

# ME451

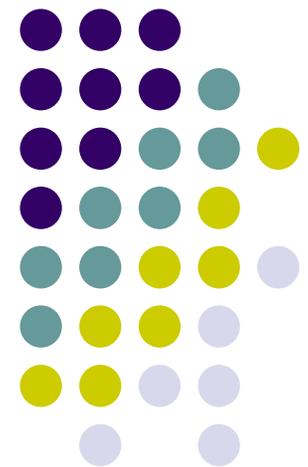
## Kinematics and Dynamics of Machine Systems

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### Introduction to Dynamics

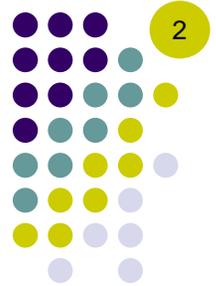
6.3.2, 6.3.3

November 20, 2014



- Quote of the day: "My personal philosophy is not to undertake a project unless it is manifestly important and nearly impossible."  
-- Edwin Land (Polaroid camera inventor)

# Before we get started...



- Last time
  - TSDA and RSDA force elements
  - Equations of motion for collections of bodies
- Today
  - Lagrange Multiplier Theorem
  - EOM for a collection of rigid bodies connected through joints
  - Example of setting up the equations of motion, slider crank example
  - Setting up initial conditions for the dynamics analysis
- Assignment posted online due on December 2
  - HW 9: 6.3.3, 6.4.2
  - MATLAB 8 – posted online
  - ADAMS 5 – posted online
- Final Project proposal was due on Nov 18 @ 11:59 PM
  - Please ping me if you haven't heard from me by the end of day today
- Project 2 assigned on Tuesday, due 12/11 at 11:59 PM



6.3.2, 6.3.3

**Lagrange Multipliers**

**Mixed Differential-Algebraic Equations of Motion**

# Short Detour

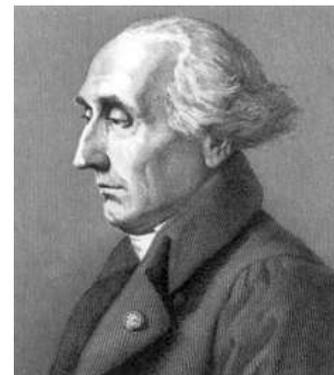
## ~ Lagrange Multiplier Theorem ~



- Theorem (my take on it, that is):

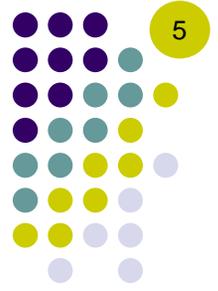
Assume that a vector  $\mathbf{b} \in \mathbb{R}^n$  and a matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$ , with  $m < n$ , are such that for **ANY** vector  $\mathbf{x} \in \mathbb{R}^n$  for which the condition  $\mathbf{A}\mathbf{x} = \mathbf{0}$  holds then the condition  $\mathbf{x}^T \mathbf{b} = 0$  also holds. In other words, any  $\mathbf{x}$  that is perpendicular to the rows of  $\mathbf{A}$  is automatically perpendicular to  $\mathbf{b}$  as well.

Then it turns out that there is a relationship between  $\mathbf{A}$  and  $\mathbf{b}$ , and in fact  $\mathbf{b}$  is a linear combination of the rows of  $\mathbf{A}$  (or of the columns of  $\mathbf{A}^T$ ). In other words, there is a so called “Lagrange Multiplier”  $\lambda$  such that  $\mathbf{b} = -\mathbf{A}^T \lambda$ , or equivalently,  $\mathbf{A}^T \lambda + \mathbf{b} = \mathbf{0}$ .



