A MATLAB® Implementation of a Set of Three Vehicle Dynamics Models

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Abstract

Three commonly used vehicle models; i.e., a bicycle model [1], an 8-DOF vehicle model [2], and a 14-DOF vehicle model [2], are discussed as implemented in MATLAB® and made available as open source in the public domain. Three different steering input commands, including sine wave steering, step steering and ramp steering, are applied in vehicle steering simulation and results are discussed insofar as their correctness is concerned. These models are envisioned to be used in conjunction with model predictive control (MPC) as applied in the context of multi-body dynamics, see [3,4]. These vehicle models are used in several other projects that investigate the use of MPC for autonomous vehicle driving, see [5,7].

Keywords: Vehicle Modeling, Tire Modeling, Steering Simulation
Contents

1 Introduction 3

2 Vehicle Dynamics Modeling 3
   2.1 Tire Modeling .................................................. 3
   2.2 Bicycle Model .................................................. 4
   2.3 14-DOF Vehicle Model .......................................... 5
   2.4 8-DOF vehicle model .......................................... 12

3 Simulation and Analysis 15
   3.1 Vehicle Model Parameters .................................... 15
   3.2 Simulation Results ............................................ 16
      3.2.1 Bicycle Model ............................................ 16
      3.2.2 8-DOF Vehicle Model .................................... 18
      3.2.3 14-DOF Vehicle Model .................................... 22

4 Conclusions and Future Work 26
1 Introduction

Vehicle dynamics modeling is a crucial basis for Autonomous Ground Vehicles, which is concerned with a mobile vehicle’s ability to drive safely and generate optimal control commands. The fidelity of a dynamic model is defined as presenting the movement and forces on a vehicle accurately with given conditions. Three models, the bicycle model, the 8-DOF vehicle model and the 14-DOF vehicle model, are generated in order to implement the model predictive control for path tracking in MATLAB®. The current work presents, through numerical simulation, the robustness and validity of three dynamic models with different inputs. In this report, these vehicle models are developed and implemented respectively and simulation results of the vehicle response are presented under the conditions of sine wave steering input, step steering input and ramp steering input.

The paper is structured as follows. Section 2 describes three vehicle dynamics models in detail. The simulation results of three different steering angle inputs are presented in Section 3. This is then followed by concluding remarks in Section 4 which discusses future research direction.

2 Vehicle Dynamics Modeling

This section describes the tire and vehicle model used for simulations. The nomenclature used in the bicycle model depicted in Figures 1. We denote by $F_l$ and $F_c$ the longitudinal and lateral tire forces respectively, $I_z$ is the car inertia along z axis, $X$ and $Y$ are the absolute car position inertial coordinates, $a$ and $b$ are the car geometry (distance of front and rear wheel from center of mass), $m$ is the car mass, $s$ is the slip ratio, $y$ and $x$ are the local longitudinal and lateral coordinates in car body frame, $\dot{x}$ and $\dot{y}$ are the vehicle longitudinal and lateral speeds, respectively, $\alpha$ is the tire slip angle, $\delta$ is the wheel steering angle, $\psi$ is the heading angle(yaw angle), $\dot{\psi}$ is yaw rate and $\beta$ is vehicle sideslip angle. C.M. is the abbreviation of Center of Mass. The lower subscripts $(\cdot)_f$ and $(\cdot)_r$ specialize a variable at the front wheel and the rear wheel, respectively [8].

2.1 Tire Modeling

Except for aerodynamics forces and gravity, all of the forces which affect vehicle behavior are provided by the tires. Because tire forces produce primary external influence and they have highly nonlinear performance, it is essential to use a realistic tire model, especially when investigating large control inputs that results in response near the limits of the linear character scale of the tire. The tire lateral and longitudinal forces are assumed to depend on normal force, slip angle, surface friction, and slip ratio. However, when the slip ratio and slip angle are limited within small values, the tire model can be simplified and generate linearied lateral force and longitudinal force [9].

Under this assumption, the tire lateral force can be given as $F_c = C_\alpha(\mu, F_z)\alpha$, where $C_\alpha$ is tire cornering stiffness related to tire normal force and tire-road friction coefficient; and
tire longitudinal force can be given as $F_l = C_x(\mu, F_z)s_{f,r}$, where $s_{f,r}$ is the slip ratio of front tire or rear tire, $C_x$ is the tire longitudinal stiffness which also related to the tire normal force and tire-road friction coefficient.

