ME451
Kinematics and Dynamics of Machine Systems

Introduction to Dynamics

6.3.4, 6.6

November 25, 2014

- Quote of the day: "Marge, don't discourage the boy! Weaseling out of things is important to learn. It's what separates us from the animals! Except the weasel."

-- Homer Simpson

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Before we get started…

- Last time
  - Lagrange Multiplier Theorem
  - EOM for a collection of rigid bodies connected through joints
  - Example of setting up the equations of motion, slider crank example
- Today
  - Setting up initial conditions for the dynamics analysis
  - Revisit the computation of reaction forces
  - Maybe start discussion of numerical integration
- Assignment posted online due on December 2
  - HW 9: 6.3.3, 6.4.2
  - MATLAB 8 – posted online
  - ADAMS 5 – posted online
- Project 2 assigned on Tuesday, due 12/11 at 11:59 PM
- Exam on Th December 4, open everything
  - Review session on Wd, December 3 at 7:15 PM (same idea like last time)
  - Room: 1163ME (next door room)

Most Important Slide of ME451

- Equations of Motion $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}\lambda = \mathbf{Q}^{A}$
- Position Constraint Equations $\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0}$
- Velocity Constraint Equations $\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\mathbf{\Phi}_{t} \triangleq \nu$
- Acceleration Constraint Equations $\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -\left(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt} \triangleq \gamma$

6.3.4

Initial Conditions

[making the simple complicated]

'Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creative.‘
-- Charles Mingus

ICs for the EOM of Constrained Planar Systems

- We must provide ICs at the initial time t_0 to "seed" the numerical solution
  - How many can/should we specify?
  - How exactly do we specify them?

- Recall that the constraint and velocity equations must be satisfied at all times (including the initial time t_0)

- In other words, we have nc generalized coordinates, but they are not independent, as they must satisfy

$$\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0}$$
$$\mathbf{\Phi}_{\mathbf{q}} \dot{\mathbf{q}} = \nu$$

Specifying Position ICs (1/2)

- We have nc generalized coordinates that must satisfy m equations, thus leaving $NDOF = nc - m$ degrees of freedom

$$\mathbf{\Phi}(\mathbf{q}_0, t_0) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}_0) \\ \mathbf{\Phi}^D(\mathbf{q}_0, t_0) \end{bmatrix} = \mathbf{0} \qquad \mathbf{q}_0 \in \mathbb{R}^{nc}, \; \mathbf{\Phi} \in \mathbb{R}^{m}$$

- To completely specify the position configuration at t_0 we must therefore provide additional $NDOF$ conditions

- How can we do this?
  - Recall what we did in Kinematics to specify driver constraints (to "take care" of the excess DOFs): provide $NDOF$ additional conditions
  - In Dynamics, to specify IC, we provide $NDOF$ additional conditions of the form

$$\mathbf{\Phi}^I(\mathbf{q}_0, t_0) = \mathbf{0} \qquad \mathbf{\Phi}^I \in \mathbb{R}^{NDOF}$$

Specifying Position ICs (2/2)

- The complete set of conditions that the generalized coordinates must satisfy at the initial time t_0 is therefore

$$\mathbf{\Phi}^*(\mathbf{q}_0, t_0) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}_0) \\ \mathbf{\Phi}^D(\mathbf{q}_0, t_0) \\ \mathbf{\Phi}^I(\mathbf{q}_0, t_0) \end{bmatrix} = \mathbf{0}$$

- How do we know that the IC we imposed are properly specified?
- Implicit Function Theorem gives us the answer: the Jacobian must be nonsingular

$$\det\left(\mathbf{\Phi}^*_{\mathbf{q}}(\mathbf{q}_0, t_0)\right) \neq 0$$

- If this is the case, we can solve the nonlinear system (using for example Newton's method)

$$\mathbf{\Phi}^*(\mathbf{q}_0, t_0) = \mathbf{0}$$

to obtain the initial configuration $\mathbf{q}_0$ at the initial time t_0

Specifying Velocity ICs (1/2)

