ME751
Advanced Computational Multibody Dynamics

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Quote of the Day

“Never mistake motion for action“

Ernest Hemingway

Before we get started…

- On Wednesday, we learned:
  - Review of co-rotational formulation
  - Kinematics of the FFR
  - Various types of attachment conditions for the FFR
  - Modes of deformation

- This lecture…
  - Attachment conditions and modes of deformation
  - We are going to start thinking we can use FE with FFR
  - Derivation of Equations of Motion of FFR
  - Inertia shape integrals
  - Modal order reduction in FFR equations
    - Eigenmodes
    - Component Mode Synthesis
    - Modern model order reduction

Floating Frame of Reference

Does FFR translation and orientation describe the rigid-body dynamics of the flexible body?

Not in general! Different ways to define the FFR attachment conditions will result in different splitting of FFR dynamics and elastic displacements w.r.t. FFR

1. Modes of Deformation

$$\boldsymbol{r}_P^i = \underbrace{\boldsymbol{R}^i}_{\text{Position of FFR}} + \underbrace{\boldsymbol{A}^i\left(\boldsymbol{\theta}^i\right)}_{\text{Orientation of FFR}} \underbrace{\bar{\boldsymbol{u}}^P}_{\text{Local position of } P}$$

This vector can be split into its reference position and its elastic displacement as follows

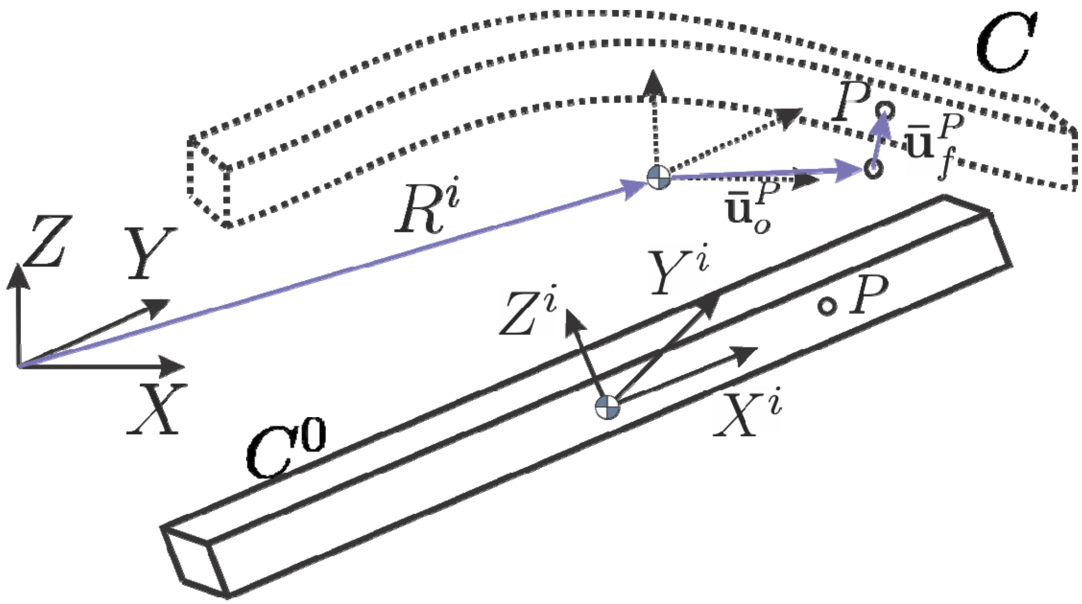
$$\bar{\mathbf{u}}^P = \bar{\mathbf{u}}_o^P + \bar{\mathbf{u}}_f^P.$$

$\bar{\mathbf{u}}_o^P$ represents the undeformed position of P, whereas its elastic displacement, measured in the FFR, is denoted by the vector $\bar{\mathbf{u}}_f^P$. The elastic displacement vector $\bar{\mathbf{u}}_f^P$ may be decomposed into the product of a space-dependent matrix and a vector of time-dependent flexible coordinates, as follows

$$\bar{\mathbf{u}}_f^P = \mathbf{S}^i\left(\bar{\mathbf{u}}_o^P\right)\mathbf{q}_f^i,$$

where $\mathbf{S}^i(\bar{\mathbf{u}}_o^P)$ is a shape function matrix referred to the reference (undeformed) configuration and $\mathbf{q}_f^i$ is the vector of flexible coordinates of body i.