### 2.2 Bicycle Model

The single-track vehicle model [10] or ‘bicycle model’ [9] [11] is often used in the controller design, the schematic of this model is depicted in Figure 1. Several simplifications and assumptions are employed as listed below:

1. Neglecting the effect of steering system, combining the left and right tires of front and rear axles respectively, taking the front wheel steering angle as the input directly, i.e., $\delta_f = \delta, \delta_r = 0$;

2. Assuming that the vehicle only moves in yaw plane, neglecting the effect of suspension, so the vertical displacement, the pitch angle and the roll angle are considered zero;

3. The full vehicle are treated as rigid body and the lateral acceleration coefficient (i.e., the ratio between lateral acceleration and gravity acceleration) is limited within 0.4;

4. Assuming that the left and right tire slip angle are equal, the front wheel steer angle is small and the lateral forces are proportional to tire slip angles; Then, the lateral force can be given as $F_c = C_\alpha \alpha$, where $C_\alpha$ denotes the tire cornering stiffness and $\alpha$ denotes the tire slip angle;

5. Assuming that the effect of aerodynamics and the load transfer between the left and right tires are neglected.

![Figure 1: Bicycle Model](9 10 11)
With the assumption 1 to 5, the bicycle model can be described as [8][9]:

\begin{align*}
m(\ddot{x} - \dot{y}\dot{\psi}) &= F_{lf}\cos\delta - F_{cf}\sin\delta + F_{lr} &\quad (1) \\
m(\ddot{y} + \dot{x}\dot{\psi}) &= F_{lf}\sin\delta + F_{cf}\cos\delta + F_{cr} \\
I_z\ddot{\psi} &= a(F_{cf}\cos\delta + F_{lf}\sin\delta) - bF_{cr}. &\quad (3)
\end{align*}

Assuming that the vehicle side slip angle and the tire slip angle are small, the approximation of linearized side slip angle and tire slip angle can be simplified as below [10]:

\[ \beta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \approx \frac{\dot{y}}{\dot{x}}, \alpha_f = \arctan(\beta + \frac{a\dot{\psi}}{\dot{x}}) - \delta \approx \beta + \frac{a\dot{\psi}}{\dot{x}} - \delta, \alpha_r = \arctan(\beta - \frac{b\dot{\psi}}{\dot{x}}) \approx \beta - \frac{b\dot{\psi}}{\dot{x}}. \]

The position of the vehicle expressed in global coordinates can be calculated as follows [8][9]:

\begin{align*}
\dot{Y} &= \dot{x}\sin\psi + \dot{y}\cos\psi \\
\dot{X} &= \dot{x}\cos\psi - \dot{y}\sin\psi.
\end{align*}

Choosing \([\dot{y}, \dot{x}, \psi, \dot{\psi}, Y, X]\) as the state variable, with Eq. (1) through Eq. (5), small angle assumption and linear tire model, then the nonlinear vehicle model equations are given as

\begin{align*}
\ddot{y} &= -\dot{x}\dot{\psi} + \frac{C_{af}(\dot{y} + a\dot{\psi})}{m} - \frac{\dot{\psi}}{m} + \frac{C_{ar}(\dot{y} - b\dot{\psi})}{m} &\quad (6) \\
\ddot{x} &= \dot{y}\dot{\psi} + \frac{s_fC_{zf} + s_rC_{zx}}{m} - \frac{C_{af}(\dot{y} + a\dot{\psi})}{m} - \delta \frac{\dot{x}}{m} - \delta &\quad (7) \\
\ddot{\psi} &= \frac{1}{I_z} \left( aC_{af}(\dot{y} + a\dot{\psi}) - bC_{ar}(\dot{y} - b\dot{\psi}) - \delta \dot{x} \right) &\quad (8) \\
\dot{Y} &= \dot{x}\sin\psi + \dot{y}\cos\psi \\
\dot{X} &= \dot{x}\cos\psi - \dot{y}\sin\psi.
\end{align*}

### 2.3 14-DOF Vehicle Model

In order to better represent the vehicle lateral and yaw dynamics as well as coupling of yaw-roll motion due to the transient lateral load transfer during extreme maneuvers, higher order model such as 8-DOF and 14-DOF are also used in rollover studies. A 14-DOF vehicle model, which considers the suspension at each corner, has the same benefits of an 8-DOF vehicle model, with the additional capabilities of predicting vehicle pitch and heave motions. It also offers the flexibility of modeling nonlinear springs and dampers and can simulate the vehicle responses to normal force inputs in the case of an active suspension system. Moreover, the 14-DOF model, unlike the 8-DOF model, can predict vehicle behavior even
after wheel lift-off and thus can be used in developing and testing the validity of rollover prediction/prevention strategies [2].

The Figure 2 shows the schematic of the two axle vehicle model used to investigate vehicle roll response to steering and torque inputs. This schematic includes 6 DOF at the vehicle lumped mass C.M. and 2 DOF at each of the four wheels, including vertical suspension travel and wheel spin. The body is modeled as being rigid, with body-fixed coordinates attached at the C.M. and aligned in principal directions (coordinate frame 1). \( u, v, w \) indicate forward, lateral, and vertical velocities, respectively, of the sprung mass [2].

