# ME451
# Kinematics and Dynamics of Machine Systems

## Introduction to Dynamics

6.3.1

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- Quote of the day: “The best way to predict the future is to invent it.”

--Alan Kay

# Before we get started…

- Last time
  - Finish discussion properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies
  - Enlarging the family of forces that can show up in the equations of motion

- Today
  - TSDA and RSDA force elements
  - Equations of motion for collections of bodies

- Project 1 – Due date: Nov 18 at 11:59 PM
  - Requires you to use `simEngine2D` in conjunction with excavator example discussed in class
  - Not trivial, requires some thinking

- Assignment posted online due on Nov. 20
  - Has pen-and-paper, ADAMS, and MATLAB components
  - Due in one week, Nov. 20.

- Final Project proposal due on Nov 18 @ 11:59 PM - post your proposal on the Forum
  - A discussion thread was started on this topic
  - Dan to provide feedback

# A Couple of Quick Things

## [Four Issues]

- Issue 1: if you can answer a Forum post, please do so
  - I don't have a TA and get overwhelmed at times

- Issue 2: please use the MATLAB debugger
  - Learn how to place a breakpoint in MATLAB
  - Learn how to step through the code and inspect the values of your variables
  - This is a <u>big deal</u>

- Issue 3: When you debug your code, figure out by hand a $\boldsymbol{q}$ that you know is a consistent configuration of your mechanism
  - If you feed that $\boldsymbol{q}$ in your code, do you get $\Phi(\boldsymbol{q}, 0) = 0$?
    - If not, you have a bug in how you compute Phi

# A Couple of Quick Things (Issue 4)

**Setup:** Assume that we want to see if the third column of the Jacobian Jac is correct. The expression of the third column is

$$\frac{\partial \boldsymbol{\Phi}(\mathbf{q}, t)}{\partial \phi_1} .$$

**Key Remark:** The sensitivity of $\boldsymbol{\Phi}$ with respect to $\phi_1$ can be approximated as

$$\frac{\partial \boldsymbol{\Phi}(\mathbf{q}, t)}{\partial \phi_1} \approx \frac{\boldsymbol{\Phi}(\hat{\mathbf{q}}, t) - \boldsymbol{\Phi}(\mathbf{q}, t)}{\epsilon} .$$

**Notation used:** $\epsilon$ some small value, say $10^{-4}$, and

$$\mathbf{q} = \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 \\ x_2 \\ \vdots \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{q}} \equiv \begin{bmatrix} x_1 \\ y_1 \\ \phi_1 + \epsilon \\ x_2 \\ \vdots \end{bmatrix} .$$

**Bottom Line:** Compare the third column of Jac that you get in MATLAB with the approximate value that you get by applying approach above. If the two column vectors are quite different then you have at least one bug. If you know that $\boldsymbol{\Phi}(\mathbf{q}, t)$ is computed correctly (after discussion in Issue 3), then the bug is in the way you compute the Jac in MATLAB.

# (TSDA)
# Translational Spring-Damper-Actuator (1/2)

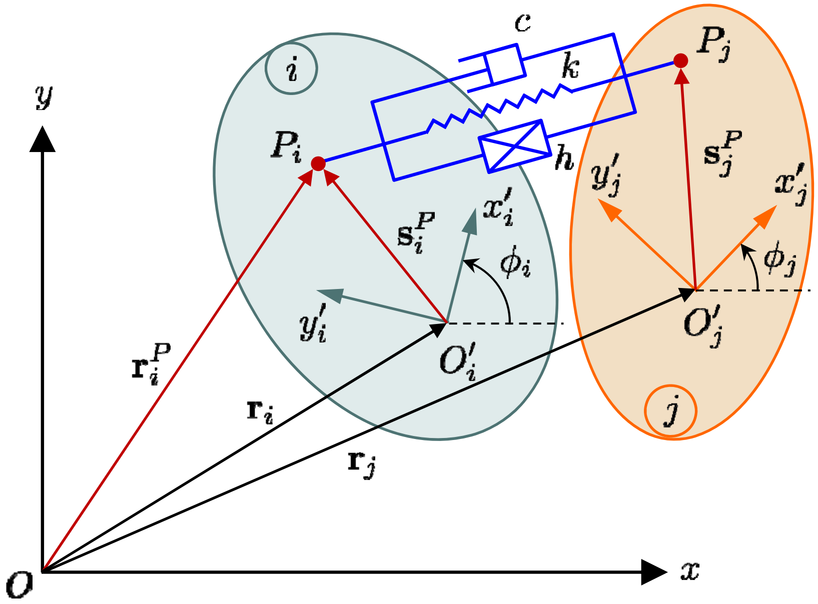
- Setup
    - Compliant connection between points $P_i$ on body $i$ and $P_j$ on body $j$
    - In its most general form it can consist of:
        - A spring with spring coefficient $k$ and free length $l_0$
        - A damper with damping coefficient $c$
        - An actuator (hydraulic, electric, etc.) which applies a force $h(l, \dot{l}, t)$