Joseph-Louis  
Lagrange  
(1736– 1813)

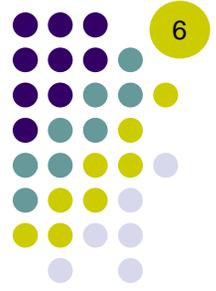
# Lagrange Multiplier Theorem: Example (1/2)



$$\text{Let } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

1. Show that  $\mathbf{x}^T \mathbf{b} = 0$  for any vector  $\mathbf{x} \in \mathbb{R}^3$  for which  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .
2. Show that there is indeed a vector  $\lambda$  such that  $\mathbf{b} + \mathbf{A}^T \lambda = \mathbf{0}$ .

# Lagrange Multiplier Theorem: Example (2/2)



$$\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

If  $\mathbf{Ax} = \mathbf{0}$ , then  $\mathbf{x}$  must be of the form  $\mathbf{x} = \begin{bmatrix} \alpha \\ -\alpha \\ \alpha \end{bmatrix}$  for some  $\alpha \in \mathbb{R}$ .

For such an  $\mathbf{x}$ , we find that  $\mathbf{x}^T \mathbf{b} = [\alpha, -\alpha, \alpha] \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \alpha - 3\alpha + 2\alpha = 0$ .

Then, according to the Lagrange Multiplier Theorem, there must exist two multipliers  $\lambda_1$  and  $\lambda_2$  such that

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad \begin{cases} 1 - \lambda_1 + 2\lambda_2 = 0 \\ 3 + 3\lambda_2 = 0 \\ 2 + \lambda_1 + \lambda_2 = 0 \end{cases}$$