![Figure 2: Schematic of 14-DOF vehicle model with one-dimensional suspension and coordinate frames](image)

The force and velocity components in the right front corner of a vehicle is depicted in Figure 3. The velocity \( u_{sr}, v_{sr}, \) and \( w_{sr} \) are the velocities of the right front strut mounting point in the longitudinal, lateral, and vertical directions, respectively, in the body-fixed coordinate frame, which is attached to the sprung mass C.M. (coordinate frame 1). These velocities can be obtained by transforming the C.M. velocities as [2]\

\[
\begin{pmatrix}
u_{sr} \\
v_{sr} \\
w_{sr}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \frac{c_f}{2} \\
0 & 0 & a \\
-\frac{c_f}{2} & -a & 0
\end{pmatrix} \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} + \begin{pmatrix}
u \\
v \\
w
\end{pmatrix}.
\] (11)

For the left front corner:

\[
\begin{pmatrix}
u_{lf} \\
v_{lf} \\
w_{lf}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \frac{-c_f}{2} \\
0 & 0 & a \\
\frac{c_f}{2} & -a & 0
\end{pmatrix} \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} + \begin{pmatrix}
u \\
v \\
w
\end{pmatrix}.
\] (12)
Figure 3: Description of forces and velocities at the right front corner of a vehicle

For the left rear corner:

\[
\begin{pmatrix}
 u_{slr} \\
 v_{slr} \\
 w_{slr}
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & -\frac{c_r}{2} \\
 0 & 0 & -b \\
 \frac{c_r}{2} & b & 0
\end{pmatrix}\begin{pmatrix}
 \omega_x \\
 \omega_y \\
 \omega_z
\end{pmatrix} + \begin{pmatrix}
 u \\
 v \\
 w
\end{pmatrix} .
\]

(13)

For the right rear corner:

\[
\begin{pmatrix}
 u_{srr} \\
 v_{srr} \\
 w_{srr}
\end{pmatrix} = \begin{pmatrix}
 0 & 0 & \frac{c_r}{2} \\
 0 & 0 & -b \\
 -\frac{c_r}{2} & b & 0
\end{pmatrix}\begin{pmatrix}
 \omega_x \\
 \omega_y \\
 \omega_z
\end{pmatrix} + \begin{pmatrix}
 u \\
 v \\
 w
\end{pmatrix} .
\]

(14)

The forces \( F_{xsr_f}, F_{ysr_f} \) and \( F_{zsr_f} \) are the forces transmitted to the sprung mass along the longitudinal, lateral, and vertical directions, respectively, of coordinate frame 1. The forces \( F_{xgsrf}, F_{ygsrf}, \) and \( F_{zgsrf} \) are the forces acting at the tire ground contact patch in the same coordinate frame 1. These forces can be written in terms of the tire forces \( F_{xgrf}, F_{ygrf}, \) and \( F_{zgrf} \) by projecting its components along coordinate frame 2 as

\[
\begin{pmatrix}
 F_{xgsrf} \\
 F_{ygsrf} \\
 F_{zgsrf}
\end{pmatrix} = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & \cos\phi & \sin\phi \\
 0 & -\sin\phi & \cos\phi
\end{pmatrix}\begin{pmatrix}
 \cos\phi & 0 & -\sin\phi \\
 0 & 1 & 0 \\
 \sin\theta & 0 & \cos\theta
\end{pmatrix}\begin{pmatrix}
 F_{xgrf} \\
 F_{ygrf} \\
 F_{zgrf}
\end{pmatrix} .
\]

(15)

The forces \( F_{xgrf} \) and \( F_{ygrf} \) are obtained by resolving the longitudinal(\( F_{xtrf} \)) and cornering(\( F_{ytrf} \))
forces at the tire contact patch as \[2\]

\[ F_{x\text{grf}} = F_{x\text{trf}} \cos \delta - F_{y\text{trf}} \sin \delta; \quad (16) \]
\[ F_{y\text{grf}} = F_{y\text{trf}} \cos \delta + F_{x\text{trf}} \sin \delta; \quad (17) \]

where the $\delta$ is the steering angle at the road wheels.

The linear tire model has been used in the development of the tire forces $F_{x\text{trf}}$ and $F_{y\text{trf}}$. The longitudinal and lateral slips used in the tire model were calculated as follows \[2\]:

\[ s_{\text{rf}} = \frac{(r_{\text{rf}} \omega_{\text{rf}} - (u_{\text{grf}} \cos \delta + v_{\text{grf}} \sin \delta))}{|(u_{\text{grf}} \cos \delta + v_{\text{grf}} \sin \delta)|}; \quad (18) \]
\[ \alpha_{\text{rf}} = \tan^{-1} \left( \frac{v_{\text{grf}}}{u_{\text{grf}}} \right) - \delta. \quad (19) \]

The $u_{\text{grf}}$ and $v_{\text{grf}}$, longitudinal and lateral velocities at the tire contact patch of the right front wheel, can be determined as \[2\]

\[ u_{\text{grf}} = \cos \theta \left( u_{\text{urf}} - \omega_y \cdot r_{\text{rf}} \right) + \sin \theta \left( w_{\text{urf}} \cos \phi + \sin \phi \left( \omega_x \cdot r_{\text{rf}} + v_{\text{urf}} \right) \right) \quad (20) \]
\[ v_{\text{grf}} = \cos \phi \left( u_{\text{urf}} + \omega_x \cdot r_{\text{rf}} \right) - w_{\text{urf}} \sin \phi. \quad (21) \]

