}, \ldots, \Delta u^*_{k+N_c-1}]^T. \]  

(74)

The first element of the control sequences is taken as the actual control input increment of the controller [10].

\[ u(t) = u(t - 1) + \Delta u^*_t. \]  

(75)

### 5.1.4 Constraint Analysis

There are constraints imposed on the control increment and the outputs. Specifically, the vehicle dynamics model, the constraints imposed on steering angle for the path-tracking problem are specified as [10]

1. The constraints for control can be given as

\[ u_{\text{min}}(t + k) \leq u(t + k) \leq u_{\text{max}}(t + k), \quad k = 0, 1, \ldots, N_c - 1 \]  

(76)

2. The constraints for control increments can be given as

\[ \Delta u_{\text{min}}(t + k) \leq \Delta u(t + k) \leq \Delta u_{\text{max}}(t + k), \quad k = 0, 1, \ldots, N_c - 1 \]  

(77)

3. The output constraints can be given as

\[ y_{\text{min}}(t + k) \leq y(t + k) \leq y_{\text{max}}(t + k), \quad k = 0, 1, \ldots, N_c - 1 \]  

(78)

In the cost function, the variables to be solved are control increments in the control horizon. The constraint conditions can only be expressed as a form of the control increment or a form of the control increment multiplied by the transformation matrix. Thus, Eq. (76)
through (78) need to be converted for obtaining the corresponding transformation matrix. The relationship is defined as [10]

\[ u(t + k) = u(t + k - 1) + \Delta u(t + k), \]  

assuming that

\[ U_t = 1_{N_c} \otimes u(k - 1) \]  

(80)

\[ A = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & \cdots & \cdots & 1 \\
\end{bmatrix} \otimes I_m \]  

(81)

where \( 1_{N_c} \) is the column vector of \( N_c \) ones, \( I_m \) is the identity matrix with a dimension of \( m \), \( \otimes \) is Kronecker product, and \( u(k - 1) \) is the previous control input [10].

Combining Eq. (79) through (81) can be converted into [10]

\[ U_{min} \leq A \Delta U_t + U_t \leq U_{max}, \]  

(82)

where \( U_{min} \) and \( U_{max} \) are the lower and upper bound of the control input, respectively.

Combining Eq. (63) and Eq. (78), the output constraints can be given as:

\[ Y_{min} \leq \Theta_m \xi(k) + \Gamma_m \Delta U_m + \Psi_mD_k \leq Y_{max}, \]  

(83)

This concludes the setting up of the constraint optimization problem induced by the model prediction control algorithm [10].

5.2 Longitudinal Dynamics Control: PID Controller

5.2.1 Longitudinal Dynamics Equation

The wheel rotational dynamics of front right and rear right wheel in the plant are given as,

\[ J_w\dot{\omega}_{rf} = 0.5T_a - 0.25T_b - r_{rf}F_{xtrf}, \]  

(84)

\[ J_w\dot{\omega}_{rr} = -0.25T_b - r_{rr}F_{xtrr}, \]  

(85)

where the \( T_a \) is total driving torque applied to the vehicle, \( T_b \) is the total braking torque applied to the vehicle. With these wheel rotational equations and simplification in Section
the longitudinal equation for the plant is given as

\[
\dot{u} = \frac{1}{m + \frac{J_w}{r_{lf}^2} + \frac{J_w}{r_{lf}^2} + \frac{J_w}{r_{tr}^2} + \frac{J_w}{r_{rr}^2}} \left( (0.5T_a - 0.25T_b) \frac{r_{lf}}{r_{lf}} + (0.5T_a - 0.25T_b) \frac{r_{rf}}{r_{rf}} + (-0.25T_b) \frac{r_{rr}}{r_{rr}} - m\omega_y w + m\omega_x v + mg\sin\theta \right). \tag{86}
\]

The longitudinal velocity and acceleration of the plant vehicle can be obtained from the above equation, which will be used in the PID speed controller.

### 5.2.2 PID Controller

More than half of the industrial controllers in use today are PID controllers or modified PID controllers. The usefulness of PID controls lies in their general applicability to most control systems. Figure 6 shows a PID control of a plant [28].