$$\lambda = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

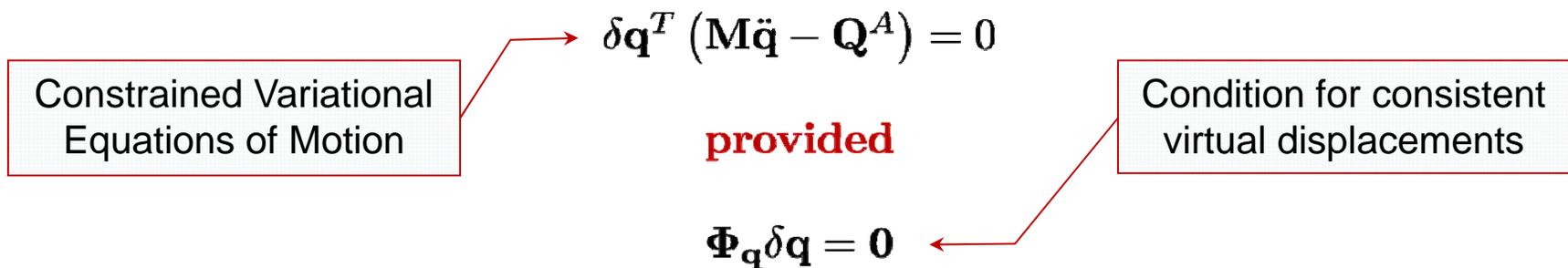
# Constrained Variational EOM



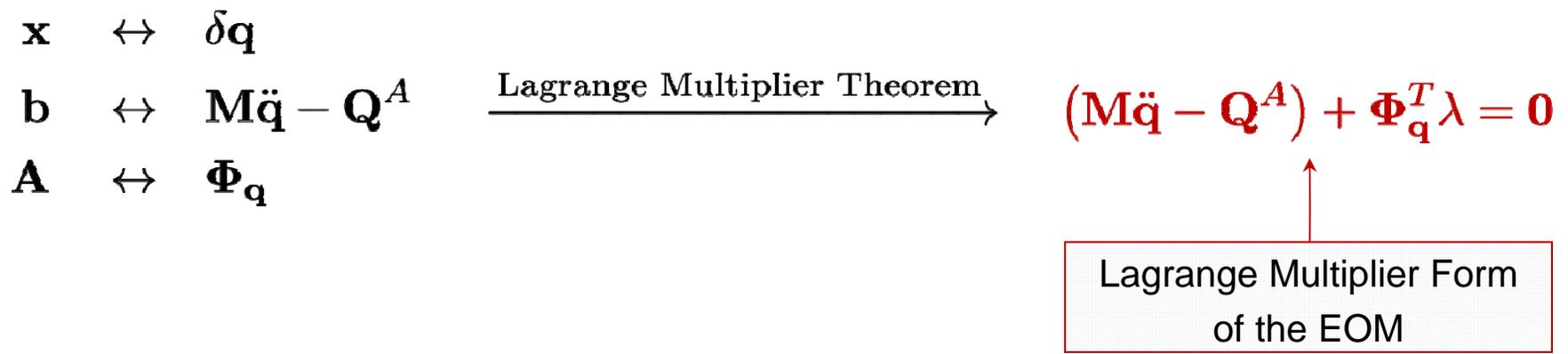
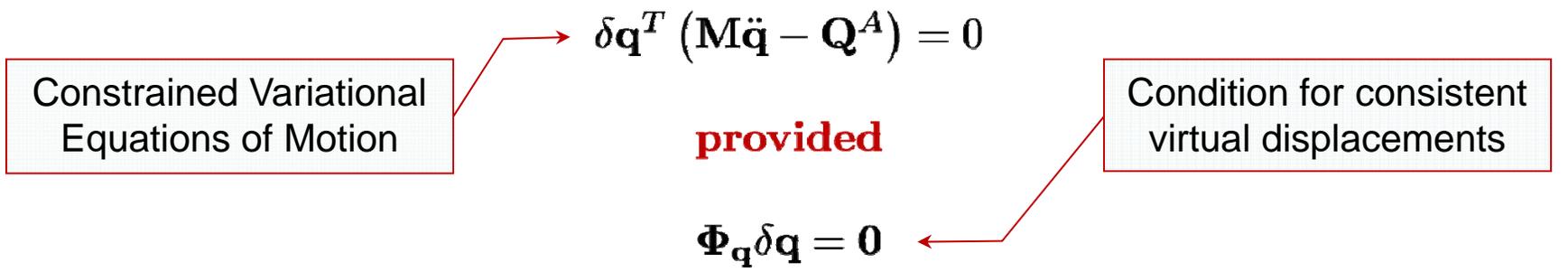
$$\delta \mathbf{q}^T (\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}) = 0 \quad \Leftrightarrow \quad \delta \mathbf{q}^T (\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A - \mathbf{Q}^C) = 0 \quad \Leftrightarrow \quad \delta \mathbf{q}^T (\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A) = 0$$

Arbitrary                      Arbitrary                      Consistent

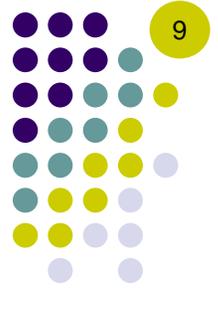
- We can eliminate the (unknown) constraint forces if we compromise to only consider virtual displacements that are **consistent** with the constraint equations



# Mixed Differential-Algebraic EOM



# Lagrange Multiplier Form of the EOM



- Equations of Motion  $\mathbf{M}\ddot{\mathbf{q}} + \Phi_{\mathbf{q}}^T \lambda = \mathbf{Q}^A$
- Position Constraint Equations  $\Phi(\mathbf{q}, t) = \mathbf{0}$
- Velocity Constraint Equations  $\Phi_{\mathbf{q}} \dot{\mathbf{q}} = -\Phi_t \triangleq \nu$
- Acceleration Constraint Equations  $\Phi_{\mathbf{q}} \ddot{\mathbf{q}} = -(\Phi_{\mathbf{q}} \dot{\mathbf{q}})_{\mathbf{q}} \dot{\mathbf{q}} - 2\Phi_{\mathbf{q}t} \dot{\mathbf{q}} - \Phi_{tt} \triangleq \gamma$

