ME 440
Intermediate Vibrations

Tu, March 12, 2009

Chapter 4: Vibration Under General Forcing Conditions

Section 4.4

Before we get started…

- Last Time:
  - Periodic Excitation
  - Special Excitation Cases

- Today:
  - HW Assigned (due March 26):
    - 4.23 (Use Eq. 4.36)
    - 4.24 (See Example 4.9 in the book, it helps…)

  - Material Covered:
    - Response under an impulsive force
    - Duhamel's Integral (convolution integral)

[New Topic: Case C (see slide beginning of lecture)]

Impulse Excitation and Response

- Impulse Excitation: What is it?
  - When a dynamical system is excited by a suddenly applied and short in duration nonperiodic excitation F(t)
    - Specifically, what does "short in duration" mean?
    - It means that it's shorter than 1/10 of the natural period ($\tau_n = 2\pi/\omega_n$)
      - This is a rule of thumb…
  - Note: steady state response not produced (forcing term is not periodic)
    - Only transient response present in the system
  - Oscillations take place at the natural and/or damped natural frequencies of the system

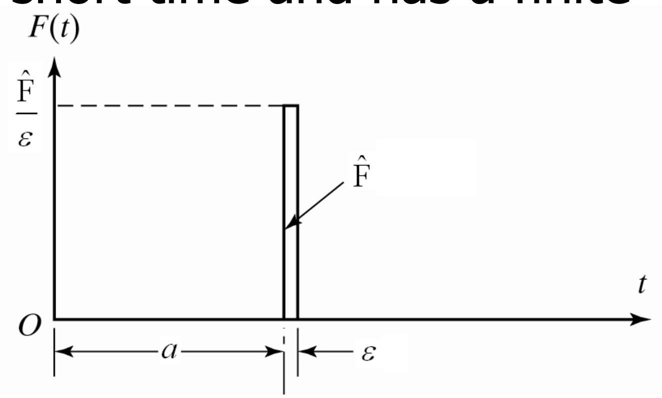
[Cntd.]

Impulse Excitation and Response

- Impulse: Time integral of the force

$$\hat{F} = \int F(t)dt$$

- Impulsive Force: A force of very large magnitude which acts for a very short time and has a finite *nonzero* impulse



$$\text{As} \quad \epsilon \rightarrow 0 \quad , \quad \frac{\hat{F}}{\epsilon} \rightarrow \infty \quad ,$$

$$\text{but} \quad \hat{F} \quad \text{is finite.}$$

- Impulse is the area of the rectangle in the figure above
  - "Spike" can be high, but area underneath is finite

Dirac Delta Function

- The type of impulsive function F(t) for which the following condition holds:

$$\hat{F} = \int F(t)dt = 1$$

- Dirac Delta Function also called unit impulse, or delta function

- A delta function at t=a is identified by the symbol δ(t-a) and has the following properties

$$\text{(I)} \qquad \delta(t-a) = 0 \quad \text{for all} \;\; \mathrm{t} \neq a$$

$$\text{(II)} \qquad \int_0^\infty \delta(t-a)dt = 1 \quad 0 < a < \infty$$

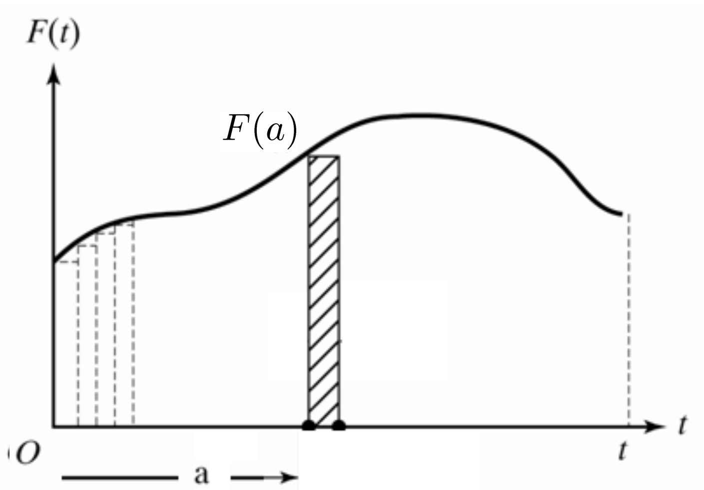
Comment on Dirac Function

(somewhat tricky…)

- If you take a function F(t) and multiply it by the Dirac Function δ(t-a), what do you get?

$$G(t) = F(t) \cdot \delta(t-a) \quad \text{for } \forall \; a \;\text{ with } \quad 0 < a < \infty$$