- Specifying a set of position ICs is not enough
- We are dealing with 2nd order differential equations and we therefore also need ICs for the generalized velocities
- The generalized velocities must satisfy the velocity equation at all times, in particular at the initial time t_0

$$\mathbf{\Phi_q}(\mathbf{q}_0, t_0)\dot{\mathbf{q}}_0 = \begin{bmatrix} \mathbf{\Phi}^K_{\mathbf{q}}(\mathbf{q}_0) \\ \\ \mathbf{\Phi}^D_{\mathbf{q}}(\mathbf{q}_0, t_0) \end{bmatrix} \dot{\mathbf{q}}_0 = \begin{bmatrix} \nu^K \\ \\ \nu^D \end{bmatrix} = \nu \qquad \dot{\mathbf{q}}_0 \in \mathbb{R}^{nc} \text{ , } \mathbf{\Phi_q} \in \mathbb{R}^{m \times nc}$$

- We have two choices:
  - Specify velocity ICs for the same generalized coordinates for which we specified initial position ICs

$$\mathbf{\Phi}^I_{\mathbf{q}}(\mathbf{q}_0, t_0)\dot{\mathbf{q}}_0 = \nu^I \qquad \mathbf{\Phi}^I_{\mathbf{q}} \in \mathbb{R}^{NDOF \times nc}$$

  - Specify velocity ICs on a completely different set of generalized coordinates

$$\mathbf{B}^I(\mathbf{q}_0, t_0)\dot{\mathbf{q}}_0 = \nu^I \qquad \mathbf{B}^I \in \mathbb{R}^{NDOF \times nc}$$

Specifying Velocity ICs (2/2)

- In either case, we must be able to find a unique solution for the initial generalized velocities $\dot{\mathbf{q}}_0$ at the initial time t_0

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}}^{K}(\mathbf{q}_0) \\ \mathbf{\Phi}_{\mathbf{q}}^{D}(\mathbf{q}_0, t_0) \\ \mathbf{\Phi}_{\mathbf{q}}^{I}(\mathbf{q}_0, t_0) \end{bmatrix} \dot{\mathbf{q}}_0 = \begin{bmatrix} \nu^K \\ \nu^D \\ \nu^I \end{bmatrix} \qquad \begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}}^{K}(\mathbf{q}_0) \\ \mathbf{\Phi}_{\mathbf{q}}^{D}(\mathbf{q}_0, t_0) \\ \mathbf{B}^{I}(\mathbf{q}_0, t_0) \end{bmatrix} \dot{\mathbf{q}}_0 = \begin{bmatrix} \nu^K \\ \nu^D \\ \nu^I \end{bmatrix}$$

- In both cases, we solve the linear system

$$\mathbf{\Phi}_{\mathbf{q}}^{*}(\mathbf{q}_0, t_0)\dot{\mathbf{q}}_0 = \nu$$

for the initial generalized velocities and therefore we must ensure that

$$\det\left(\mathbf{\Phi}_{\mathbf{q}}^{*}(\mathbf{q}_0, t_0)\right) \neq 0$$

Initial Conditions: Conclusions

- The IC problem is actually simple if we remember what we did in Kinematics regarding driver constraints

- We only do this at the initial time t_0 to provide a starting configuration for the mechanism. Otherwise, the dynamics problem is underdefined

- Initial conditions can be provided either by
  - Specifying a consistent initial configuration (that is a set of nc generalized coordinates and nc generalized velocities that satisfy the constraint and velocity equations at t_0)
    - This is what you should do for **`simEngine2D`**

  - Specifying additional $NDOF$ conditions (that are independent of the existing kinematic and driver constraints) and relying on the Kinematic solver to compute the consistent initial configuration
    - This is what a general purpose solver might do, such as ADAMS

Specifying ICs in `simEngine2D`

- Recall a typical body definition in an ADM file (JSON format)