**Most Important Slide in ME451**

# Mixed Differential-Algebraic EOM



- Combine the EOM and the Acceleration Equation

$$\begin{bmatrix} \mathbf{M} & \Phi_{\mathbf{q}}^T \\ \Phi_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^A \\ \gamma \end{bmatrix}$$

to obtain a **mixed system of differential-algebraic equations**

- The constraint equations and velocity equation must also be satisfied

$$\Phi(\mathbf{q}, t) = \mathbf{0}$$

$$\Phi_{\mathbf{q}} \dot{\mathbf{q}} = \nu$$

- **Question:** Under what conditions can we uniquely calculate the generalized accelerations and Lagrange multipliers?
  - Remember that we're always after computing the accelerations

# Constrained Dynamic Existence Theorem



**Theorem:** Let the Jacobian have full row rank and let the kinetic energy of the system be positive for any nonzero consistent virtual velocity. Then the generalized accelerations and Lagrange multipliers are uniquely determined.

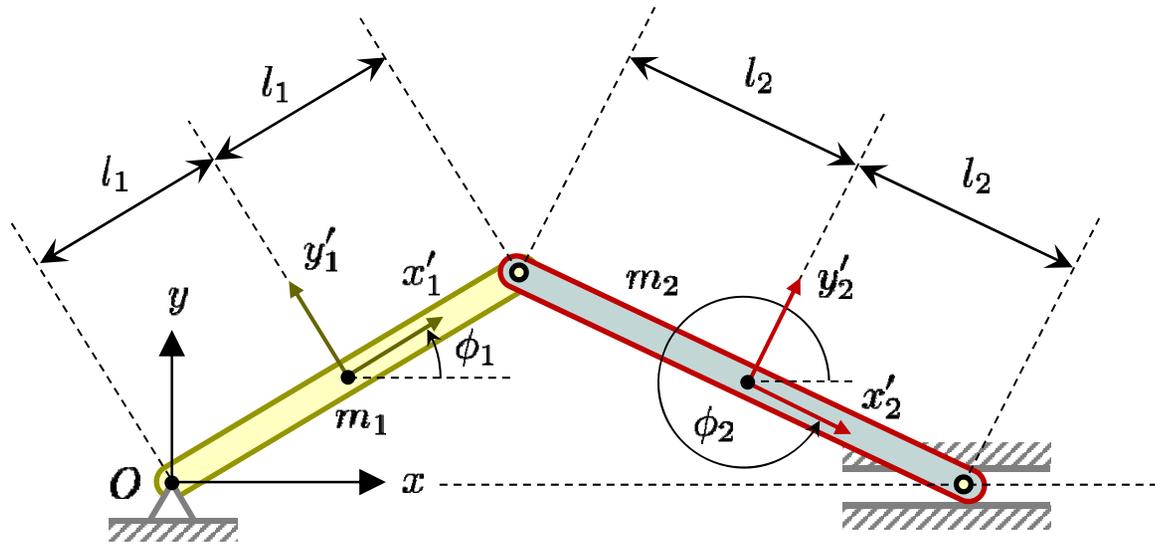
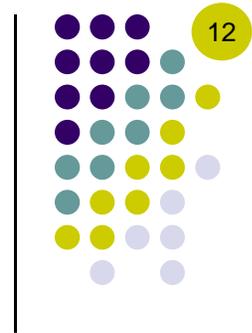
In other words:

$$\left. \begin{array}{l} \text{rank}(\Phi_{\mathbf{q}}) = nh \\ \delta \dot{\mathbf{q}}^T \mathbf{M} \delta \dot{\mathbf{q}} > 0, \forall \delta \dot{\mathbf{q}} \neq 0 \text{ such that } \Phi_{\mathbf{q}} \delta \dot{\mathbf{q}} = \mathbf{0} \end{array} \right\} \Rightarrow \begin{bmatrix} \mathbf{M} & \Phi_{\mathbf{q}}^T \\ \Phi_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \text{ nonsingular}$$

Under these conditions, we can therefore solve a linear system to obtain

- **Generalized accelerations**  $\ddot{\mathbf{q}}$   
which can be integrated to obtain generalized velocities and positions
- **Lagrange multipliers**  $\lambda$   
which can be used to compute the joint reaction forces