Modes of Deformation



$$\mathbf{r}_P^i = \mathbf{R}^i + \mathbf{A}^i \left(\bar{\mathbf{u}}_o^P + \underbrace{\mathbf{S}^i}_{} \mathbf{q}_f^i \right)$$

What are these modes of deformation or mode shapes?

Modes of Deformation

Modes of deformation

- Are assumed shapes that the structure can take
- Selection of mode shapes limits admissible deformation
- Many methods to obtain a set of modes from an FE model
- Some of feature to watch:
  - MS must be consistent with boundary conditions
    - Constrained displacements or angles
    - External loads
  - MS must capture desired structure's dynamics
- Example of modes selection: Eigenvalue Analysis

Eigenvalue Analysis

A structure (flexible body) is assumed to have the following linear dynamics equation:

$$\mathbf{m}_{ff}\ddot{\mathbf{q}}_f + \mathbf{K}_{ff}\mathbf{q}_f = \mathbf{0}.$$

Probing a trial solution: $\mathbf{q}_f = \mathbf{a}\mathbf{e}^{i\omega t}$, we get

$$\mathbf{K}_{ff}\mathbf{a} = \omega^2\mathbf{m}_{ff}\mathbf{a} \tag{1}$$

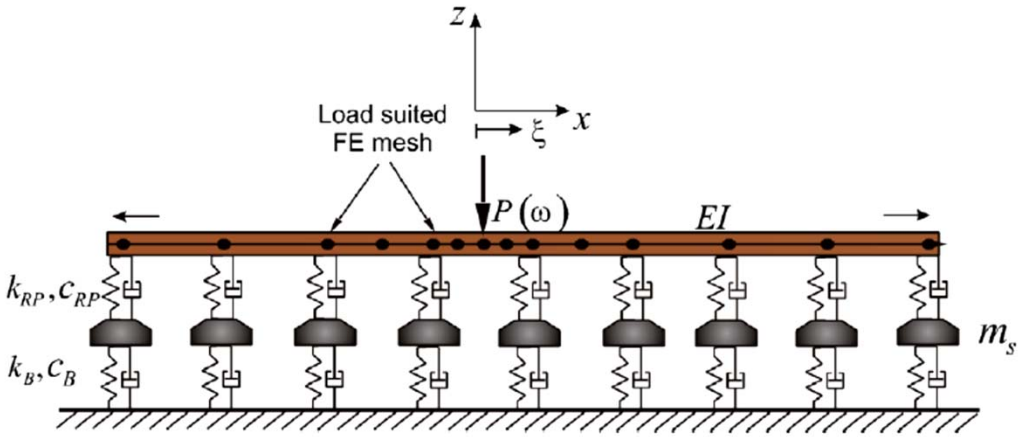
This expression is a generalized eigenvalue problem whose solution yields a set of n_f eigenvalues $(\omega_k)^2$ and their corresponding eigenvectors $\mathbf{a}_k$, $k = 1, 2, ..., n_f$

The eigenvectors are also called *normal modes* or *mode shapes*. As a first straightforward approximation, the order of the FE model can be reduced by selecting a number of modes $n_m \ll n_f$

$$\mathbf{S}^i = \underbrace{\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_{n_m}\end{bmatrix}}$$