![Figure 6: PID control of a plant](image)

The speed tracking error is defined as

\[
e_u = u_d - u \tag{87}
\]

\[
e_{\dot{u}} = \dot{u}_d - \dot{u} \tag{88}
\]

\[
e_{s_s} = \int_{t_1}^{t_2} e_u dt = e_u T_s, \tag{89}
\]

where \(u_d\) refers to the desired speed, \(u\) refers to the speed of the plant; \(\dot{u}_d\) refers to the desired longitudinal acceleration, \(\dot{u}\) refers to the longitudinal acceleration of the plant. \(T_s\) is the sample time and when using the arc length dependent model, \(T_s = \Delta s / \dot{s}\).

In the PID controller, when the speed of the plant is smaller than the desired speed, then \(T_a = K_P e_u + K_I e_{s_s} + K_D e_{\dot{u}}, \quad T_b = 0\); when the speed of the plant is greater than the desired speed, then \(T_b = K_P e_u + K_I e_{s_s} + K_D e_{\dot{u}}, \quad T_a = 0\); when the speed of the plant equals to the desired speed, then \(T_a = 0, \quad T_b = 0\).

### 6 Simulation Results

#### 6.1 Model Parameters

The parameters for these vehicle models are listed as below:
Table 1: Parameters for the vehicle dynamics modeling

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>Sprung mass</td>
<td>1400(kg)</td>
</tr>
<tr>
<td>$J_x$</td>
<td>Sprung mass roll moment of inertia</td>
<td>900(kgm$^2$)</td>
</tr>
<tr>
<td>$J_y$</td>
<td>Sprung mass yaw moment of inertia</td>
<td>2000(kgm$^2$)</td>
</tr>
<tr>
<td>$J_z$</td>
<td>Sprung mass pitch moment of inertia</td>
<td>2420(kgm$^2$)</td>
</tr>
<tr>
<td>$a$</td>
<td>Distance of sprung mass C.M. from front axle</td>
<td>1.14(m)</td>
</tr>
<tr>
<td>$b$</td>
<td>Distance of sprung mass C.M. from rear axle</td>
<td>1.4(m)</td>
</tr>
<tr>
<td>$h$</td>
<td>Sprung mass C.M. height</td>
<td>0.75(m)</td>
</tr>
<tr>
<td>$c_f$</td>
<td>Front track width</td>
<td>1.5 (m)</td>
</tr>
<tr>
<td>$c_r$</td>
<td>Rear track width</td>
<td>1.5 (m)</td>
</tr>
<tr>
<td>$k_{sf}$</td>
<td>Front suspension stiffness</td>
<td>35000 (N/m)</td>
</tr>
<tr>
<td>$k_{sr}$</td>
<td>Rear suspension stiffness</td>
<td>30000 (N/m)</td>
</tr>
<tr>
<td>$b_{sf}$</td>
<td>Front suspension damping coefficient</td>
<td>2500 (Ns/m)</td>
</tr>
<tr>
<td>$b_{sr}$</td>
<td>Rear suspension damping coefficient</td>
<td>2000 (Ns/m)</td>
</tr>
<tr>
<td>$m_{uf}$</td>
<td>Front unsprung mass</td>
<td>80 (kg)</td>
</tr>
<tr>
<td>$m_{ur}$</td>
<td>Rear unsprung mass</td>
<td>80 (kg)</td>
</tr>
<tr>
<td>$k_{tf}$</td>
<td>Front tire stiffness</td>
<td>200000 (N/m)</td>
</tr>
<tr>
<td>$k_{tr}$</td>
<td>Rear tire stiffness</td>
<td>200000 (N/m)</td>
</tr>
<tr>
<td>$C_{αf}$</td>
<td>Front right tire cornering stiffness</td>
<td>44000 (N/rad)</td>
</tr>
<tr>
<td>$C_{αr}$</td>
<td>Rear right tire cornering stiffness</td>
<td>47000 (N/rad)</td>
</tr>
<tr>
<td>$C_{xf}$</td>
<td>Front right tire longitudinal stiffness</td>
<td>5000 (N)</td>
</tr>
<tr>
<td>$C_{xr}$</td>
<td>Rear right tire longitudinal stiffness</td>
<td>5000 (N)</td>
</tr>
<tr>
<td>$r_0$</td>
<td>Nominal tire radius</td>
<td>0.285 (m)</td>
</tr>
<tr>
<td>$J_w$</td>
<td>Tire/wheel roll inertia</td>
<td>1 (kgm$^2$)</td>
</tr>
<tr>
<td>$h_{rcf}$</td>
<td>Front roll center distance below sprung mass C.M.</td>
<td>0.65 (m)</td>
</tr>
<tr>
<td>$h_{rcr}$</td>
<td>Rear roll center distance below sprung mass C.M.</td>
<td>0.6 (m)</td>
</tr>
<tr>
<td>$μ$</td>
<td>Tire-road friction coefficient</td>
<td>0.9</td>
</tr>
</tbody>
</table>