# Slider-Crank Example (1/4)



Body 1:  $m_1, J'_1 = \frac{m_1 l_1^2}{3}$

Body 2:  $m_2, J'_2 = \frac{m_2 l_2^2}{3}$

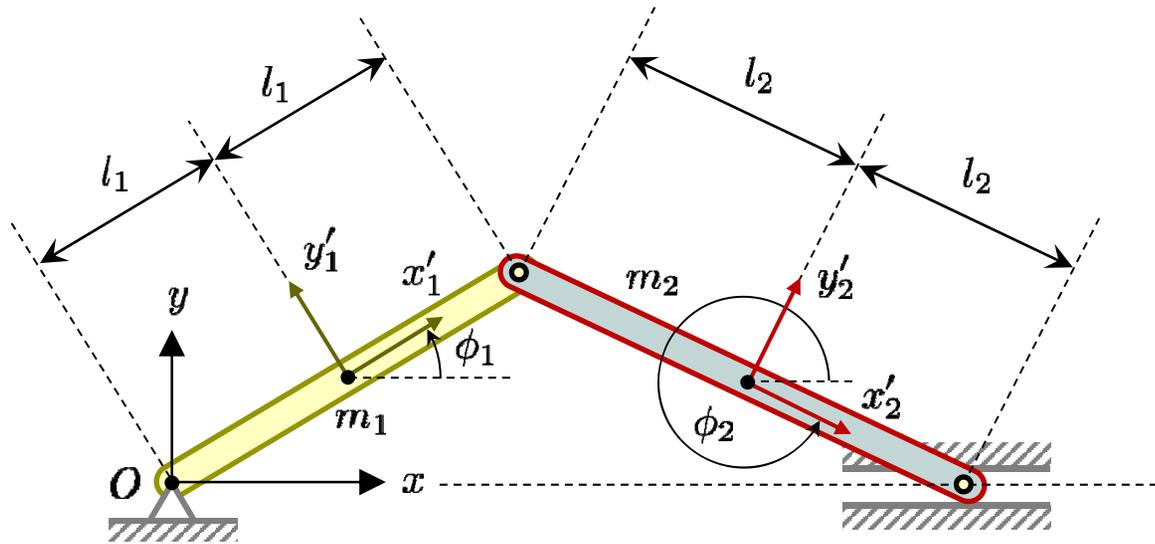
Virtual work of applied forces:

$$\delta W = [\delta x_1 \quad \delta y_1 \quad \delta \phi_1 \quad \delta x_2 \quad \delta y_2 \quad \delta \phi_2] \begin{bmatrix} 0 \\ -m_1 g \\ 0 \\ 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

Generalized applied forces:

$$\mathbf{Q}^A = \begin{bmatrix} 0 \\ -m_1 g \\ 0 \\ 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

# Slider-Crank Example (2/4)



Body 1:  $m_1, J'_1 = \frac{m_1 l_1^2}{3}$

Body 2:  $m_2, J'_2 = \frac{m_2 l_2^2}{3}$

Constraint Equations:

$$\Phi(\mathbf{q}) = \begin{bmatrix} x_1 - l_1 \cos \phi_1 \\ y_1 - l_1 \sin \phi_1 \\ (x_2 - l_2 \cos \phi_2) - (x_1 + l_1 \cos \phi_1) \\ (y_2 - l_2 \sin \phi_2) - (y_1 + l_1 \sin \phi_1) \\ y_2 + l_2 \sin \phi_2 \end{bmatrix}$$

Jacobian:

$$\Phi_{\mathbf{q}} = \begin{bmatrix} 1 & 0 & l_1 \sin \phi_1 & 0 & 0 & 0 \\ 0 & 1 & -l_1 \cos \phi_1 & 0 & 0 & 0 \\ -1 & 0 & l_1 \sin \phi_1 & 1 & 0 & l_2 \sin \phi_2 \\ 0 & -1 & -l_1 \cos \phi_1 & 0 & 1 & -l_2 \cos \phi_2 \\ 0 & 0 & 0 & 0 & 1 & l_2 \cos \phi_2 \end{bmatrix}$$

