ME451
Kinematics and Dynamics of Machine Systems

Introduction to Dynamics

6.1
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- Quote of the day: "Noise makes little good; good makes little noise."
-- St François de Sales (1567-1622)

Before we get started…

- Last time (before the exam)
  - Started working towards deriving the equations of motion of a rigid body

- Today
  - Wrap up the Newton-Euler equations of motion for a rigid body
  - Properties of the Centroid and Mass Moment of Inertia, Inertial Properties of Composite Bodies

- Project 1 – Due date: Nov 18 at 11:59 PM
  - Requires you to use `simEngine2D` in conjunction with excavator example discussed in class
  - Not trivial, requires some thinking

- HW due on Th includes ADAMS, MATLAB, pen-and-paper
  - Probably the toughest assignment this semester

Midterm Exam 1

- Highest score: 100
- Average: 85
- Standard Deviation: 16.88

- Problem A: KDOF question caused lots of problems

- Problem B: Two different ways of modeling the same mechanism
  - Scores not as high as I was hoping
  - People not reading what was asked (provided the equations as well)

- Problem 3: overall, good understanding of how to pose the set of position, velocity, and acceleration constraint equations
  - People seem to have understood Newton-Raphson

D'Alembert's Principle

For **consistent** virtual displacements $\delta \mathbf{r}^P$

$$\int_m (\delta \mathbf{r}^P)^T \ddot{\mathbf{r}}^P dm(P) = \int_m (\delta \mathbf{r}^P)^T \mathbf{f}_d(P) dm(P)$$

This is D'Alembert's principle for the motion of a rigid body. D'Alemebert's principle is an extension of the *Principle of Virtual Work* to the case of accelerated motion.

Jean-Baptiste d'Alembert
(1717– 1783)

Virtual Displacements in terms of Variations in Generalized Coordinates (1/2)

We have concluded that the following:

$$\int_m (\delta \mathbf{r}^P)^T \ddot{\mathbf{r}}^P dm(P) = \int_m (\delta \mathbf{r}^P)^T \mathbf{f}_d(P) dm(P)$$

must hold for all virtual displacements $\delta \mathbf{r}^P$ that are consistent with the constraints imposing rigid-body motion.

Next Step: Express the virtual displacements $\delta \mathbf{r}^P$ using variations in the generalized coordinates $\mathbf{q}$

- In this step we keep in mind that $\delta \mathbf{r}^P$ must be **consistent** with the rigid-body virtual displacement, that is, we are dealing with a **rigid body** here

Virtual Displacements in terms of Variations in Generalized Coordinates (2/2)

Recall that

$$\delta \mathbf{r}^P = \delta \mathbf{r} + \delta\phi \mathbf{B}\mathbf{s}'^P \quad \text{and} \quad \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{B}\mathbf{s}'^P - \dot{\phi}^2\mathbf{A}\mathbf{s}'^P$$

Use these relations to expand

$$\int_m (\delta \mathbf{r}^P)^T \ddot{\mathbf{r}}^P dm(P) = \int_m (\delta \mathbf{r}^P)^T \mathbf{f}_d(P) dm(P)$$

(which only holds for **consistent** $\delta \mathbf{r}^P$) to

$$\begin{aligned} \delta \mathbf{r}^T \ddot{\mathbf{r}} \int_m dm(P) + \left(\delta \mathbf{r}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) + \delta\phi \ddot{\mathbf{r}}^T \mathbf{B} \right) \int_m \mathbf{s}'^P dm(P) \\ + \delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) \mathbf{s}'^P dm(P) \\ = \delta \mathbf{r}^T \int_m \mathbf{f}_d(P) dm(P) + \delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{f}_d(P) dm(P) \end{aligned}$$

(which holds for **arbitrary** variations $\delta \mathbf{r}$ and $\delta\phi$!)

6.1.2, 6.1.3

Variational EOM with Centroidal Coordinates
Newton-Euler Differential EOM

Centroidal Reference Frames

- The variational EOM for a single rigid body can be significantly simplified if we pick a special LRF
- A centroidal reference frame is an LRF located at the center of mass
- How is such an LRF special?