The longitudinal ($u_{\text{urf}}$) and lateral ($v_{\text{urf}}$) velocities of the unsprung mass in the body coordinate frame used in the earlier equations can simply be written as \[2\]

\[ u_{\text{urf}} = u_{\text{srf}} - l_{\text{srf}} \omega_y \quad (22) \]
\[ v_{\text{urf}} = v_{\text{srf}} + l_{\text{srf}} \omega_x, \quad (23) \]

where $l_{\text{srf}}$ is the instantaneous length of the strut. The derivative of $u_{\text{urf}}$ and $v_{\text{urf}}$ is given as

\[ \dot{u}_{\text{urf}} = \dot{u}_{\text{srf}} - \left( \omega_{\text{srf}} - \omega_{\text{urf}} \right) \omega_y + l_{\text{srf}} \dot{\omega}_y \quad (24) \]
\[ \dot{v}_{\text{urf}} = \dot{v}_{\text{srf}} + \left( \omega_{\text{srf}} - \omega_{\text{urf}} \right) \omega_y + l_{\text{srf}} \dot{\omega}_y. \quad (25) \]

The unsprung mass vertical velocity $w_{\text{urf}}$ represents the degree of freedom corresponding to the suspension deflection and can be expressed by applying Newton’s law for the vertical motion of the unsprung mass as \[2\]

\[ m_u \ddot{w}_{\text{urf}} = \cos \phi \left( \cos \theta \left( F_{x\text{grf}} - m_u g \right) + \sin \theta F_{x\text{grf}} \right) - \sin \phi F_{y\text{grf}} - F_{dz\text{rf}} - x_{\text{srf}} \cdot k_{\text{sf}} - \dot{x}_{\text{srf}} \cdot b_{\text{sf}} - m_u \left( v_{\text{urf}} \cdot \omega_x - u_{\text{urf}} \cdot \omega_y \right), \quad (26) \]

where $k_{\text{sf}}$ is the suspension stiffness, $b_{\text{sf}}$ the suspension damping coefficient, and $x_{\text{srf}}$ the instantaneous compression of the right front suspension spring. The force, $F_{dz\text{rf}}$, represents the additional load transfer that occurs at the wheels through the suspension links because of the reaction force to the force transmitted to the sprung mass through the roll center \[2\].
The instantaneous suspension spring deflection \( x_{srf} \) is given as \(^2\)
\[
\dot{x}_{srf} = -w_{srf} + w_{urf}. \tag{27}
\]

The vertical force \( F_{zgrf} \) acting at the tire ground contact patch in coordinate frame 2 can be written in terms of the tire stiffness \( (k_{tf}) \) and the instantaneous tire deflection \( (x_{trf}) \) as \(^2\)
\[
F_{zgrf} = F_{ztvf} = x_{trf}k_{tf}. \tag{28}
\]

The instantaneous tire deflection \( x_{trf} \) in equation (28) is given as \(^2\)
\[
\dot{x}_{trf} = w_{grf} - w_{uirf} = w_{grf} - \left( \cos \theta \left( w_{urf} \cos \phi + v_{urf} \sin \phi \right) - u_{urf} \sin \theta \right), \tag{29}
\]
where \( w_{uirf} \) is the vertical velocity of the wheel center in the inertial coordinate frame. For the simulations in this article, it is assumed that the vertical velocity \( w_{grf} \) at the tire contact patch is zero (smooth road). It should be noted that even though the tire is assumed to remain at a fixed angle with the strut, the vertical stiffness of the tire, \( k_{tf} \), is always considered to be normal to the ground between the ground and the wheel center \(^2\).

The instantaneous tire radius is then determined as \(^2\)
\[
r_{rf} = \frac{r_0 - x_{trf}}{\cos \theta \cos \phi}. \tag{30}
\]

To account for the wheel lift-off, when the tire radial compression becomes less than zero, the tire normal force \( F_{zgrf} \) is set equal to zero. In addition, the instantaneous tire radius is considered equal to the nominal tire radius until it returns to the road surface \(^2\).

\[
If \quad x_{trf} < 0 \quad then \quad F_{zgrf} = 0 \quad and \quad r_{rf} = r_0. \tag{31}
\]

As the tire normal force becomes zero, no lateral \( (F_{ygrf}) \) and longitudinal \( (F_{xgrf}) \) tire forces are developed at that contact patch. Thus, the only forces acting at that suspension corner are the weight and inertia forces of the unsprung mass, which are very small in magnitude. As the cardan angles and appropriate coordinate transformations between the body-fixed coordinate frame 1 and the coordinate frame 2 at the tire-ground contact patch are considered for all the forces and velocities in the system, the model is able to simulate vehicle behavior after wheel lift-off and during the rollover event with just the modification mentioned in (31) \(^2\).

The instantaneous length of the strut \( l_{srf} \) is given as \(^2\)
\[
l_{srf} = l_{sif} - (x_{srf} - x_{sif}), \tag{32}
\]
where \( l_{sif} \) is the initial length of strut and \( x_{sif} \) is the initial suspension spring deflection.