# Slider-Crank Example (3/4)



$$(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}^A) + \Phi_{\mathbf{q}}^T \lambda = \begin{bmatrix} m_1 \ddot{x}_1 \\ m_1 \ddot{y}_1 + m_1 g \\ \frac{m_1 l_1^2}{3} \ddot{\phi}_1 \\ m_2 \ddot{x}_2 \\ m_2 \ddot{y}_2 \\ \frac{m_2 l_2^2}{3} \ddot{\phi}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ l_1 \sin \phi_1 & -l_1 \cos \phi_1 & l_1 \sin \phi_1 & -l_1 \cos \phi_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & l_2 \sin \phi_2 & -l_2 \cos \phi_2 & l_2 \cos \phi_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{bmatrix} = \mathbf{0}$$

Lagrange Multiplier Form of the EOM

$$\Phi(\mathbf{q}) = \begin{bmatrix} x_1 - l_1 \cos \phi_1 \\ y_1 - l_1 \sin \phi_1 \\ (x_2 - l_2 \cos \phi_2) - (x_1 + l_1 \cos \phi_1) \\ (y_2 - l_2 \sin \phi_2) - (y_1 + l_1 \sin \phi_1) \\ y_2 + l_2 \sin \phi_2 \end{bmatrix} = \mathbf{0}$$

Constraint Equations

$$\Phi_{\mathbf{q}} \ddot{\mathbf{q}} = \gamma \equiv \begin{bmatrix} -l_1 \dot{\phi}_1^2 \cos \phi_1 \\ -l_1 \dot{\phi}_1^2 \sin \phi_1 \\ -l_2 \dot{\phi}_2^2 \cos \phi_2 - l_1 \dot{\phi}_1^2 \cos \phi_1 \\ -l_2 \dot{\phi}_2^2 \sin \phi_2 - l_1 \dot{\phi}_1^2 \sin \phi_1 \\ -l_2 \dot{\phi}_2^2 \sin \phi_2 \end{bmatrix} = \mathbf{0}$$

Acceleration Equation

# Slider-Crank Example (4/4)



Mixed Differential-Algebraic Equations of Motion



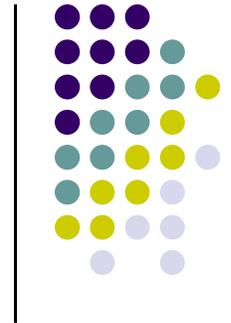
$$\begin{bmatrix}
 m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
 0 & 0 & \frac{m_1 l_1^2}{3} & 0 & 0 & 0 & l_1 \sin \phi_1 & -l_1 \cos \phi_1 & l_1 \sin \phi_1 & -l_1 \cos \phi_1 & 0 \\
 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & \frac{m_2 l_2^2}{3} & 0 & 0 & l_2 \sin \phi_2 & -l_2 \cos \phi_2 & l_2 \cos \phi_2 \\
 \hline
 1 & 0 & l_1 \sin \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -l_1 \cos \phi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & l_1 \sin \phi_1 & 1 & 0 & l_2 \sin \phi_2 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & -l_1 \cos \phi_1 & 0 & 1 & -l_2 \cos \phi_2 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & l_2 \cos \phi_2 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 \ddot{x}_1 \\
 \ddot{y}_1 \\
 \ddot{\phi}_1 \\
 \ddot{x}_2 \\
 \ddot{y}_2 \\
 \ddot{\phi}_2 \\
 \lambda_1 \\
 \lambda_2 \\
 \lambda_3 \\
 \lambda_4 \\
 \lambda_5
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 -m_1 g \\
 0 \\
 0 \\
 -m_2 g \\
 0 \\
 -l_1 \dot{\phi}_1^2 \cos \phi_1 \\
 -l_1 \dot{\phi}_1^2 \sin \phi_1 \\
 -l_2 \dot{\phi}_2^2 \cos \phi_2 - l_1 \dot{\phi}_1^2 \cos \phi_1 \\
 -l_2 \dot{\phi}_2^2 \sin \phi_2 - l_1 \dot{\phi}_1^2 \sin \phi_1 \\
 -l_2 \dot{\phi}_2^2 \sin \phi_2
 \end{bmatrix}$$