  By definition of the center of mass (more on this later) is the point where the following integral vanishes:

$$\int_m \mathbf{s}'^P \, dm(P) = 0$$

Variational EOM with Centroidal LRF (1/3)

$$\delta \mathbf{r}^T \ddot{\mathbf{r}} \int_m dm(P) + \left(\delta \mathbf{r}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) + \delta\phi \ddot{\mathbf{r}}^T \mathbf{B}\right) \int_m \mathbf{s}'^P dm(P)$$
$$+ \delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) \mathbf{s}'^P dm(P)$$
$$= \delta \mathbf{r}^T \int_m \mathbf{f}_d(P) dm(P) + \delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{f}_d(P) dm(P)$$

- Total body mass:

  $\int_m dm(P) = m$

- Definition of centroid:

  $\int_m \mathbf{s}'^P dm(P) = \mathbf{0}$

$$LHS = m\, \delta \mathbf{r}^T\, \ddot{\mathbf{r}} + J'\, \delta\phi\, \ddot{\phi}$$

- Definition of mass moment of inertia:

  $\int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{B} \mathbf{s}'^P \ddot{\phi} dm(P) = \ddot{\phi} \int_m (\mathbf{s}'^P)^T \mathbf{s}'^P dm(P)$

  $J' \triangleq \int_m (\mathbf{s}'^P)^T \mathbf{s}'^P dm(P)$

- Direct expansion:

  $\int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{A} \mathbf{s}'^P dm(P) = 0$

Variational EOM with Centroidal LRF (2/3)

$$\delta \mathbf{r}^T \ddot{\mathbf{r}} \int_m dm(P) + \left(\delta \mathbf{r}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) + \delta\phi \ddot{\mathbf{r}}^T \mathbf{B}\right) \int_m \mathbf{s}'^P dm(P)$$
$$+\delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T (\ddot{\phi}\mathbf{B} - \dot{\phi}^2 \mathbf{A}) \mathbf{s}'^P dm(P)$$
$$= \delta \mathbf{r}^T \int_m \mathbf{f}_d(P) dm(P) + \delta\phi \int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{f}_d(P) dm(P)$$

- Resultant of all forces acting on the body:

  $\mathbf{F} = \int_m \mathbf{f}_d(P) dm(P)$

- Moment of all forces acting on the body (about the COM):

  $n = \int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{f}_d(P) dm(P)$

$$RHS = \delta \mathbf{r}^T \, \mathbf{F} + \delta\phi \, n$$

EOM for a Single Rigid Body: Newton-Euler Equations

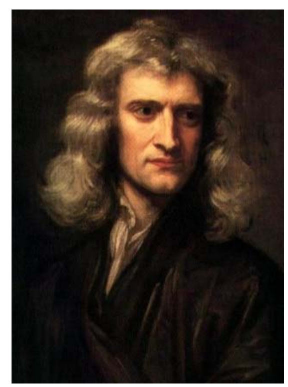
- The variational EOM of a rigid body with a centroidal body-fixed reference frame were obtained as:

$$\delta \mathbf{r}^T \left(m\ddot{\mathbf{r}} - \mathbf{F} \right) + \delta\phi \left(J'\ddot{\phi} - n \right) = 0 \quad \Leftrightarrow \quad \begin{bmatrix} \delta\mathbf{r}^T & \delta\phi \end{bmatrix} \begin{bmatrix} m\ddot{\mathbf{r}} - \mathbf{F} \\ J'\ddot{\phi} - n \end{bmatrix} = 0$$

- Assume all forces acting on the body have been accounted for.
- Since $\delta\mathbf{r}$ and $\delta\phi$ are arbitrary, using the orthogonality theorem, we get:

$$\begin{bmatrix} m\ddot{\mathbf{r}} - \mathbf{F} \\ J'\ddot{\phi} - n \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{matrix} m\ddot{\mathbf{r}} = \mathbf{F} \\ J'\ddot{\phi} = n \end{matrix}$$

- Important: The Newton-Euler equations are valid only if all force effects have been accounted for
  - This includes both applied forces/torques and constraint forces/torques (from interactions with other bodies).