- You get a function G(t), which has two interesting properties:
  - G(t) is zero everywhere, except at t=a
  - For any 0<a<1, its integral from zero to 1 assumes the value F(a):



$$\int_0^{\infty} G(t) = \int_0^{\infty} F(t) \cdot \delta(t-a) dt = F(a) \quad \text{for} \quad 0 < a < \infty$$

Going Back To Mechanical Systems…

- Recall Newton's Second Law: $F(t)\,dt = m\,dv$
- Then, $\int_0^T F(t)dt = m\int_0^T dv \quad \Rightarrow \quad v(T) = v(0) + \frac{\hat{F}}{m}$
- Assume now that F(t) is an impulsive force applied at time t=0 with the goal of "kicking" the system in motion. Then,

$$v(0^+) = v(0) + \frac{\hat{F}}{m}$$

- If v(0)=0, then $v(0^+) = \frac{\hat{F}}{m}$
- Important observation: an impulsive force will lead to a sudden change in velocity without appreciable change in its displacement

Response To Impulsive Force ~ Undamped Systems ~

- Impulsive force applied to undamped system, free* response
- In Chapter 2, we saw that

$$x_h(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

- Assume zero initial conditions, but system is kicked in motion by an impulsive force. Then,

$$x_0 = 0 \qquad \dot{x}_0 = \frac{\hat{F}}{m}$$

- Response ends up looking like this:

$$x_h(t) = \frac{\hat{F}}{m\omega_n} \sin \omega_n t \qquad \text{where} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

Response To Impulsive Force ~ Underdamped Systems ~

- Impulsive force applied to underdamped system, free* response
- In Chapter 2, we saw that

$$x_h(t) = e^{-\zeta\omega_n t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

- Assume zero initial conditions, but system is kicked in motion by an impulsive force. Then,

$$x_0 = 0 \qquad \dot{x}_0 = \frac{\hat{F}}{m}$$

- Response ends up looking like this:

$$x(t) = \frac{\hat{F}}{m\omega_d}\, e^{-\zeta\omega_n t}\, \sin\omega_d t$$

Follow Up Discussion, Concluding Remark

- Important observation, which simplifies the picture
  - Because we are dealing with linear systems, to find their response to any impulsive force, first find the response of the system to the Dirac Function
  - Response of the system to the Dirac Function is denoted by g(t)
  - Then, for any impulsive function with $\hat{F}$, the response is simply going to be a scaling of the response obtained for the Dirac Function:

$$x(t) = \hat{F}\, g(t)$$

- Look back at the previous two slides to see that the response of an underdamped or undamped system is indeed obtained as indicated above…

[New Topic:]

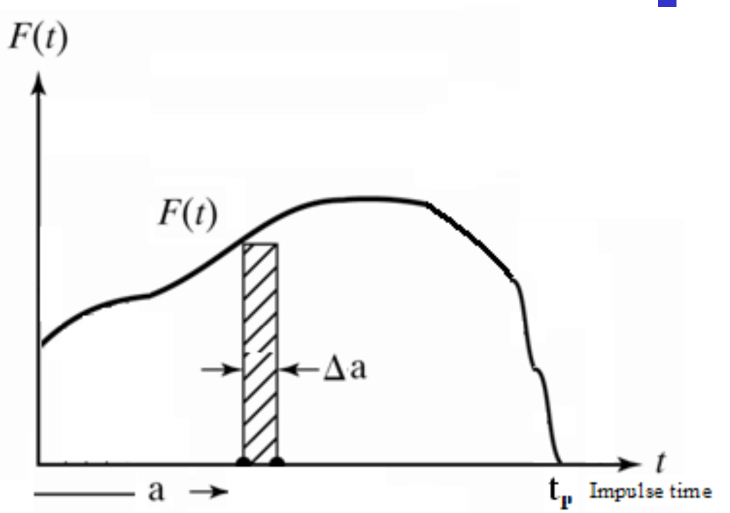
Arbitrary Excitation

- To understand this topic, you need to keep in mind three things:
  - First, we are dealing with linear systems, and the principle of superposition holds
  - Second, recall that an integral is defined as a sum (a special sum, that is)
  - Third, recall that g(t) represents the response of the system to a Dirac Delta function. For any arbitrary impulse, the response is just a scaling of g(t) to obtain x(t)
- The claim is that having the expression of g(t) is a very important step in understanding the response of the system to *any* arbitrary external excitation F(t)

[Cntd.]