$$\begin{bmatrix}
 x_1 - l_1 \cos \phi_1 \\
 y_1 - l_1 \sin \phi_1 \\
 (x_2 - l_2 \cos \phi_2) - (x_1 + l_1 \cos \phi_1) \\
 (y_2 - l_2 \sin \phi_2) - (y_1 + l_1 \sin \phi_1) \\
 y_2 + l_2 \sin \phi_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Constraint Equations

$$\begin{bmatrix}
 1 & 0 & l_1 \sin \phi_1 & 0 & 0 & 0 \\
 0 & 1 & -l_1 \cos \phi_1 & 0 & 0 & 0 \\
 -1 & 0 & l_1 \sin \phi_1 & 1 & 0 & l_2 \sin \phi_2 \\
 0 & -1 & -l_1 \cos \phi_1 & 0 & 1 & -l_2 \cos \phi_2 \\
 0 & 0 & 0 & 0 & 1 & l_2 \cos \phi_2
 \end{bmatrix}
 \begin{bmatrix}
 \dot{x}_1 \\
 \dot{y}_1 \\
 \dot{\phi}_1 \\
 \dot{x}_2 \\
 \dot{y}_2 \\
 \dot{\phi}_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}$$

Velocity Equation



6.3.4

## **Initial Conditions**

**[making the simple complicated]**

'Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creative.'

-- Charles Mingus

# A Philosophical Take



- What are we talking about here?
  - Problems where you know the rate of change for a system and you are asked where the system will be, for instance, one year from now
- Pretty obvious thing:
  - If you know the rate at which change occurs, you need to know where the system is now in order to know where it'll be in the future
  - That is, you need initial conditions