Isaac Newton
(1642 – 1727)

Leonhard Euler
(1707 – 1783)

Newton-Euler EOM

- Here's where we are: the EOM for a centroidal LRF (CLRF)

$$\begin{bmatrix} m\ddot{\mathbf{r}} - \mathbf{F} \\ J'\ddot{\phi} - n \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{matrix} m\ddot{\mathbf{r}} = \mathbf{F} \\ J'\ddot{\phi} = n \end{matrix}$$

- Got these second order differential equations starting from "first principles"
  - Newton's laws for a particle
  - The rigid body assumption

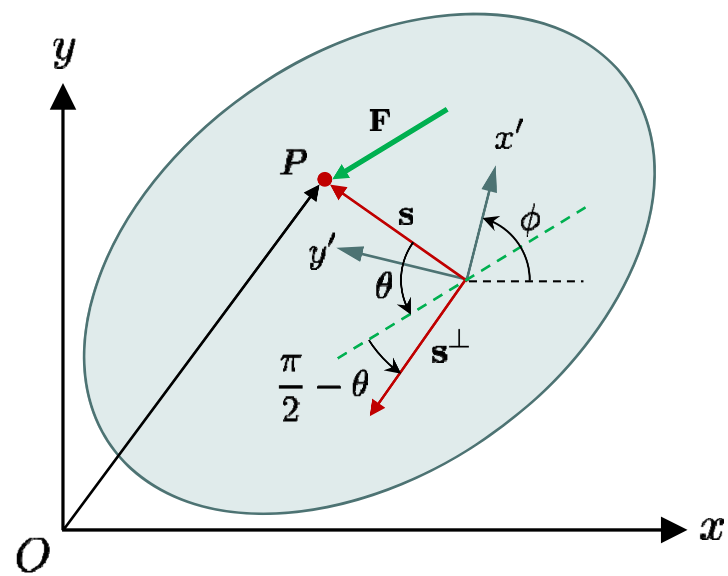
- They tell us what the acceleration of the CLRF slapped on the body looks like
  - Recall that this is what we were after: figuring out what the acceleration is
    - Integrate once to get velocity
    - Integrate once again to get positions
    - (easier said than done)

Variational EOM with Centroidal LRF (3/3)

Why do we say that the quantity $\int_m (\mathbf{s}'^P)^T \mathbf{B}^T \mathbf{f}_d(P) dm(P)$ is the *torque* (or *moment*) n of the forces acting on the body?

Consider a force $\vec{F}$ acting on the body at point P which is located by the vector $\vec{s}$. Then, the torque of this force about an axis perpendicular to the $x-y$ plane and passing through the origin of the LRF has magnitude equal to:

$$
\begin{aligned}
n &= \|\vec{s} \times \vec{F}\| \\
&= \|\vec{s}\| \cdot \|\vec{F}\| \cdot \sin\theta \\
&= \|\vec{s}\| \cdot \|\vec{F}\| \cdot \cos\left(\frac{\pi}{2} - \theta\right) \\
&= \vec{s}^{\perp} \cdot \vec{F}
\end{aligned}
$$



Using algebraic vectors, we get

$$
\begin{aligned}
n &= \left(\mathbf{s}^{\perp}\right)^T \mathbf{F} = (\mathbf{R}\mathbf{s})^T \mathbf{F} \\
&= (\mathbf{R}\mathbf{A}\mathbf{s}')^T \mathbf{F} = (\mathbf{B}\mathbf{s}')^T \mathbf{F}
\end{aligned}
$$

Roadmap: Check Progress

What have we done so far?

- Derived the variational and differential EOM for a single rigid body

$$\begin{bmatrix} m\ddot{\mathbf{r}} - \mathbf{F} \\ J'\ddot{\phi} - n \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{matrix} m\ddot{\mathbf{r}} = \mathbf{F} \\ J'\ddot{\phi} = n \end{matrix}$$

What is left?