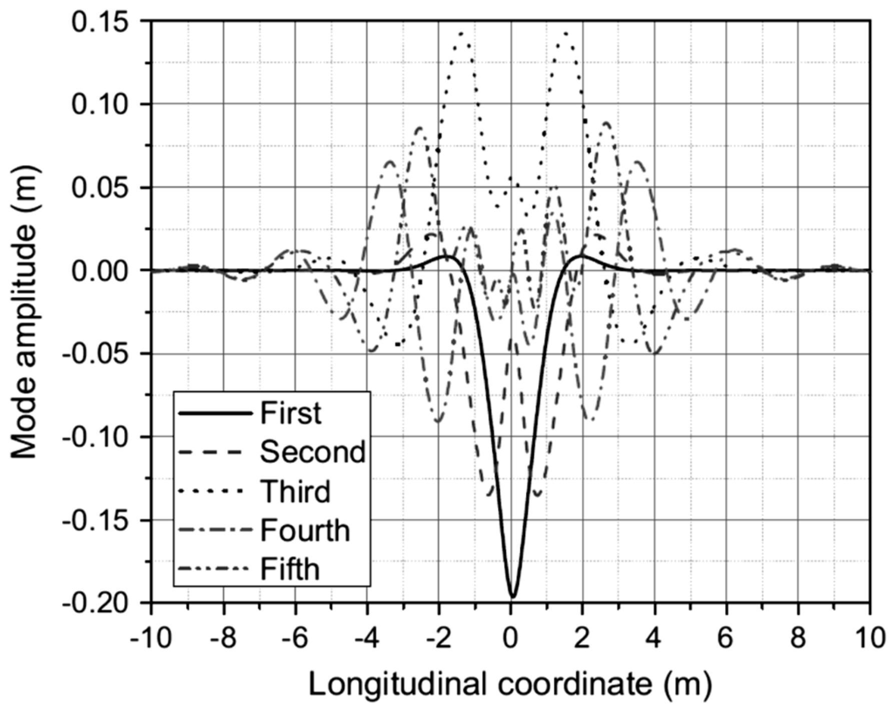
How does this look like?

Eigenvalue Analysis



Finite element model:

Thousands of degrees of freedom



A few mode shapes (dofs)

Eigenvalue Analysis

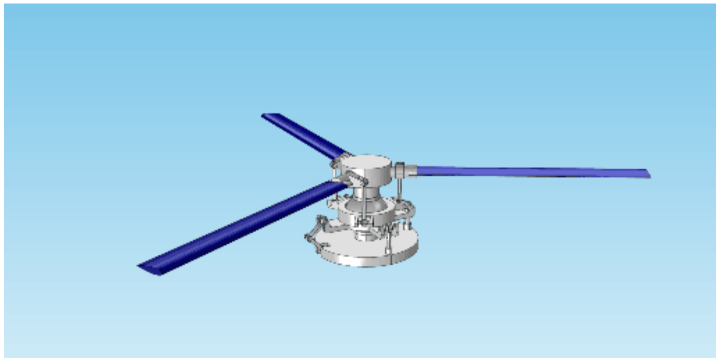
A few more words on model order reduction

- Selection of modes is an art
- There is a multitude of methods to do so
- In many cases, the order of the system can be reduced very significantly –orders of magnitude

We will get back to this topic…

Example of Coupling: Geometric stiffening

- Typical phenomenon of helicopter blades
  - Natural flap mode frequency depends on the angular velocity
  - Coupling between angular velocity (rigid body) and bending and stretch (flexible body) are key in describing geometric stiffening
- Bending (flap) and torsional (lead-lag) modes depend on angular velocity
  - Larger angular velocities result in stiffer blade's dynamics
- For a typical helicopter blade
  - Natural frequency of flap mode at rest: 2.64 rad/s
  - Natural frequency of flap mode at 27 rad/s: 27.83 rad/s
  - This phenomenon may be visible to the naked eye when blades start spinning: Coriolis forces cause stresses that stiffen the flexible body