- The vector between points $P_i$ and $P_j$ is defined as

$$\mathbf{d}_{ij} = \mathbf{r}_j^P - \mathbf{r}_i^P$$
$$= \mathbf{r}_j + \mathbf{A}_j \mathbf{s}'^P_j - \mathbf{r}_i - \mathbf{A}_i \mathbf{s}'^P_i$$

and has a length of

$$l^2 = \mathbf{d}_{ij}^T \mathbf{d}_{ij}$$



# (TSDA)
# Translational Spring-Damper-Actuator (2/2)

- General Strategy
  - Write the virtual work produced by the force element in terms of an appropriate virtual displacement

$$\delta W = -f \, \delta l$$

Note: positive $\delta l$ separates the bodies

where

$$f = k(l - l_0) + c\dot{l} + h(l, \dot{l}, t)$$

Note: tension defined as positive

  - Express the virtual work in terms of the generalized virtual displacements $\delta \mathbf{q}_i$ and $\delta \mathbf{q}_j$

$$\delta l = \left( \frac{\mathbf{d}_{ij}}{l} \right)^T \left( \delta \mathbf{r}_j + \delta \phi_j \mathbf{B}_j \mathbf{s'}_j^P - \delta \mathbf{r}_i - \delta \phi_i \mathbf{B}_i \mathbf{s'}_i^P \right)$$

$$\delta W = -\frac{f}{l} \mathbf{d}_{ij}^T \left( \delta \mathbf{r}_j + \delta \phi_j \mathbf{B}_j \mathbf{s'}_j^P - \delta \mathbf{r}_i - \delta \phi_i \mathbf{B}_i \mathbf{s'}_i^P \right)$$

  - Identify the generalized forces (coefficients of $\delta \mathbf{q}_i$ and $\delta \mathbf{q}_j$)