- Properties of the mass moment of inertia ⬅
- Define a general strategy for including external forces in $\mathbf{F}$ above
- Treatment of constraint forces
- Derive the variational and differential EOM for systems of constrained bodies

6.1.4, 6.1.5

Properties of the Centroid and Mass Moment of Inertia
Inertial Properties of Composite Bodies

Location of the Center of Mass (1/2)

- The center of mass is the point on the body where the weighted relative position of the distributed mass sums to zero:

$$\int_m \mathbf{s}'^P dm(P) = 0$$

- Question: How can we calculate the location $\boldsymbol{\rho}''$ of the COM with respect to an LRF $O''x''y''$?

$$\begin{aligned} \mathbf{0} &= \int_m \mathbf{s}'^P dm(P) \\ &= \int_m \left(\mathbf{s}''^P - \rho''\right) dm(P) \\ &= \int_m \mathbf{s}''^P dm(P) - m\rho'' \end{aligned}$$

where we have defined the total body mass as:

$$m = \int_m dm(P)$$

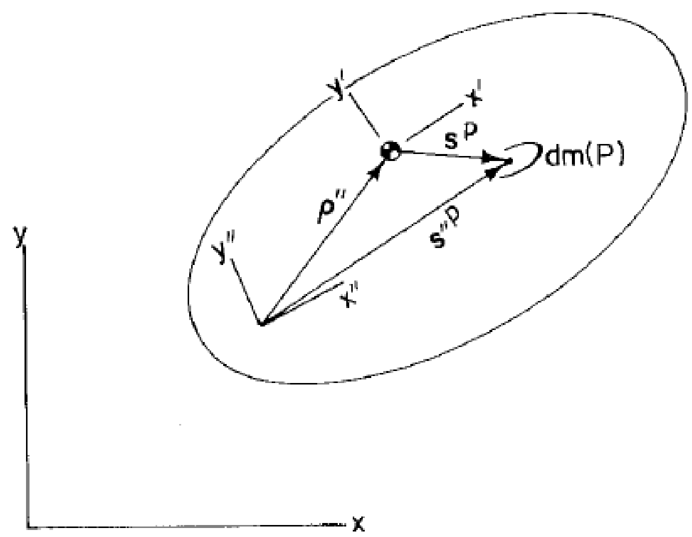


Figure 6.1.3 Location of a centroid.

$$\rho'' = \frac{1}{m}\int_m \mathbf{s}''^P dm(P)$$

Location of the Center of Mass (2/2)

- For a rigid body, the COM is fixed with respect to the body
- If the body has constant density, the COM coincides with the centroid of the body shape
- If the rigid body has a line of symmetry, then the COM is somewhere along that axis

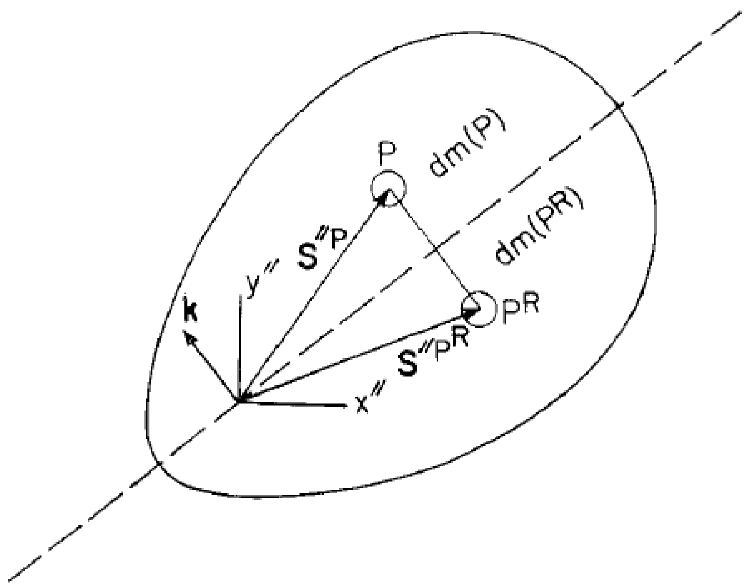


Figure 6.1.4 Body with axis of symmetry.

Notes:

- Here, symmetry axis means that **both** mass distribution and geometry are symmetric with respect to that axis
- If the rigid body has two axes of symmetry, the centroid is on each of them, and therefore is at their intersection