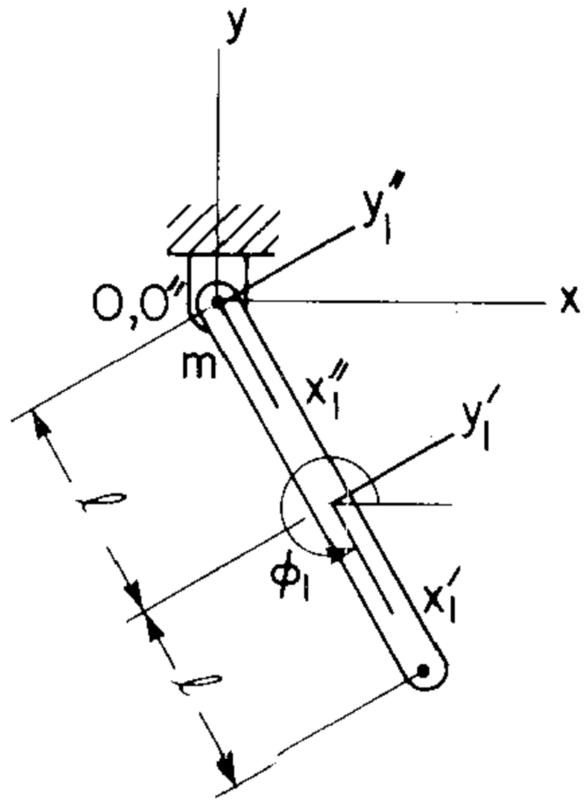
```
{
    "name": "slider",
    "id": 1,
    "mass": 2,
    "jbar": 0.3,
    "q0": [2, 0, 0],
    "qd0": [0, 0, 0]
}
```

- In other words, we include in the definition of a body its initial position and velocity (values for the generalized coordinates and velocities at the initial time, which we will always assume to be $t_0 = 0$)

- Make sure that the values **q0** and **qd0** are such that $\mathbf{\Phi}(\mathbf{q}_0, 0) = \mathbf{0}$ and $\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}_0 = \boldsymbol{\nu}$, and then you have a well defined Dynamic Analysis

ICs for a Simple Pendulum [handout]

- Specify ICs for the simple pendulum such that
  - it starts from a vertical configuration (hanging down), and
  - it has an initial angular velocity $2\pi \frac{rad}{s}$.
- Assume that $l = 0.2\ m$



6.6

Constraint Reaction Forces

[somewhat hard to grasp]

Reaction Forces

- Remember that we jumped through some hoops to get rid of the reaction forces that develop in joints

- Now, we want to go back and recover them, since they are important:
  - Durability analysis
  - Stress/Strain analysis
  - Selecting bearings in a mechanism
  - Etc.

- The key ingredient needed to compute the reaction forces in all joints is the set of Lagrange multipliers λ

Reaction Forces: The Basic Idea

- Recall the partitioning of the total force acting on the mechanical system

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \vdots \\ \mathbf{Q}_{nb} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A + \mathbf{Q}_1^C \\ \mathbf{Q}_2^A + \mathbf{Q}_2^C \\ \vdots \\ \mathbf{Q}_{nb}^A + \mathbf{Q}_{nb}^C \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A \\ \mathbf{Q}_2^A \\ \vdots \\ \mathbf{Q}_{nb}^A \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_1^C \\ \mathbf{Q}_2^C \\ \vdots \\ \mathbf{Q}_{nb}^C \end{bmatrix} = \mathbf{Q}^A + \mathbf{Q}^C$$

- Applying a variational approach (principle of virtual work) we ended up with this equation of motion

$$\delta\mathbf{q}^T \left(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}\right) = 0 \quad \Leftrightarrow \quad \delta\mathbf{q}^T \left(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A - \mathbf{Q}^C\right) = 0 \quad \Rightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A - \mathbf{Q}^C = \mathbf{0}$$

- After jumping through hoops, we ended up with this:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^T \lambda = \mathbf{Q}^A \quad \Leftrightarrow \quad \mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A + \mathbf{\Phi}_{\mathbf{q}}^T \lambda = \mathbf{0}$$

- It's easy to see that

$$\mathbf{Q}^C = -\mathbf{\Phi}_{\mathbf{q}}^T \lambda$$

Reaction Forces: Its Net Effect [1/6]