Arbitrary Excitation

- The fundamental idea:
  - Consider the arbitrary excitation F(t) to be just a train of successive impulsive excitations…
  - … then apply the principle of superposition to find the response as the sum of the responses to this train of impulsive excitations



- Examine one of the impulses
  - Cross-hatched in the figure
  - At t=a its strength is (the cross-hatched area):

$$\hat{F} = F(a)\Delta a$$

  - The system’s response at any time t, where t>a, is dependent upon the elapsed time (t – a):

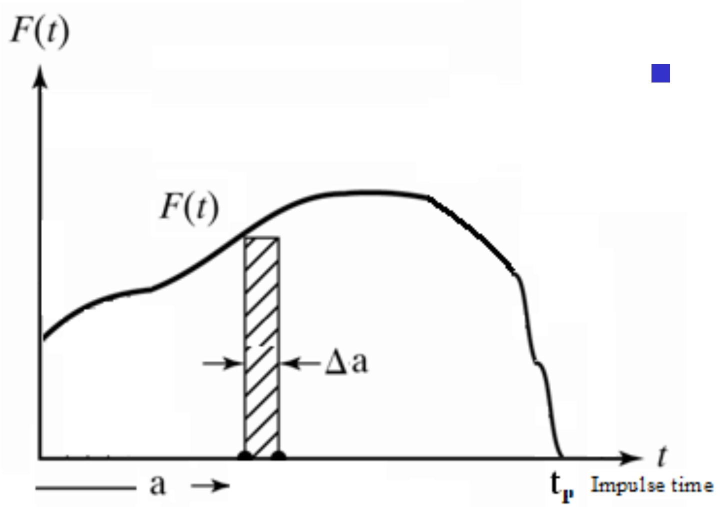
$$F(a)\ \Delta a\ g(t-a)$$

[Cntd.]

Arbitrary Excitation

- The fundamental idea:
  - Consider the arbitrary excitation F(t) to be just a train of successive impulsive excitations…
  - … then apply the principle of superposition to find the response as the sum of the responses to this train of impulsive excitations

- Now sum up all the responses, for every single value of a…
- This is where the integral part comes into play:



- Examine one of the impulses
  - Cross-hatched in the figure
  - At t=a its strength is (the cross-hatched area):

$$\hat{F} = F(a)\Delta a$$

  - The response at a time t is then obtained by combining all these contributions, for each t>a:

$$x_{Cnv}(t) = \sum F(a)\; g(t-a)\; \Delta a = \int_0^t F(a)g(t-a)da$$

[Cntd.]

Arbitrary Excitation

- Express in standard form: $$x_{Cnv}(t) = \int_0^t F(\tau) g(t-\tau) d\tau$$

- Integral above: the "convolution integral" or "Duhamel's integral"

- Another equivalent form in which it's known (do change of variables…):

$$x_{Cnv}(t) = \int_0^t F(t-\lambda) g(\lambda) d\lambda$$

- Because τ and λ are dummy integration variables, I can say that

$$x_{Cnv}(t) = \int_0^t F(\tau) g(t-\tau) d\tau = \int_0^t F(t-\tau) g(\tau) d\tau$$

[Concluding Remarks]

Arbitrary Excitation

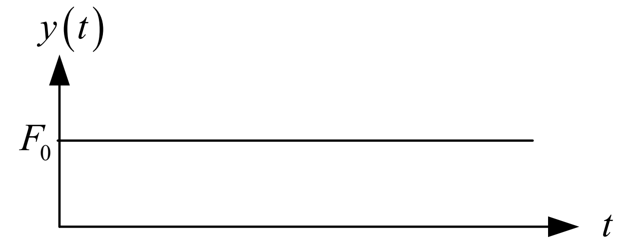
- You have a system and you are in a position to tell how it responds to a Dirac impulse (this response has been denoted by g(t))
- You have some arbitrary excitation F(t) applied to the system

- What have we just accomplished?
  - We can find the response $x_{Cnv}(t)$ of the system to the arbitrary excitation F(t):

$$x_{Cnv}(t) = \int_0^t F(\tau) g(t-\tau) d\tau = \int_0^t F(t-\tau) g(\tau) d\tau$$

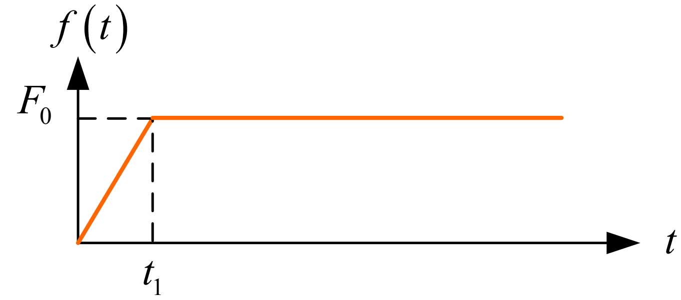
- In practical applications, when you have to evaluate one of these two integrals, shift the simpler of the two functions
  - See Examples…

Example [AO]



- Determine the response of a 1DOF system to the step function
  - Work with an <u>undamped</u> system

Example [AO]



- Determine undamped response for a step function with rise time t_1
- Use a trick to break excitation function into two parts: