ME451
Kinematics and Dynamics of Machine Systems

Introduction to Dynamics

November 13, 2014

- Quote of the day: "I am the greatest, I said that even before I knew I was."
-- Muhammad Ali

Dan Negrut
ME451, Fall 2014
University of Wisconsin-Madison

Before we get started…

- Last time
  - Wrapped up the Newton-Euler equations of motion for a rigid body
  - Started properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies
- Today
  - Finish discussion properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies
  - Enlarging the family of forces that can show up in the equations of motion
- Project 1 – Due date: Nov 18 at 11:59 PM
  - Requires you to use `simEngine2D` in conjunction with excavator example discussed in class
  - Not trivial, requires some thinking
- New assignment posted online
  - Has pen-and-paper, ADAMS, and MATLAB components
  - Due in one week, Nov. 20.
- Final Project proposal due on Nov 18 @ 11:59 PM - post your proposal on the Forum
  - A discussion thread was started on this topic
  - Dan to provide feedback

Take Away Slide: Newton-Euler EOM

- Here's what we derived last time: the EOM for a centroidal LRF (CLRF)

$$\begin{bmatrix} m\ddot{\mathbf{r}} - \mathbf{F} \\ J'\ddot{\phi} - n \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{matrix} m\ddot{\mathbf{r}} = \mathbf{F} \\ J'\ddot{\phi} = n \end{matrix}$$

- Got these second order differential equations starting from "first principles"
  - Newton's laws for a particle
  - The rigid body assumption

- They tell us what the acceleration of the CLRF slapped on the body looks like
  - Recall that this is what we were after: figuring out what the acceleration is
    - Integrate once to get velocity
    - Integrate once again to get positions
    - (easier said than done)

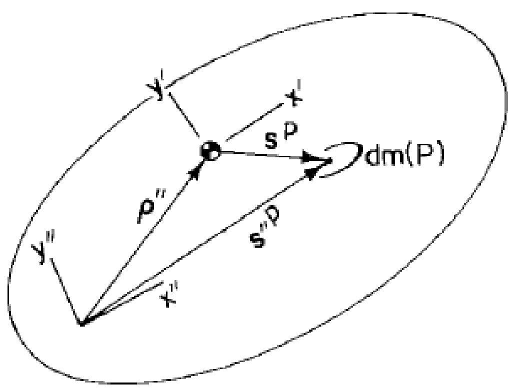
Mass Moment of Inertia (MMI) Parallel Axis Theorem

- Also called Polar Moment of Inertia (PMI)
- The MMI with respect to some LRF $O'x'y'$ is by definition the following integral

$$J' = \int_m (\mathbf{s}'^P)^T \mathbf{s}'^P dm(P)$$

- Question: Given the value J' calculated with respect to the centroidal frame, what is the value of this integral with respect to the LRF $O''x''y''$?

- Parallel Axis Theorem (Steiner's Theorem)



$$J'' = \int_m (\mathbf{s}''^P)^T \mathbf{s}''^P dm(P)$$
$$= J' + m\left(\rho''^T \rho''\right)$$

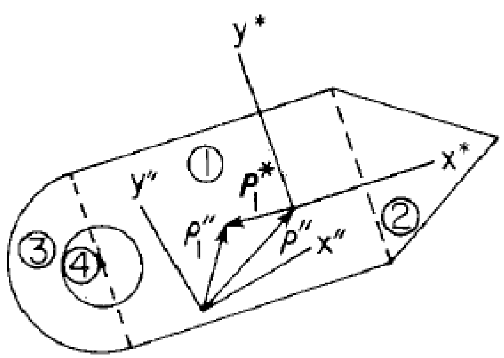
Jakob Steiner
(1796– 1863)

Inertial Properties of Composite Bodies

- Masses, centroid locations, and MMI for rigid bodies with constant density and of simple shapes can be easily calculated
- Question: how do we calculate these quantities for bodies made up of rigidly attached subcomponents?
  - Step 1: Calculate the total body mass $m = \sum_{i=1}^{k} m_i$
  - Step 2: Compute the centroid location of the composite body $\rho'' = \frac{1}{m} \sum_{i=1}^{k} m_i \rho_i''$
  - Step 3: For each subcomponent, apply the parallel axis theorem to include the MMI of that subcomponent with respect to the newly computed centroid, to obtain the MMI of the composite body

$$J^* = \sum_{i=1}^{k} \left(J_i' + m_i \rho_i^{*T} \rho_i^* \right)$$

ρ_i^* represents the vector from the composite body CM to body i CM



  - Note: if holes are present in the composite body, it is ok to add and subtract material

Roadmap: Check Progress

What have we accomplished so far?

- Derived the variational and differential EOM for a single rigid body
  - These equations assume their simplest form in a centroidal RF
- Properties of the mass moment of inertia, figuring out center of mass, etc.

What's left at this point?

- Define a general methodology for including external forces, concentrated at a given point P on the body
  - Virtual work and generalized forces
- Elaborate on the nature of these concentrated forces. These can be:
  - Models of common force elements (TSDA and RSDA)
  - Reaction (constraint) forces, modeling the interaction with other bodies
- Derive EOM for systems of constrained bodies

6.2

Virtual Work and Generalized Force

Including Concentrated Forces (1/3)

Setup:

- A single rigid body
- Absolute (Cartesian) generalized coordinates using a centroidal frame
- A concentrated force $\mathbf{F}$ acts on the body at point P, located by $\mathbf{s}'^P$

Question:

- How do we include the force $\mathbf{F}$ in the EOM?

Solution:

- A general methodology is to use D'Alembert's Principle
- The key steps to derive the generalized force produced by $\mathbf{F}$ are
  - Write down the virtual work produced by $\mathbf{F}$
  - Add this virtual work to the balance of virtual work that shows up in 'Alembert's principle

Including Concentrated Forces (2/3)

- Rearrange the variational EOM as:

$$\delta\mathbf{r}^T\left(\mathbf{F} - m\ddot{\mathbf{r}}\right) + \delta\phi\left(n - J'\ddot{\phi}\right) = 0$$
$$\Updownarrow$$
$$\left(\delta\mathbf{r}^T\ \mathbf{F} + \delta\phi\, n\right) + \left(\delta\mathbf{r}^T\ \left(-m\ddot{\mathbf{r}}\right) + \delta\phi\left(-J'\ddot{\phi}\right)\right) = 0$$

  and read this as "the virtual work of the applied (external) forces and the inertial forces is zero for any *virtual variations* $(\delta\mathbf{r}\,, \delta\phi)$ of the generalized coordinates" (D'Alembert's Principle)

- A more compact and convenient form uses matrix-vector notation

$$\underbrace{\begin{bmatrix}\delta\mathbf{r}^T & \delta\phi\end{bmatrix}}_{\delta\mathbf{q}^T}\underbrace{\begin{bmatrix}\mathbf{F}\\ n\end{bmatrix}}_{\mathbf{Q}} - \underbrace{\begin{bmatrix}\delta\mathbf{r}^T & \delta\phi\end{bmatrix}}_{\delta\mathbf{q}^T}\underbrace{\begin{bmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & J'\end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix}\ddot{\mathbf{r}}\\ \ddot{\phi}\end{bmatrix}}_{\ddot{\mathbf{q}}} = 0$$

called **Generalized Force**

$$\delta\mathbf{q}^T\left(\mathbf{Q} - \mathbf{M}\ddot{\mathbf{q}}\right) = 0$$

Including Concentrated Forces (3/3)

$$\delta\mathbf{q}^T\left(\mathbf{Q}-\mathbf{M}\ddot{\mathbf{q}}\right)=0$$

- Nomenclature:
  - $\ddot{\mathbf{q}}$ generalized accelerations
  - $\delta\mathbf{q}$ generalized virtual displacements
  - $\mathbf{M}$ generalized mass matrix
  - $\mathbf{Q}$ generalized forces

- Recipe for including a concentrated force in the EOM:
  - Write the virtual work of the given force or torque
  - Express this virtual work in terms of the generalized virtual displacements
  - Identify the generalized force $\mathbf{Q}$ (gather the terms that multiply $\delta\mathbf{r}^{\mathrm{T}}$ and $\delta\phi$)
  - Include the generalized force in the matrix form of the variational EOM

$$\mathbf{F}\ \xrightarrow[\text{to}]{\text{leads}}\ \delta W=\cdots=\delta\mathbf{r}^T\,\mathbf{Q}_1+\delta\phi\,Q_2\quad\rightarrow\quad\mathbf{Q}=\begin{bmatrix}\mathbf{Q}_1\\ Q_2\end{bmatrix}$$

(understanding this process will come in handy in your assignment)

Where the Rubber Hits the Road: $\mathbf{Q}$ Produced by a Point Force (1/2)

Consider a point force $\mathbf{F}^P$ acting on the body at point P located, with respect to the centroidal LRF, by the vector $\mathbf{s}^P$.