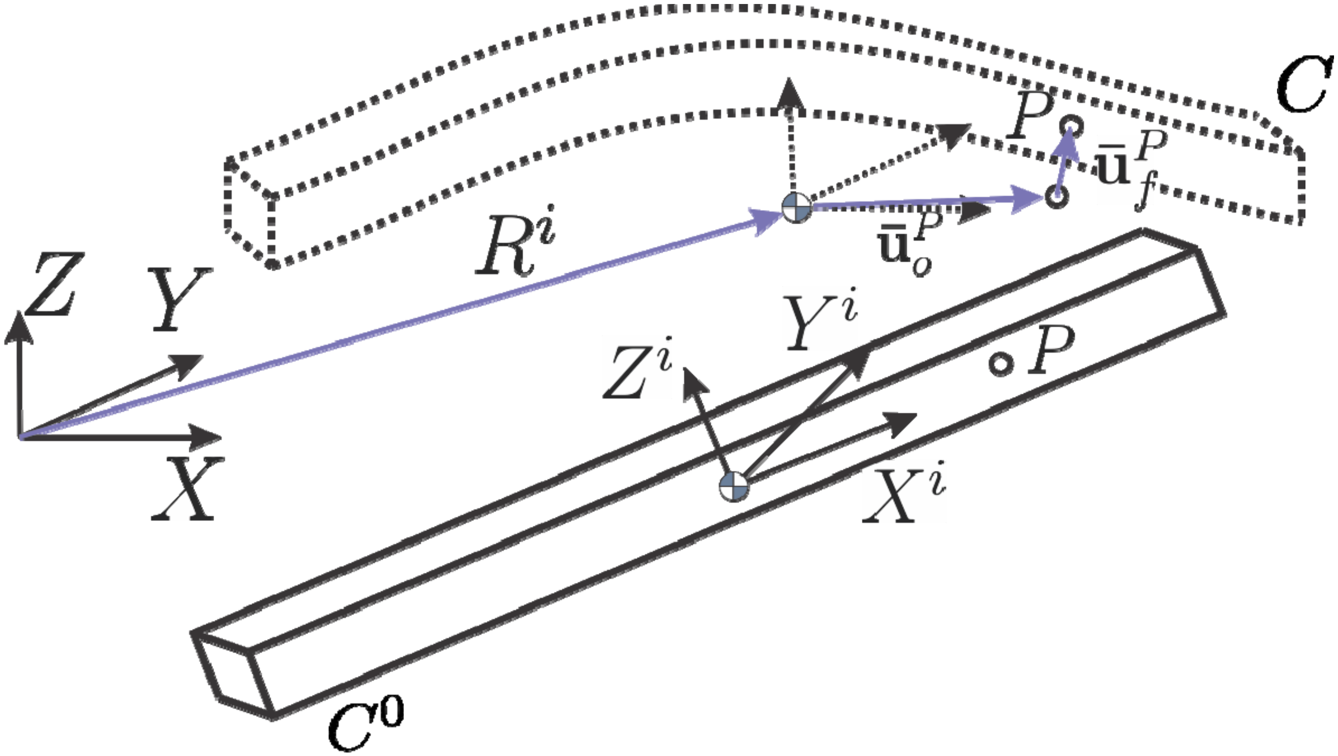
Source:

https://www.comsol.com/blogs/modeling-a-helicopter-swashplate-mechanism/

FFR can capture geometric stiffening… But how?