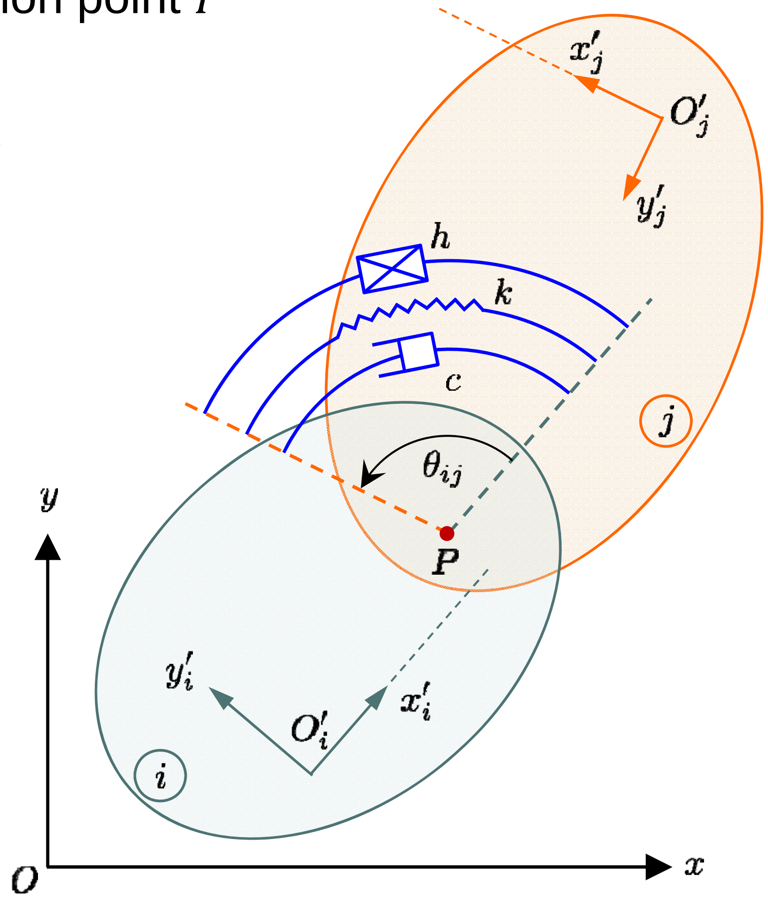
$$\mathbf{Q}_i = \frac{f}{l} \begin{bmatrix} \mathbf{d}_{ij} \\ \mathbf{d}_{ij}^T \mathbf{B}_i \mathbf{s'}_i^P \end{bmatrix} \qquad \mathbf{Q}_j = \frac{f}{l} \begin{bmatrix} \mathbf{d}_{ji} \\ \mathbf{d}_{ji}^T \mathbf{B}_j \mathbf{s'}_j^P \end{bmatrix} = -\frac{f}{l} \begin{bmatrix} \mathbf{d}_{ij} \\ \mathbf{d}_{ij}^T \mathbf{B}_j \mathbf{s'}_j^P \end{bmatrix}$$

# (RSDA)
# Rotational Spring-Damper-Actuator (1/2)

- Setup
  - Bodies $i$ and $j$ connected by a revolute joint at $P$
  - Torsional compliant connection at the common point $P$
  - In its most general form it can consist of:
    - A torsional spring with spring coefficient $k$ and free angle $\theta_0$
    - A torsional damper with damping coefficient $c$
    - An actuator (hydraulic, electric, etc.) which applies a torque $h(\theta_{ij}, \dot{\theta}_{ij}, t)$

- The angle $\theta_{ij}$ from $x'_i$ to $x'_j$ (positive counterclockwise) is

$$\boldsymbol{\theta_{ij} = \phi_j - \phi_i}$$



# (RSDA)
# Rotational Spring-Damper-Actuator (2/2)

- General Strategy
  - Write the virtual work produced by the force element in terms of an appropriate virtual displacement

$$\delta W = -n\,\delta\theta_{ij}$$

Note: positive $\delta\theta_{ij}$ separates the axes

where

$$n = k(\theta_{ij} - \theta_0) + c\dot{\theta}_{ij} + h(\theta_{ij}, \dot{\theta}_{ij}, t)$$

Note: tension defined as positive

  - Express the virtual work in terms of the generalized virtual displacements $\delta\mathbf{q}_i$ and $\delta\mathbf{q}_j$

$$\delta\theta_{ij} = \delta\phi_j - \delta\phi_i$$

$$\delta W = -n\,(\delta\phi_j - \delta\phi_i)$$

  - Identify the generalized forces (coefficients of $\delta\mathbf{q}_i$ and $\delta\mathbf{q}_j$)

$$\mathbf{Q}_i = \begin{bmatrix} \mathbf{0} \\ n \end{bmatrix} \qquad \mathbf{Q}_j = -\begin{bmatrix} \mathbf{0} \\ n \end{bmatrix}$$

# Why the Minus Sign in Front of $\delta l \cdot f$ ?

- Focus discussion on TSDA (RSDA is just the same)

- Remark 1: Tension in TSDA element; i.e., $l \geq l_0$ , leads to positive force $f$
- Remark 2: If virtual displacement leads to positive $\delta l$, then length of element increases to $l + \delta l$, that is, a stretched TSDA gets more stretched

- Our question: how will the virtual work balance have to change when a spring experiences a positive TSDA force $f$ and then $\delta l$ further increases the length of the TSDA?
  - To have this happen, energy needs to come from somewhere to make a stretched TSDA stretch even further
  - This explains the negative sign in front of the $\delta W$ associated with a TSDA – the net effect is to “drain” energy from somewhere else