- Write the virtual work done by $\mathbf{F}^P$ over a virtual displacement $\delta\mathbf{r}^P$ of point P:

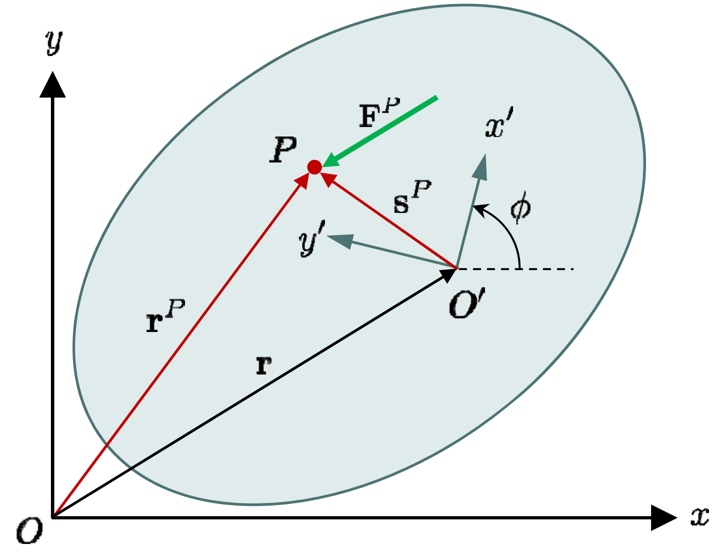
$$\delta W = (\delta\mathbf{r}^P)^T\mathbf{F}^P$$

- Use $\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \quad \Rightarrow \quad \delta\mathbf{r}^P = \delta\mathbf{r} + \delta\phi\mathbf{B}\mathbf{s}'^P$
- Express this virtual work in terms of $\delta\mathbf{q}$:

$$\begin{aligned}\delta W &= (\delta\mathbf{r} + \delta\phi\mathbf{B}\mathbf{s}'^P)^T\mathbf{F}^P \\ &= \delta\mathbf{r}^T\mathbf{F}^P + \delta\phi(\mathbf{s}'^P)^T\mathbf{B}^T\mathbf{F}^P\end{aligned}$$

- Identify the generalized force $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix}\mathbf{F}^P \\ (\mathbf{s}'^P)^T\mathbf{B}^T\mathbf{F}^P\end{bmatrix} = \begin{bmatrix}\mathbf{I}_2 \\ (\mathbf{B}\mathbf{s}'^P)^T\end{bmatrix}\mathbf{F}^P$$



Where the Rubber Hits the Road: $\mathbf{Q}$ Produced by a Point Force (1/2)

What if the point force is better expressed in the LRF?

Use $\mathbf{F}^P = \mathbf{A}\mathbf{F'}^P$ to obtain:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{F}^P \\ (\mathbf{s'}^P)^T \mathbf{B}^T \mathbf{F}^P \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{F'}^P \\ (\mathbf{s'}^P)^T \mathbf{B}^T \mathbf{A}\mathbf{F'}^P \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}\mathbf{F'}^P \\ (\mathbf{s'}^P)^T \left(\mathbf{A}\mathbf{R}\right)^T \mathbf{A}\mathbf{F'}^P \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{F'}^P \\ (\mathbf{s'}^P)^T \mathbf{R}^T \mathbf{F'}^P \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} \\ \left(\mathbf{R}\mathbf{s'}^P\right)^T \end{bmatrix} \mathbf{F'}^P$$

Important: A concentrated force applied at a point away from the center of mass (that is $\mathbf{s'}^P \neq \mathbf{0}$) also induces a torque, that is a component in the generalized force that affects the rotational degree of freedom of the body:

$$Q_2 = (\mathbf{s'}^P)^T \mathbf{B}^T \mathbf{F}^P = \left(\mathbf{R}\mathbf{s'}^P\right)^T \mathbf{F'}^P$$

(see bottom of slide 10 for meaning of Q_2)

Including a Torque

What is the generalized force induced by applying a torque n to the body?

Note: no need to specify the point of application for a torque (unlike a force, see previous two slides)

Writing the virtual work done by the torque n and expressing it in terms of the generalized virtual displacements gives:

$$\delta W = \delta\phi \, n$$

and therefore

$$\mathbf{Q} = \begin{bmatrix} \mathbf{0} \\ n \end{bmatrix}$$

Tractor Model [Example 6.1.1]

- Derive EOM under the following assumptions:
  - Traction (driving) force T_r generated at rear wheels
  - Small angle assumption (on the pitch angle ϕ_1)
  - Tire forces depend linearly on tire deflection (in reality both tire and terrain deform):

$$F_t = f(d_t) = \begin{cases} kd_t\,, & d_t \geq 0 \\ 0\,, & d_t < 0 \end{cases}$$