- What we obtain by multiplying the transposed Jacobian of a constraint, $\mathbf{\Phi}_{\mathbf{q}}^{T}$, with the computed corresponding Lagrange multiplier(s), $\boldsymbol{\lambda}$, is the constraint reaction force expressed as a generalized force:

$$\mathbf{Q}^{C} = -\mathbf{\Phi}_{\mathbf{q}}^{T}\lambda$$

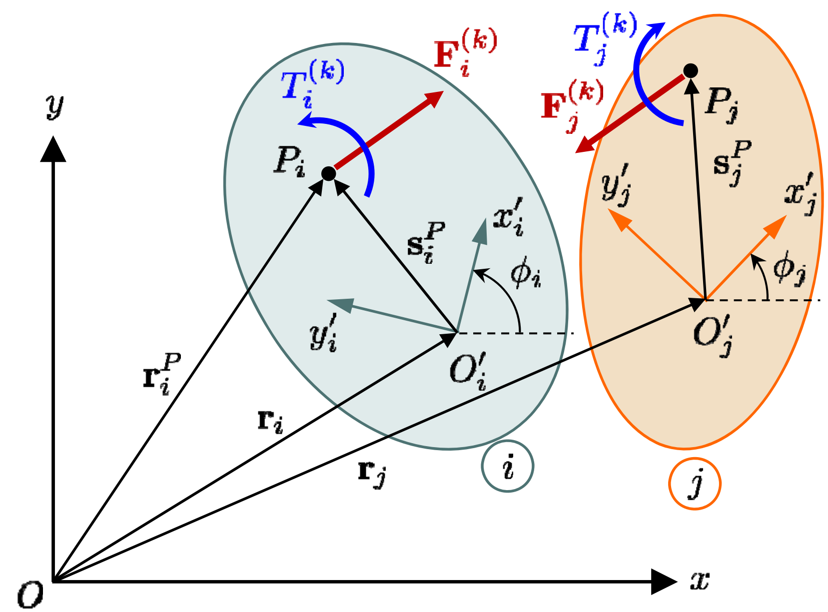
- Important Observation: One might want a physical representation of this generalized force
  - We would like to find F_x, F_y, and a torque T due to the constraint
  - We would like to report these quantities as acting at some point P on a body

- In other words: Look for a fictitious force which, when acting on the body at the point P, would lead to a generalized force equal to $\mathbf{Q}^{C}$

Reaction Forces: Framework [2/6]

- Assume that the k-th joint in the system constrains points P_i on body i and P_j on body j

- We are interested in finding the reaction forces and torques $\mathbf{F}_i^{(k)}$ and $T_i^{(k)}$ acting on body i at point P_i, as well as $\mathbf{F}_j^{(k)}$ and $T_j^{(k)}$ acting on body j at point P_j



- The book complicates the formulation for no good reason by expressing these reaction forces with respect to some arbitrary body-fixed RFs attached at the points P_i and P_j, respectively.

- It is much easier to derive the reaction forces and torques in the GRF and, if desired, re-express them in any other frame by using the appropriate rotation matrices

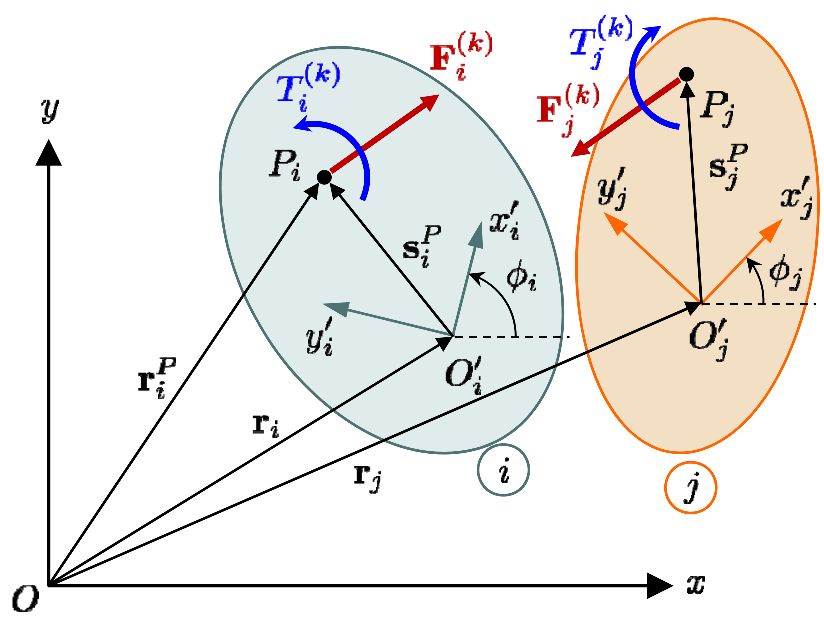
Reaction Forces: Setup [3/6]

- Let the m_k constraint equations defining the k-th joint be

$$\mathbf{\Phi}^{(k)}(\mathbf{q}_i, \mathbf{q}_j, t) = \mathbf{0}\,, \quad \mathbf{\Phi}^{(k)} \in \mathbb{R}^{m_k}$$

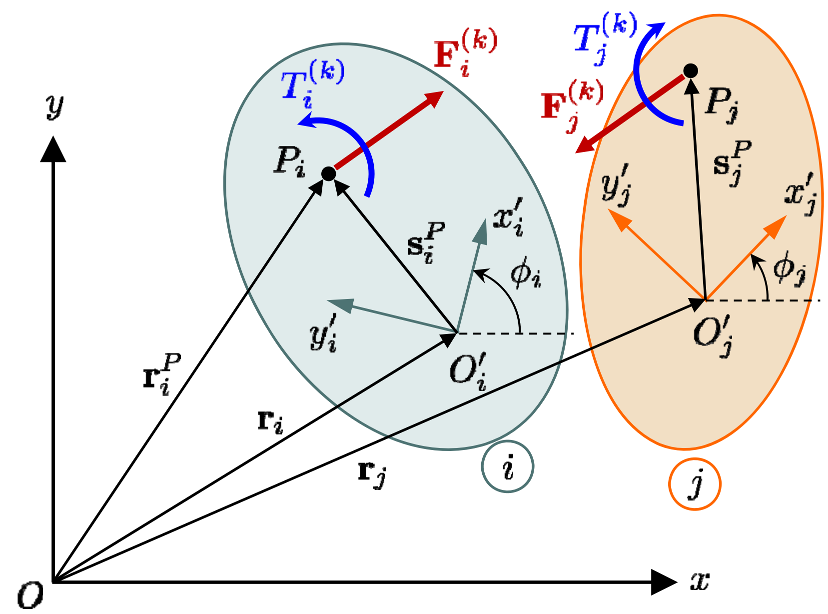
- Let the m_k Lagrange multipliers associated with this joint be

$$\lambda^{(k)} \in \mathbb{R}^{m_k}$$



Reaction Force: Closer Look [4/6]

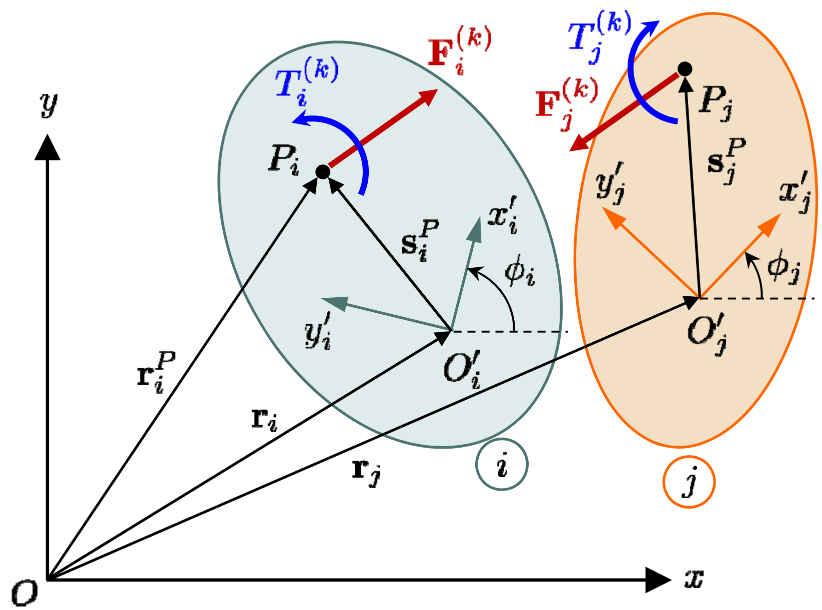
- For the sake of this discussion assume that $m_k = 1$
  - In other words, look at a constraint and not a joint such as revolute, translational, etc. – which has $m_k = 2$
  - Discussion is simpler this way – carries over to $m_k = 2$ as well