# Generalized Forces: Summary

- Question: How do we specify the terms $\mathbf{F}$ and $n$ in the EOM?

$$\begin{array}{lll} m\ddot{\mathbf{r}} = \mathbf{F} & \text{EOM for translation} & \text{(Newton)} \\ J'\ddot{\phi} = n & \text{EOM for rotation} & \text{(Euler)} \end{array}$$

- Answer: Recall where these terms come from…

$$m\mathbf{a} = \mathbf{f}$$

Integral manipulations (use rigid-body assumptions)

$$\delta\mathbf{r}^T\left(m\ddot{\mathbf{r}} - \mathbf{F}\right) + \delta\phi\left(J'\ddot{\phi} - n\right) = 0$$

Redefine in terms of generalized forces and virtual displacements

$$\delta\mathbf{q}^T\left(\mathbf{Q} - \mathbf{M}\ddot{\mathbf{q}}\right) = 0$$

Explicitly identify virtual work of generalized forces

$$\underbrace{\delta\mathbf{q}^T\mathbf{Q}}_{\text{Virtual work of generalized external forces}} + \underbrace{\delta\mathbf{q}^T\left(-\mathbf{M}\ddot{\mathbf{q}}\right)}_{\text{Virtual work of generalized inertial forces}} = 0$$

D'Alembert's Principle effectively says that, upon including a new external force, the body's generalized accelerations must change to preserve the balance of virtual work.

As such, to include a new force (or torque), we are interested in the contribution of this force on the virtual work balance.

# Roadmap: Check Progress

What have we accomplished so far?

- Derived the variational and differential EOM for a single rigid body
- Defined how to calculate inertial properties
- Defined a general strategy for including external forces
    - Concentrated (point) forces
    - Forces from compliant elements (TSDA and RSDA)

What is there left?

- Treatment of constraint forces
- Derive the variational and differential EOM for systems of constrained bodies

## 6.3.1

# Variational Equations of Motion for Planar Systems

# Variational and Differential EOM for a Single Rigid Body $i$

- The variational EOM of a rigid body $i$ with a centroidal body-fixed reference frame:

$$\delta\mathbf{r}_i^T\left(m\ddot{\mathbf{r}}_i - \mathbf{F}_i\right) + \delta\phi_i\left(J_i'\ddot{\phi}_i - n_i\right) = 0 \quad \Leftrightarrow \quad \left[\delta\mathbf{r}_i^T\ ,\ \delta\phi_i\right]\begin{bmatrix} m\ddot{\mathbf{r}}_i - \mathbf{F}_i \\ J_i'\ddot{\phi}_i - n_i \end{bmatrix} = 0$$

- Since $\delta\mathbf{r}_i$ and $\delta\phi_i$ are arbitrary, using the orthogonality theorem, we got:

$$\begin{bmatrix} m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i \\ J_i'\ddot{\phi}_i - n_i \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{matrix} m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i \\ J_i'\ddot{\phi}_i = n_i \end{matrix}$$

- Important: Above equations valid only if all force effects have been accounted for
  - This includes both applied forces/torques and constraint forces/torques (from interactions with other bodies).

# Matrix Form of the EOM for a Single Body *i*

$$\begin{bmatrix} \delta \mathbf{r}_i^T & \delta \phi_i \end{bmatrix} \begin{bmatrix} m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i \\ J_i' \ddot{\phi}_i - n_i \end{bmatrix} = 0$$

$$\Updownarrow$$

$$\delta \mathbf{q}_i^T \left( \mathbf{M}_i \ddot{\mathbf{q}}_i - \mathbf{Q}_i \right) = 0$$

Generalized Virtual Displacement

(arbitrary)

Generalized Mass Matrix

Generalized Accelerations

Generalized Force; includes all forces acting on body $i$:

This includes all applied forces and all reaction forces

$$\delta \mathbf{q}_i = \begin{bmatrix} \delta \mathbf{r}_i \\ \delta \phi_i \end{bmatrix}$$

$$\mathbf{M}_i \triangleq \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & J_i' \end{bmatrix}$$

$$\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \ddot{\phi}_i \end{bmatrix}$$

$$\mathbf{Q}_i \triangleq \begin{bmatrix} F_{x,i} \\ F_{y,i} \\ n_i \end{bmatrix}$$

# [Side Trip]
# A Vector-Vector Multiplication Trick

- Assume you have two vectors $\mathbf{a}$ and $\mathbf{b}$**,** each made up of $nb$ “sub-vectors”, each of which has dimension 3

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{nb} \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^3 \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_{nb} \end{bmatrix}, \quad \mathbf{b}_i \in \mathbb{R}^3$$

- The dot product of $\mathbf{a}$ and $\mathbf{b}$ can be expressed in terms of sub-vectors:

$$\mathbf{a}^T\mathbf{b} = \mathbf{a}_1^T\mathbf{b}_1 + \mathbf{a}_2^T\mathbf{b}_2 + \cdots + \mathbf{a}_{nb}^T\mathbf{b}_{nb}$$

# Variational EOM for the Entire System (1/2)

- Consider a system made up of $nb$ bodies
- Write the EOM (in matrix form) for each individual body

$$\begin{gathered}\delta\mathbf{q}_1^T\left(\mathbf{M}_1\ddot{\mathbf{q}}_1-\mathbf{Q}_1\right)=0\\ \delta\mathbf{q}_2^T\left(\mathbf{M}_2\ddot{\mathbf{q}}_2-\mathbf{Q}_2\right)=0\\ \vdots\\ \delta\mathbf{q}_{nb}^T\left(\mathbf{M}_{nb}\ddot{\mathbf{q}}_{nb}-\mathbf{Q}_{nb}\right)=0\end{gathered}$$

- Sum them up:

$$\delta\mathbf{q}_1^T\left(\mathbf{M}_1\ddot{\mathbf{q}}_1-\mathbf{Q}_1\right)+\delta\mathbf{q}_2^T\left(\mathbf{M}_2\ddot{\mathbf{q}}_2-\mathbf{Q}_2\right)+\cdots+\delta\mathbf{q}_{nb}^T\left(\mathbf{M}_{nb}\ddot{\mathbf{q}}_{nb}-\mathbf{Q}_{nb}\right)=0$$

- Use trick on previous slide to express this dot product as

$$\left[\delta\mathbf{q}_1^T\,,\,\delta\mathbf{q}_2^T\,,\,\ldots\,,\,\delta\mathbf{q}_{nb}^T\right]\begin{bmatrix}\mathbf{M}_1\ddot{\mathbf{q}}_1-\mathbf{Q}_1\\ \mathbf{M}_2\ddot{\mathbf{q}}_2-\mathbf{Q}_2\\ \vdots\\ \mathbf{M}_{nb}\ddot{\mathbf{q}}_{nb}-\mathbf{Q}_{nb}\end{bmatrix}=0$$

# Variational EOM for the Entire System (2/2)

## [Repackaging result on previous page, introduce new notation]

- Matrix form of the variational EOM for a system made up of $nb$ bodies

$$\delta\mathbf{q}^T\left(\mathbf{M}\ddot{\mathbf{q}}-\mathbf{Q}\right)=0$$

Generalized Virtual Displacement

$$\delta\mathbf{q}=\begin{bmatrix}\delta\mathbf{q}_1\\\delta\mathbf{q}_2\\\vdots\\\delta\mathbf{q}_{nb}\end{bmatrix}$$

Generalized Mass Matrix

$$\mathbf{M}=\begin{bmatrix}\mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0}\\\mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{0}\\\vdots & \vdots & \ddots & \vdots\\\mathbf{0} & \mathbf{0} & \cdots & \mathbf{M}_{nb}\end{bmatrix}$$

Generalized Accelerations

$$\ddot{\mathbf{q}}=\begin{bmatrix}\ddot{\mathbf{q}}_1\\\ddot{\mathbf{q}}_2\\\vdots\\\ddot{\mathbf{q}}_{nb}\end{bmatrix}$$

Generalized Force

$$\mathbf{Q}=\begin{bmatrix}\mathbf{Q}_1\\\mathbf{Q}_2\\\vdots\\\mathbf{Q}_{nb}\end{bmatrix}$$

# A Closer Look at the Generalized Forces

- Total force acting on a body is sum of applied (external) and constraint (internal to the system) forces:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \vdots \\ \mathbf{Q}_{nb} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A + \mathbf{Q}_1^C \\ \mathbf{Q}_2^A + \mathbf{Q}_2^C \\ \vdots \\ \mathbf{Q}_{nb}^A + \mathbf{Q}_{nb}^C \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^A \\ \mathbf{Q}_2^A \\ \vdots \\ \mathbf{Q}_{nb}^A \end{bmatrix} + \begin{bmatrix} \mathbf{Q}_1^C \\ \mathbf{Q}_2^C \\ \vdots \\ \mathbf{Q}_{nb}^C \end{bmatrix} = \mathbf{Q}^A + \mathbf{Q}^C$$

- Goal: get rid of constraint forces $\mathbf{Q}_i^C$, which (at least for now) are unknown

- To do this, we need to compromise and give up something…

# Constraint Forces

- Constraint Forces
  - Forces that develop in the physical joints present in the system: (revolute, translational, distance constraint, etc.)
  - They are the forces that ensure the satisfaction of the constraints (they are such that the motion stays compatible with the kinematic constraints)
  - Assumption: there is no friction in these idealized joints
    - If friction is present, these friction forces should be accounted for separately, like external forces

- **KEY OBSERVATION**: The net virtual work produced by the constraint forces present in the system as a result of a set of consistent virtual displacements is zero
  - We have to account for the work of all reaction forces present in the system
  - This is the same observation we used to eliminate the internal interaction forces when deriving the EOM for a single rigid body

- Therefore

$$\delta\mathbf{q}^T\mathbf{Q}^C = 0$$

provided δ**q** is a consistent virtual displacement

# Consistent Virtual Displacements: What does this mean?

What does it take for a virtual displacement to be consistent (with the set of constraints) at a given, fixed time $t^*$?

- Start with a consistent configuration $\mathbf{q}$; i.e., a configuration that satisfies the constraint equations:

$$\mathbf{\Phi}(\mathbf{q}, t^*) = 0$$

- A consistent virtual displacement $\delta\mathbf{q}$ is a virtual displacement which ensures that the configuration $\mathbf{q} + \delta\mathbf{q}$ is also consistent:

$$\mathbf{\Phi}(\mathbf{q} + \delta\mathbf{q}, t^*) = 0$$

- Apply a Taylor series expansion and assume small variations:

$$\mathbf{\Phi}(\mathbf{q} + \delta\mathbf{q}, t^*) \approx \mathbf{\Phi}(\mathbf{q}, t^*) + \mathbf{\Phi}_{\mathbf{q}}\delta\mathbf{q} \quad \Rightarrow \quad \mathbf{\Phi}_{\mathbf{q}}\delta\mathbf{q} = 0$$

# Constrained Variational EOM

$$\delta\mathbf{q}^T\left(\mathbf{M}\ddot{\mathbf{q}}-\mathbf{Q}\right)=0 \quad\Leftrightarrow\quad \delta\mathbf{q}^T\left(\mathbf{M}\ddot{\mathbf{q}}-\mathbf{Q}^A-\mathbf{Q}^C\right)=0 \quad\Leftrightarrow\quad \delta\mathbf{q}^T\left(\mathbf{M}\ddot{\mathbf{q}}-\mathbf{Q}^A\right)=0$$

Arbitrary

Arbitrary

Consistent

- We can eliminate the (unknown) constraint forces if we compromise to only consider virtual displacements that are consistent with the constraint equations

Constrained Variational Equations of Motion

$$\delta\mathbf{q}^T\left(\mathbf{M}\ddot{\mathbf{q}}-\mathbf{Q}^A\right)=0$$

**provided**

$$\mathbf{\Phi}_{\mathbf{q}}\delta\mathbf{q}=\mathbf{0}$$

Condition for consistent virtual displacements