6.2 Tracking Objectives Definition

We implemented the simulation of MPC-based path tracking with PID speed control in two cases considering time-minimum speed profile design. In case A, the reference trajectory and curvature are shown in Fig. 7 and Fig. 8. In case B, the reference trajectory and curvature are shown in Fig. 9 and Fig. 10.
Figure 7: Tracking path: Case A

Figure 8: Reference curvature profile: Case A
In the model predictive path tracking controller, we chose the heading angle \( \psi \), the lateral position \( Y \) and the longitudinal position of vehicle C.M. in global coordinates as the objectives. The reference lateral displacement \( Y \) of vehicle C.M., the reference longitudinal displacement \( X \) of vehicle C.M. and the reference heading angle \( \psi \) are given as follows:
Figure 11: Reference lateral position of vehicle C.M. (Case A)

Figure 12: Reference longitudinal position of vehicle C.M. (Case A)
Figure 13: Reference heading angle (Case A)

Figure 14: Reference lateral position of vehicle C.M. (Case B)
Figure 15: Reference longitudinal position of vehicle C.M. (Case B)

Figure 16: Reference heading angle (Case B)
6.3 Path tracking and speed control considering minimum-time speed profile along the entire path

6.3.1 Case A

**Minimum-time travel speed profile generation** Given the path curvature, we generate a speed profile using the approach according to Section 3 for the vehicle to track the path as fast as possible.

The first step of speed profile generation aims to find the maximum permissible steady-state vehicle speed given zero longitudinal force. There is a boundary limit $35m/s$ for the longitudinal velocity. The next step is a forward integration step, where the velocity of a given point is determined by the velocity of the previous points and the available longitudinal force for acceleration constrained by the lateral force. The third step is a backward integration step, where the available longitudinal force for deceleration is again constrained by the lateral force demand on all tires. In the forward and backward integration step, the value of $U_x(s)$ is compared to the corresponding value in the Eq. (51) and Eq. (53) respectively and the minimum value is taken. The reference speed is the one in red solid line shown in Fig. 17.

![Image of speed profile generation](image)

**Figure 17: Speed profile generation**

**Constraints definition** According to the tire modeling, the relationship between the tire slip angle and cornering force is linear when the tire slip angle is small. With small angle assumption for the linear tire models, the front road wheel steering angle is limited as $-10^\circ \leq \alpha \leq 10^\circ$, and the control increment is limited in $-4.25^\circ \leq \Delta \alpha \leq 4.25^\circ$.

The constraint imposed on the lateral position of the vehicle C.M. in global coordinates is given as $-100(m) \leq Y \leq 100(m)$; The constraint imposed on the heading angle of the vehicle...
is given as $-3(rad) \leq \psi \leq 3(rad)$; The constraint imposed on the longitudinal position of the vehicle C.M. is given as $-60(m) \leq X \leq 800(m)$.

**Path tracking and speed control results** After designing the minimum-time travel speed profile, we used the MPC-based path tracking with PID speed control to follow the given path and achieve the designed speed profile. The results are shown as follows:
Figure 20: Comparison of vehicle C.M. lateral position

Figure 21: Comparison of vehicle heading angle
Figure 22: Comparison of vehicle C.M. longitudinal position

Figure 23: Steering angle input
6.3.2 Case B

Minimum-time travel speed profile generation  Given the reference trajectory and curvature, we designed a speed profile according to Section 3 for the vehicle to track the path as fast as possible. The curvature of the given path is shown in Fig. 10. The reference speed profile for case B is the one in red solid line shown in Fig. 25.
**Constraints definition**  For the constraints on the control input signal, with small angle assumption for the linear tire models, the front road wheel steering angle is limited as $-10^\circ \leq \alpha \leq 10^\circ$, and the control increment is limited in $-4.25^\circ \leq \Delta \alpha \leq 4.25^\circ$.

The constraint imposed on the lateral position of the vehicle C.M. in global coordinates is given as $-100(m) \leq Y \leq 10(m)$; The constraint imposed on the heading angle of the vehicle is given as $-6(rad) \leq \psi \leq 3(rad)$; The constraint imposed on the longitudinal position of the vehicle C.M. is given as $-40(m) \leq X \leq 60(m)$.