- Start by taking a closer look at the expression of the constraint reaction force induced by the presence of the kinematic constraint k

$$[\mathbf{Q}_i^{(k)}]^C = -[\Phi_{\mathbf{q}_i}^{(k)}]^T \lambda^{(k)} = \begin{bmatrix} -[\Phi_{\mathbf{r}_i}^{(k)}]^T \\ -[\Phi_{\phi_i}^{(k)}]^T \end{bmatrix} \lambda^{(k)} = \begin{bmatrix} -[\Phi_{\mathbf{r}_i}^{(k)}]^T \cdot \lambda^{(k)} \\ -[\Phi_{\phi_i}^{(k)}]^T \cdot \lambda^{(k)} \end{bmatrix}$$

Reaction Forces: Main Result [5/6]



- The presence of the k-th joint leads to the following reaction force and torque at point P_i on body i

$$\mathbf{F}_i^{(k)} = -\left(\mathbf{\Phi}_{\mathbf{r}_i}^{(k)}\right)^T \lambda^{(k)}$$

$$T_i^{(k)} = \left(\mathbf{\Phi}_{\mathbf{r}_i}^{(k)} \mathbf{B}_i \mathbf{s}_i'^P - \mathbf{\Phi}_{\phi_i}^{(k)}\right)^T \lambda^{(k)}$$

Reaction Forces: Comments [6/6]

- For constraint equations that act between two bodies i and j, there will also be a $\mathbf{F}_j$, T_j pair associated with such constraints, representing the constraint reaction forces on body j
  - To get $\mathbf{F}_i$ and T_i, respectively

$$\mathbf{F}_j^{(k)} = -\left(\mathbf{\Phi}_{\mathbf{r}_j}^{(k)}\right)^T \lambda^{(k)}$$

$$T_j^{(k)} = \left(\mathbf{\Phi}_{\mathbf{r}_j}^{(k)}\mathbf{B}_j\mathbf{s}'^P_j - \mathbf{\Phi}_{\phi_j}^{(k)}\right)^T \lambda^{(k)}$$

- Note that the only thing that we had to do was to replace i with j.
  - There is nothing special about i relative to j.

Reaction Forces: Comments

- Note that there is one Lagrange multiplier associated with each constraint equation
  - Number of Lagrange multipliers in mechanism is equal to number of constraints
  - Example: the revolute joint brings along a set of two kinematic constraints and therefore there will be two Lagrange multipliers associated with this joint
- Each Lagrange multiplier produces (leads to) a reaction force/torque combo
- Therefore, to each constraint equation corresponds a reaction force/torque pair that "enforces" the satisfaction of the constraint, throughout the time evolution of the mechanism
- If the system is kinematically driven (meaning there are driver constraints), the same approach is applied to obtain reaction forces associated with such constraints
  - In this case, we obtain the force/torque required to impose that driving constraint

Reaction Forces: Summary

- Each constraint in the system has a Lagrange multiplier associated with it

- The Lagrange multiplier results in the following reaction force and torque

$$\mathbf{F}_i^{(k)} = -\left(\mathbf{\Phi}_{\mathbf{r}_i}^{(k)}\right)^T \lambda^{(k)}$$

$$T_i^{(k)} = \left(\mathbf{\Phi}_{\mathbf{r}_i}^{(k)} \mathbf{B}_i \mathbf{s}'^P_i - \mathbf{\Phi}_{\phi_i}^{(k)}\right)^T \lambda^{(k)}$$

Note: The expression of $\mathbf{\Phi}$ for all the usual joints is known, so a boiler plate approach can be used to obtain the reaction force in all these joints

- An alternative expression for the reaction torque is

$$T_i^{(k)} = -\left(\mathbf{B}_i \mathbf{s}'^P_i\right)^T \mathbf{F}_i^{(k)} - \left(\mathbf{\Phi}_{\phi_i}^{(k)}\right)^T \lambda^{(k)}$$

Exam Question

- What is the expression of the reaction force/torque on body i induced by the constraint k if the point P is chosen to be the COM for body i ?

Reaction force in a Revolute Joint [Example 6.6.1]

Consider the following kinematically driven simple pendulum, with

$\phi_1 = 2\pi t + \frac{3\pi}{2}$

1. Find the reaction force in the revolute joint that connects the pendulum to ground.
2. Express the reaction force in the $O''x''y''$ reference frame