The initial length of the strut \( l_{sif} \) is taken such that \(^2\)
\[
l_{sif} = h - (r_0 - x_{tisf}), \tag{33}
\]

9
where $x_{tlf}$ is the initial tire compression.

The initial spring compression $x_{sif}$ and the initial tire compression $x_{tif}$ are determined from the static conditions as \[2\]
\[
x_{sif} = \frac{mb}{2(a + b)k_{sf}} \tag{34}
\]
\[
x_{tif} = \frac{(mb/2(a + b) + m_{uf})}{k_{tf}} \tag{35}
\]

For the left front corner, the left rear corner and the right rear corner, the instantaneous length of the strut, the initial length of strut and the initial suspension spring deflection, the initial length of the strut and the initial spring compression and the initial tire compression are the same as those of the right front corner.

The forces $F_{xsr}$ and $F_{ysr}$ transmitted to the sprung mass along the $u-$ and $v-$axes of the body-fixed coordinate frame are obtained after subtracting the components of the unsprung mass weight and inertia forces from the corresponding forces $F_{xgsr}$ and $F_{ygsr}$ acting at the tire contact patch as \[2\]
\[
F_{xsr} = F_{xgsr} + m_u g \sin\theta - m_u \dot{u}_{urf} + m_u \omega_z v_{urf} - m_u \omega_y w_{urf} \tag{36}
\]
\[
F_{ysr} = F_{ygsr} - m_u g \sin\phi \cos\theta - m_u \dot{v}_{urf} + m_u \omega_x w_{urf} - m_u \omega_z u_{urf} \tag{37}
\]

The vertical force $F_{zsr}$ transmitted to the sprung mass through the strut is given as \[2\]
\[
F_{zsr} = x_{sr} k_{sr} + \dot{x}_{sr} b_{sr} \tag{38}
\]

Figure 4 shows the forces and velocities in the roll plane of, for example, the front suspension. Generally, the roll center height is defined with reference to the ground. However, for the development of this model, the front and rear roll centers are assumed to be fixed at distances $h_{rcf}$ and $h_{rcr}$, respectively, below the sprung mass C.M. along the negative $w$-axis of the body-fixed coordinate frame 1. Moreover, the roll center is simply considered to be a point of application of the forces transmitted to the sprung mass through the suspension links and not as a kinematic constraint. In the figure 4, $F_{zslf}$ and $F_{zsr}$ are the forces transmitted to the sprung mass through the struts. $F_{yslf}$ and $F_{ysrf}$ represent the lateral forces transmitted to the sprung mass through the suspension links. In the absence of a roll center, i.e., when the roll center is assumed to be in the ground plane, the total roll moment transmitted to the sprung mass at, for example, the right front corner along the $\omega_x$ direction is given as \[2\]
\[
M_{xrf} = F_{ygser} (l_{sr} + r_{rf}) - (m_u g \sin\phi \cos\theta + m_u \dot{u}_{urf} - m_u \omega_z w_{urf} + m_u \omega_y u_{urf}) \cdot l_{srf} \tag{39}
\]
for the left front corner:
\[
M_{xlf} = F_{ygslf} (l_{sl} + r_{lf}) - (m_u g \sin\phi \cos\theta + m_u \dot{u}_{ulf} - m_u \omega_z w_{ulf} + m_u \omega_y u_{ulf}) \cdot l_{slf} \tag{40}
\]
When a roll center is modeled as shown in Fig. 4, the roll moment $M_{xrf}$ transmitted to the sprung mass by the right front corner suspension is given as

$$M_{xrf} = F_{ysrf} h_{rcf}. \quad (41)$$

Thus, the inclusion of a roll center reduces the total roll moment transferred to the sprung mass by the front suspension. The difference between the roll moments in the absence of the roll center and when the roll center is considered acts directly on the unsprung mass and is responsible for the link load transfer forces (jacking forces), $F_{dzlf}$ and $F_{dzrf}$. These forces can be estimated as

$$F_{dzrf} = -F_{dzlf} = F_{yggrf} r_{rf} + F_{ysrf} l_{rf} + F_{yyrf} l_{zf} - (F_{ysrf} + F_{ysrf}) h_{rcf}. \quad (42)$$

For the rear suspension, the total roll moment transmitted to the sprung mass at the left rear and right rear corner has the same form as the front suspension. Thus, the forces of $F_{dzlr}$ and $F_{dzrr}$ can be estimated as

$$F_{dzrr} = -F_{dzlr} = F_{ygstr} r_{tr} + F_{ystlr} l_{str} + F_{yysrr} r_{rr} + F_{ysrr} l_{srr} - (F_{ystlr} + F_{ysrr}) h_{rcr}. \quad (45)$$
The equation of motion for the 6 DOF of the sprung mass model can now be derived from the direct application of Newton’s laws for the system as \[2\]

\[
m(\ddot{u} + \omega_y v - \omega_z w) = \sum (F_{xij}) + mgsin\theta \tag{46}
\]

\[
m(\ddot{v} + \omega_z u - \omega_x w) = \sum (F_{yij}) - mgsin\phi cos\theta \tag{47}
\]