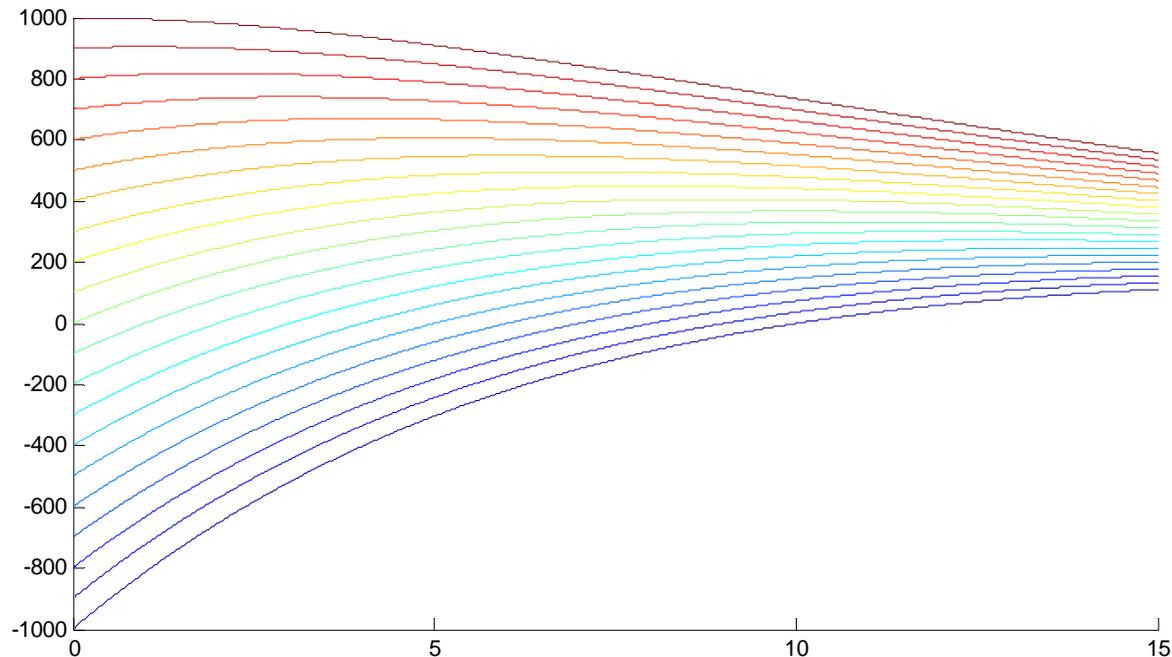
# An Example

$$\text{IVP: } \begin{cases} \text{ODE: } \dot{y} = -0.1y + 100e^{-0.1t} \\ \text{IC: } y(t_0) = y_0 \end{cases}$$

Note: An Ordinary Differential Equation (ODE) is a DE involving a single independent variable (time in this case).

What's known: you know  $t$  and state  $y$ . What you can figure out is the rate of change  $\dot{y}$  of the state  $y$ .

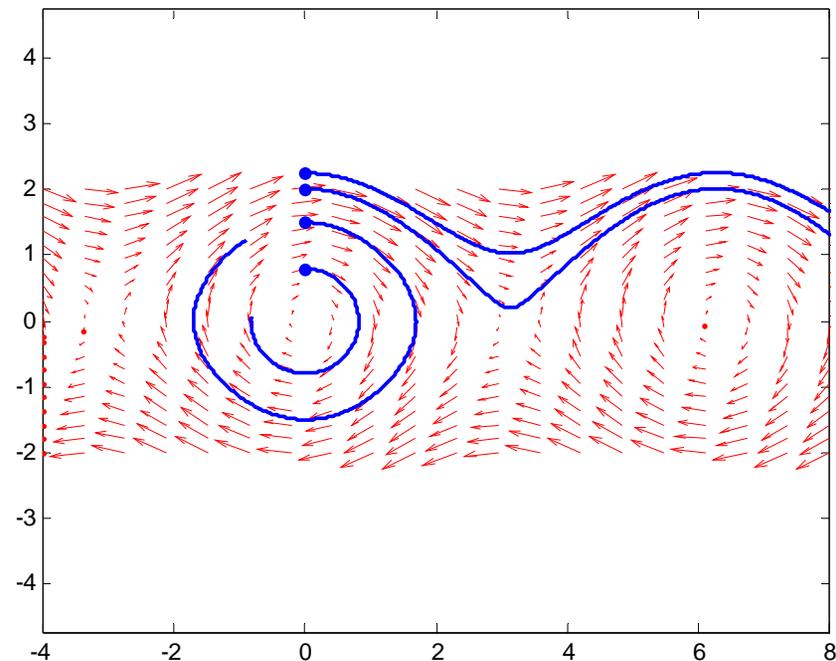
Consider several ICs:  $y_0 = [-1000 : 100 : 1000]$



# The Need for Initial Conditions



- A general solution of a differential equation (DE) of order  $k$  will contain  $k$  arbitrary independent constants of integration
- A particular solution is obtained by setting these constants to particular values. This is done by choosing a set of **initial conditions** (ICs) → Initial Value Problem (IVP)
- Informally, consider an ordinary differential equation with 2 states
  - The differential equation specifies a “velocity” field in 2D
  - An IC specifies a starting point in 2D
  - Solving the IVP simply means finding a curve in 2D that starts at the specified IC and is always tangent to the local velocity field



# ICs for the EOM of Constrained Planar Systems



- We must provide ICs at the initial time  $t_0$  to “seed” the numerical solution
  - How many can/should we specify?
  - How exactly do we specify them?
- Recall that the constraint and velocity equations must be satisfied at **all** times (including the initial time  $t_0$ )
- In other words, we have  $nc$  generalized coordinates, but they are **not independent**, as they must satisfy

$$\Phi(\mathbf{q}, t) = \mathbf{0}$$

$$\Phi_{\mathbf{q}} \dot{\mathbf{q}} = \nu$$