**Path tracking and speed control results**  After designing the minimum-time travel speed profile along the entire reference trajectory, we used the MPC-based path tracking with PID speed control to follow the given path and achieve the designed speed profile. The results of tracking performance are shown as follows:

![Figure 26: Comparison of Trajectory](image)
Figure 27: Comparison of vehicle longitudinal velocity

Figure 28: Comparison of vehicle C.M. lateral position
Figure 29: Comparison of vehicle heading angle

Figure 30: Comparison of vehicle C.M. longitudinal position
6.4 Path tracking and speed control within one moving prediction horizon

For case B, we further consider the speed profile generation just in one prediction horizon and implement the model predictive path tracking. After path tracking implementation
in this window, the control moves onto the next prediction horizon. This procedure is implemented in a receding horizon method until the vehicle arrives at its destination. Given that the general trajectory information $X, Y, \psi$ and curvature $K$ are known along the entire trajectory, the reference speed profile and path tracking implementation in one window are given as follows:

Step 1: At point $s_1$, given the curvature in $N_p$ prediction horizon ($N_p$ stands for reference speed generation), generate the reference speed in one window ($N_p$ prediction horizon) according to Section 3;

Step 2: For the lateral control: assuming the speed in $N_p$ prediction horizon ($N_p$ for model predictive path tracking) is the same as that of $s_1$, the path tracking uses MPC controller to generate steering angle within $N_p$ prediction horizon;

Step 3: For longitudinal control: using PID comparing the plant and the reference at point $s_1$ to generate the wheel torque;

Step 4: Input the wheel torque and steering angle simultaneously to the plant and return the vehicle states for the next point $s_1 + ds$;

Step 5: Moving onto the next point $s_1 + ds$, and repeat the steps (2)∼(4);

Step 6: After implementing path tracking in this window, move onto the next prediction horizon and repeat steps (1) through (5).

**Constraints definition**  For the constraints on the control input signal, with small angle assumption for the linear tire models, the front road wheel steering angle is limited as $-20^\circ \leq \alpha \leq 20^\circ$, and the control increment is limited in $-8.5^\circ \leq \Delta \alpha \leq 8.5^\circ$.

The constraint imposed on the lateral position of the vehicle C.M., the longitudinal position of the vehicle C.M. and the heading angle of the vehicle is the same as those in case B.

**Path tracking performance results**  After designing the minimum-time travel speed profile, we used the MPC-based path tracking with PID speed control to follow the given path and achieve the designed speed profile. The results of tracking performance are shown as follows:
Figure 33: Comparison of Trajectory

Figure 34: Comparison of vehicle longitudinal velocity
Figure 35: Comparison of vehicle C.M. lateral position

Figure 36: Comparison of vehicle heading angle
This technical contribution represents a step up in relation to work reported in [1]. Therein, the trajectory tracked was an 8-shaped curve with constant curvature and the optimization step in the model predictive control (MPC) algorithm was carried out without bound constraints. This contribution addresses these limitations: we employed MPC with constraints on control input and state outputs. Moreover, we worked with paths of variable curvature. For the lateral control, we used an MPC controller to generate the optimal road wheel steering angle; for longitudinal control, we used a PID controller embedded in the MPC to generate the accelerating or braking wheel torque. The minimum-time speed profile based on the curvature and vehicle models was also generated for these two trajectories in order to follow the path as fast as possible. In addition, for case B, we considered formulating the MPC algorithm for path tracking with PID speed control within one moving window in a receding horizon way. The simulation results illustrate that the vehicle tracked the reference path well based on the proposed controller. Compared with the reference heading angle $\psi$, the lateral position $Y$, the longitudinal position $X$ and the reference speed, the tracking errors are relatively small. Furthermore, the tracking performance is relatively improved for case B using the MPC formulation in one moving prediction horizon.

In future work, we will consider the path tracking and speed control in follow aspects:

1. During the MPC formulation within one moving prediction horizon, for smooth tracking purpose, the velocity profile generation should guarantee the continuity of the longitudinal acceleration. Hence, the speed profile design method may need to be reconsidered in relation to the parametric velocity profile generation method.
(2) Since the MPC planning approach can take into account the updating of the dynamic environment states, we may consider integrating the path tracking and local path planning using an MPC framework that simultaneously addresses the planning and tracking aspects.

(3) Safety and stability concerns should be also taken into account in the MPC formulation in order to ensure that the AVs follow the desired path while staying inside the stable handling and environment envelops.

References