\[
m(\ddot{w} + \omega_x v - \omega_y u) = \sum (F_{zij} + F_{dzij}) - mgsin\phi cos\theta \tag{48}
\]

\[
J_x \dot{\omega}_x = \sum (M_{xij}) + \frac{(F_{zslf} - F_{zsrj})c_f + (F_{zslr} - F_{zsrj})c_r}{2} \tag{49}
\]

\[
J_y \dot{\omega}_y = \sum (M_{yij}) + (F_{zslf} + F_{zsrj})b - (F_{zslf} + F_{zsrj})a \tag{50}
\]

\[
J_z \dot{\omega}_z = \sum (M_{zij}) + (F_{yslf} + F_{ysrf})a - (F_{yslf} + F_{ysrf})b \tag{51}
\]

\[
\frac{(-F_{zslf} + F_{zsrj})c_f + (-F_{zslr} + F_{zsrj})c_r}{2}.
\]

where \(m\) is the sprung mass and the subscript ‘ij’ denotes left front (lf), right front (rf), left rear (lr), and right rear (rr).

The cardan angles \(\theta, \psi, \phi\) needed in the aforementioned equations are obtained by performing the integration of the following equations \[2\],

\[
\dot{\theta} = \omega_y \cos\phi - \omega_z \sin\phi \tag{52}
\]

\[
\dot{\psi} = \frac{\omega_y \sin\phi}{\cos\theta} + \frac{\omega_z \cos\phi}{\cos\theta} \tag{53}
\]

\[
\dot{\phi} = \omega_x + \omega_y \sin\phi \tan\theta + \omega_z \cos\phi \tan\theta. \tag{54}
\]

The rotational dynamics for each wheels can be given as \[12\]

\[
J_w \dot{\omega}_w = T_{dwr} - T_{brw} - r_{rf}F_{xrf}, \tag{55}
\]

where \(T_{dwr}\) is the driving torque transmitted to the wheel, \(T_{brw}\) is the wheel braking torque.

### 2.4 8-DOF vehicle model

An 8-DOF full vehicle model is often used as a simplified lower order model for studying vehicle handling in scenarios which do not involve significant longitudinal accelerations. In this section, a formulation for the 8-DOF model, adapted from various references \[2\] \[12\], that can match the 14-DOF model reasonably accurately is presented \[2\].

The schematic of the 8-DOF full vehicle model is given in Figure 5. The model has four degrees of freedom for the chassis velocities and one degree of freedom at each of the four wheels representing the wheel spin dynamics. The chassis velocities include the longitudinal velocity, \(u\), the lateral velocity, \(v\), the roll angular velocity, \(\omega_x\), and the yaw angular velocity, \(\omega_z\). The pitch and heave motions are not modeled and the front and rear suspensions are
represented simply by their respective equivalent roll stiffness ($k_{\psi f}/k_{\psi r}$) and roll damping coefficients ($b_{\psi f}/b_{\psi r}$) \cite{2}.

The equations of motion for the chassis velocities are obtained as given subsequently \cite{2}:

\begin{align}
    m_t (\dot{u} - \omega_z v) &= \sum F_{xgij} + (m_{uf} a - m_{ur} b) \omega_z^2 - 2 h_{rc} m \omega_z \omega_x \tag{56} \\
    m_t (\dot{v} + \omega_z u) &= \sum F_{ygi} + (m_{ur} b - m_{uf} a) \dot{\omega}_z + h_{rc} m \dot{\omega}_x \tag{57} \\
    J_z \ddot{\omega}_z + J_{xz} \ddot{\omega}_x &= (F_{ygf} + F_{ygr}) a - (F_{yglr} + F_{ygrr}) b + \frac{(F_{xgrf} - F_{xglf}) c_f}{2} \nonumber \\
    &+ \frac{(F_{xgrr} - F_{xglr}) c_r}{2} + (m_{ur} b - m_{uf} a) (\dot{v} + \omega_z u) \tag{58} \\
    (J_x + mh_{rc}^2) \ddot{\omega}_x + J_{xz} \ddot{\omega}_z &= mgh_{rc} \phi - (k_{\phi f} + k_{\phi r}) \dot{\phi} - (b_{\phi f} + b_{\phi r}) \ddot{\phi} + h_{rc} m (\dot{v} + \omega_z u). \tag{59}
\end{align}
where

\[ h_{rc} = \frac{h_{rcf}b + h_{rcr}a}{a + b}. \] (60)

In these equations, the forces \( F_{xgij} \) and \( F_{ygij} \) are the longitudinal and lateral forces at the four tire contact patches and the subscript \( \text{ij} \) denotes \( lf, rf, lr, \) and \( rr \). As before, \( h_{rcf} \) and \( h_{rcr} \) are the vertical distances of the front and rear roll centers below the sprung mass C.M., and thus \( h_{rc} \) is the vertical distance from the sprung mass C.M. to the vehicle roll center. It should be noted that as equation for the roll degree of freedom is written by considering moments acting about the vehicle roll center. It should be noted that as equation (59) for the roll degree of freedom is written by considering moments acting about the vehicle roll center rather than the sprung mass C.M., the roll inertia of the sprung mass about the vehicle roll center \( (J_x + m h_{rc}^2) \) is considered in equation (59) [2].