FFR Kinematics

- Basic kinematics of a flexible body



- Position of an arbitrary point *P* in a deformed flexible body *i*

$$\boldsymbol{r}_P^i = \underbrace{\boldsymbol{R}^i}_{\text{Position of FFR}} + \underbrace{\boldsymbol{A}^i\left(\boldsymbol{\theta}^i\right)}_{\text{Orientation of FFR}} \underbrace{\bar{\boldsymbol{u}}^P}_{\text{Local position of } P}$$

2. Equations of FFR

The expression of the deformed position of the point P in body i may be written as

$$\mathbf{r}_P^i = \mathbf{R}^i + \mathbf{A}^i \left(\bar{\mathbf{u}}_o^P + \mathbf{S}^i \mathbf{q}_f^i \right)$$

Velocity expressions of a deformed point P within the FFR formulation are given below:

$$\dot{\bar{\mathbf{u}}}_f^P = \mathbf{S}^i \left(\bar{\mathbf{u}}_o^P \right) \dot{\mathbf{q}}_f^i,$$

$$\dot{\mathbf{r}}_P^i = \dot{\mathbf{R}}^i + \dot{\mathbf{A}}^i \left(\bar{\mathbf{u}}_o^P + \mathbf{S}^i \mathbf{q}_f^i \right) + \mathbf{A}^i \mathbf{S}^i \dot{\mathbf{q}}_f^i.$$

Equations of FFR

Before moving on with derivations, we arrange, for convenience, the velocity vector in the following form:

$$\dot{\mathbf{r}}_P^i = \begin{bmatrix} \mathbf{I} & \mathbf{B}^i & \mathbf{A}^i\mathbf{S}^i \end{bmatrix} \begin{bmatrix} \dot{\mathbf{R}}^i \\ \dot{\boldsymbol{\theta}}^i \\ \dot{\mathbf{q}}_f^i \end{bmatrix} = \mathbf{L}^i\dot{\mathbf{q}}^i,$$

where matrix $\mathbf{B}^i$ is a linear operator that acts on the rotation coordinates, as follows $\dot{\mathbf{A}}^i\bar{\mathbf{u}}^P = \mathbf{B}^i\dot{\boldsymbol{\theta}}^i$, and $\mathbf{B}^i = -\mathbf{A}^i\tilde{\bar{\mathbf{u}}}^P\bar{\mathbf{G}}^i$. The tilde $\tilde{\star}$ denotes the skew symmetric matrix operation on a vector $\star$, and $\mathbf{G}^i$ is a linear operator that transforms time derivatives of rotation parameters to the angular velocity vector of the reference motion of body i. The form of $\mathbf{G}^i$ depends upon the selection of rotational parameters.

Equations of FFR

Matrix **G** is used to go from rotation parameters to angular velocity:
$\boldsymbol{\omega} = \boldsymbol{G}\dot{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{G}}\dot{\boldsymbol{\theta}}$

For Euler parameters, these matrices take the forms:

$$\mathbf{G} = 2\begin{bmatrix} -\theta_1 & \theta_0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & \theta_0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & \theta_0 \end{bmatrix}, \text{ and } \bar{\mathbf{G}} = 2\begin{bmatrix} -\theta_1 & \theta_0 & \theta_3 & -\theta_2 \\ -\theta_2 & -\theta_3 & \theta_0 & \theta_1 \\ -\theta_3 & \theta_2 & -\theta_1 & \theta_0 \end{bmatrix}$$

And the skew symmetric matrix of a local deformed position of a point P

$$\tilde{\bar{\mathbf{u}}}^P = \begin{bmatrix} 0 & -\bar{u}_3^P & \bar{u}_2^P \\ \bar{u}_3^P & 0 & -\bar{u}_1^P \\ -\bar{u}_2^P & \bar{u}_1^P & 0 \end{bmatrix}$$

Equations of FFR

Lagrange Equations

The equations of motion of the FFR formulations are to be derived using Lagrange's equations. For a flexible body i described using a floating frame, these equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^i}{\partial \dot{\mathbf{q}}^i}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \mathbf{q}^i}\right)^{\mathrm{T}} + \mathbf{C}_{\mathbf{q}^i}^{\mathrm{T}}\lambda = \mathbf{Q}_e^i + \mathbf{Q}_a^i,$$

where T is the kinetic energy, $\mathbf{q}^i$ and $\dot{\mathbf{q}}^i$ are the generalized coordinates and velocities of body i, respectively, $\mathbf{C}_{\mathbf{q}^i}$ is the Jacobian of the constraint equations, λ is the vector of Lagrange multipliers, and $\mathbf{Q}_e^i$ and $\mathbf{Q}_a^i$ are the vectors of elastic and applied forces, respectively. Note that, in general, the vectors of Lagrange multipliers and applied forces may link the flexible body i with other (flexible) bodies and/or other interacting systems.