The equations for the wheel dynamics and the longitudinal and lateral tire forces are the same as those used in the 14-DOF model. The longitudinal and lateral velocities at, for example, the right front tire contact patch required in these equations are given as [2]

\[ u_{grf} = u + \frac{\omega_z c_f}{2} \] (61)
\[ v_{grf} = v + \omega_z a. \] (62)

The normal forces at the four tires are determined as [2]

\[ F_{zglf} = \frac{mb}{2(a + b)} + \frac{m_u f g}{2} - \left( \frac{m_u h_{uf} c_f}{c_f} + \frac{mb(h_{cg} - h_{rcf})}{c_f(a + b)} \right) \left( \dot{v} + \omega_z u \right) - \frac{(k_{\phi f} + b_{\phi f})}{c_f} \left( \dot{\phi}_f \right) \] (63)
\[ F_{zgrf} = \frac{mg}{2(a + b)} + \frac{m_u f g}{2} + \left( \frac{m_u h_{uf} c_f}{c_f} + \frac{mb(h_{cg} - h_{rcf})}{c_f(a + b)} \right) \left( \dot{v} + \omega_z u \right) + \frac{(k_{\phi f} + b_{\phi f})}{c_f} \left( \dot{\phi}_f \right) \] (64)
\[ F_{zglr} = \frac{mg}{2(a + b)} + \frac{m_u r g}{2} - \left( \frac{m_u h_{ur} c_r}{c_r} + \frac{ma(h_{cg} - h_{rcr})}{c_r(a + b)} \right) \left( \dot{v} + \omega_z u \right) - \frac{(k_{\phi r} + b_{\phi r})}{c_r} \left( \dot{\phi}_r \right) \] (65)
\[ F_{zgrr} = \frac{mg}{2(a + b)} + \frac{m_u r g}{2} + \left( \frac{m_u h_{ur} c_r}{c_r} + \frac{ma(h_{cg} - h_{rcr})}{c_r(a + b)} \right) \left( \dot{v} + \omega_z u \right) + \frac{(k_{\phi r} + b_{\phi r})}{c_r} \left( \dot{\phi}_r \right) \] (66)
These equations are fairly simple and linearized. It is possible to include several additional terms in the equations for the chassis velocities as well as the tire forces. However, in our experience, these terms have very little effect on the vehicle responses and can be ignored 2.

3 Simulation and Analysis

3.1 Vehicle Model Parameters

Parameters used for vehicle model simulations are listed as below:

Table 1: Parameters for vehicle dynamics modeling

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>Sprung mass</td>
<td>1400(kg)</td>
</tr>
<tr>
<td>$J_x$</td>
<td>Sprung mass roll inertia</td>
<td>900(kgm$^2$)</td>
</tr>
<tr>
<td>$J_y$</td>
<td>Sprung mass yaw inertia</td>
<td>2000(kgm$^2$)</td>
</tr>
<tr>
<td>$J_z$</td>
<td>Sprung mass pitch inertia</td>
<td>2420(kgm$^2$)</td>
</tr>
<tr>
<td>$J_{xz}$</td>
<td>Sprung mass inertia</td>
<td>90(kgm$^2$)</td>
</tr>
<tr>
<td>$a$</td>
<td>Distance of sprung mass C.M. from front axle</td>
<td>1.14(m)</td>
</tr>
<tr>
<td>$b$</td>
<td>Distance of sprung mass C.M. from rear axle</td>
<td>1.4(m)</td>
</tr>
<tr>
<td>$h$</td>
<td>Sprung mass C.M. height</td>
<td>0.75(m)</td>
</tr>
<tr>
<td>$c_f$</td>
<td>Front track width</td>
<td>1.5 (m)</td>
</tr>
<tr>
<td>$c_r$</td>
<td>Rear track width</td>
<td>1.5 (m)</td>
</tr>
<tr>
<td>$k_{sf}$</td>
<td>Front suspension stiffness</td>
<td>35000 (N/m)</td>
</tr>
<tr>
<td>$k_s$</td>
<td>Rear suspension stiffness</td>
<td>30000 (N/m)</td>
</tr>
<tr>
<td>$b_{sf}$</td>
<td>Front suspension damping coefficient</td>
<td>2500 (Ns/m)</td>
</tr>
<tr>
<td>$b_{sr}$</td>
<td>Rear suspension damping coefficient</td>
<td>2000 (Ns/m)</td>
</tr>
<tr>
<td>$m_{uf}$</td>
<td>Front unsprung mass</td>
<td>80 (kg)</td>
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<tr>
<td>$m_{ur}$</td>
<td>Rear unsprung mass</td>
<td>80 (kg)</td>
</tr>
<tr>
<td>$k_f$</td>
<td>Front tire stiffness</td>
<td>200000 (N/m)</td>
</tr>
<tr>
<td>$k_r$</td>
<td>Rear tire stiffness</td>
<td>200000 (N/m)</td>
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<td>$C_{αf}$</td>
<td>Front right tire cornering stiffness</td>
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<td>$C_{αr}$</td>
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<td>47000 (N/rad)</td>
</tr>
<tr>
<td>$C_{xf}$</td>
<td>Front right tire longitudinal stiffness</td>
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</tr>
<tr>
<td>$C_{xr}$</td>
<td>Rear right tire longitudinal stiffness</td>
<td>5000 (N)</td>
</tr>
<tr>
<td>$r_0$</td>
<td>Nominal tire radius</td>
<td>0.285 (m)</td>
</tr>
<tr>
<td>$J_w$</td>
<td>Tire/wheel roll inertia</td>
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</tr>
<tr>
<td>$h_{rcf}$</td>
<td>Front roll center distance below sprung mass C.M.</td>
<td>0.65 (m)</td>
</tr>
<tr>
<td>$h_{rcr}$</td>
<td>Rear roll center distance below sprung mass C.M.</td>
<td>0.6 (m)</td>
</tr>
</tbody>
</table>
3.2 Simulation Results

Assuming the longitudinal velocity is constant in the simulation, three different steering input signals are implemented to three vehicle models and the responses are studied to understand the effect of simplifications.

3.2.1 Bicycle Model

1. Take sinusoidal steering wheel angle with amplitude at 0.0087(rad), i.e. 0.5(deg) as input, and fix the vehicle longitudinal velocity at 33.73(m/s), the output of the vehicle model includes lateral velocity, yaw angle and yaw rate, which are listed as follows,

Figure 6: Bicycle model response of road wheel sine wave steer input at a speed of 33.73m/s

2. Take step steering wheel angle as input with amplitude 0.0087(rad) and the same vehicle longitudinal velocity as before, the output of the vehicle model includes lateral velocity, yaw angle and yaw rate, which are listed as follows,
3. Take ramp steering wheel angle as input with gradient at 3(deg/s). The vehicle longitudinal velocity is fixed at 50(km/h), the output of the vehicle model includes lateral velocity, yaw angle and yaw rate, which are listed as follows,
3.2.2 8-DOF Vehicle Model

1. Take sinusoidal steering wheel angle with amplitude at $0.0087\text{rad}$ as input, and maintain the vehicle longitudinal velocity at $33.73\text{m/s}$, the output of the vehicle model includes roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which are listed as follows,
Figure 9: 8-DOF Vehicle response of road wheel sine wave steer input at a speed of 33.73m/s
2. Take step steering wheel angle as input with amplitude at $0.0087\, (rad)$ and the vehicle longitudinal velocity at $33.73\, (m/s)$, the outputs of the vehicle model include roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which are listed as follows,
3. Take ramp steering wheel angle as input with gradient at $3\, (deg/s)$ and the vehicle longitudinal velocity fixed at $50\, (km/h)$, the outputs of the vehicle model include roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which list as follows,
Figure 11: 8-DOF Vehicle response of road wheel ramp steer 3deg/s input at a speed of 50km/h

### 3.2.3 14-DOF Vehicle Model

1. Take sinusoidal steering wheel angle as input with amplitude $0.0087 (rad)$ and the vehicle longitudinal velocity is $33.73 (m/s)$, the output of the vehicle model includes roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which list as follows,
Figure 12: 14-DOF Vehicle response of road wheel sine wave steer input at a speed of 33.73 m/s
2. Take step steering wheel angle as input with amplitude $0.0087\text{ (rad)}$ and the vehicle longitudinal velocity is $33.73\text{ (m/s)}$, the outputs of the vehicle model include roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which list as follows,
Figure 13: 14-DOF Vehicle response of road wheel step steer input at a speed of 33.73m/s

3. Take ramp steering wheel angle as input with gradient at $3(\text{deg/s})$ and the vehicle longitudinal velocity at $50(\text{km/h})$, the outputs of the vehicle model include roll angle, yaw angle, yaw rate, lateral velocity, lateral acceleration, which list as follows,
Figure 14: 14-DOF Vehicle response of road wheel ramp steer 3deg/s input at a speed of 50km/h

4 Conclusions and Future Work

A bicycle model, an 8-DOF vehicle model and a 14-DOF vehicle model were developed and implemented in MATLAB®. The bicycle model, assuming the vehicle only moves in yaw plane and neglecting the load transfer between the left and right tires, is often used in the vehicle controller design due to its simplicity. The 8-DOF vehicle considered the longitudinal, the lateral, the roll and the yaw motion of the chassis, and four wheel spin dynamics. The 14-DOF vehicle model, which considered the suspension at each corner, has the same benefits of an 8-DOF vehicle model, with the additional capabilities of predicting vehicle pitch and heave motions. Different front road wheel steering angles are taken as input signals and their output responses are studied and analyzed.
In the future work, model predictive control for path following will be implemented by using lower fidelity vehicle model as the prediction model in the controller and using higher fidelity vehicle model as the plant.

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References