Equations of FFR

Mass Matrix The total kinetic energy of body i may be expressed as a volume integral:

$$T^i = \frac{1}{2}\int_{V^i} \rho^i \dot{\mathbf{r}}_P^{i\mathrm{T}} \dot{\mathbf{r}}_P^i \mathrm{d}V^i$$

which, in terms of the generalized coordinates, may be written as

$$T^i = \frac{1}{2}\mathbf{q}^{i\mathrm{T}} \boxed{\mathbf{M}^i} \mathbf{q}^i.$$

Is the mass matrix constant?

The mass matrix $\mathbf{M}^i$ is nonlinear in the rotation coordinates and takes the following form

Translational

Rotational

$$\mathbf{M}^i = \int_{V^i} \rho^i \begin{bmatrix} \mathbf{I} & \mathbf{B}^i & \mathbf{A}^i\mathbf{S}^i \\ & \mathbf{B}^{i\mathrm{T}}\mathbf{B}^i & \mathbf{B}^{i\mathrm{T}}\mathbf{A}^i\mathbf{S}^i \\ sym. & & \mathbf{S}^{i\mathrm{T}}\mathbf{S}^i \end{bmatrix} \mathrm{d}V^i = \begin{bmatrix} \boxed{\mathbf{m}_{RR}} & \boxed{\mathbf{m}_{R\theta}} & \boxed{\mathbf{m}_{Rf}} \\ & \boxed{\mathbf{m}_{\theta\theta}} & \boxed{\mathbf{m}_{\theta f}} \\ sym. & & \boxed{\mathbf{m}_{ff}} \end{bmatrix}^i$$

Flexible coordinates

where submatrices have been named for convenience. Cross terms in the mass matrix, e.g. $\mathbf{m}_{Rf}$ and $\mathbf{m}_{\theta f}$, indicate that coupling between reference and flexible coordinates is captured.

Equations of FFR

The Quadratic Velocity Vector The FFR formulation introduces complex inertia terms which represent Coriolis and centrifugal forces. Deriving the inertia terms of Lagrange equations, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^i}{\partial \dot{\mathbf{q}}^i}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \mathbf{q}^i}\right)^{\mathrm{T}} = \mathbf{M}^i\ddot{\mathbf{q}}^i + \underbrace{\dot{\mathbf{M}}^i\dot{\mathbf{q}}^i - \left[\frac{\partial}{\partial \mathbf{q}^i}\left(\dot{\mathbf{q}}^{i\mathrm{T}}\mathbf{M}^i\dot{\mathbf{q}}^i\right)\right]^{\mathrm{T}}}_{\mathbf{Q}_v^i\,(\text{Quadratic velocity vector})}$$

The final expression for the quadratic velocity vector $\mathbf{Q}_v^i = \left[(\mathbf{Q}_v)_R^{i\mathrm{T}} \quad (\mathbf{Q}_v)_\theta^{i\mathrm{T}} \quad (\mathbf{Q}_v)_f^{i\mathrm{T}}\right]^{\mathrm{T}}$, in terms of generalized coordinates, reads... [next slide]

Equations of FFR

$$\left(\mathbf{Q}_v^i\right)_R = -\mathbf{A}^i \rho^i \int\limits_{V^i} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i\right] \mathrm{d}V^i$$

$$\left(\mathbf{Q}_v^i\right)_\theta = \bar{\mathbf{G}}^{i\mathrm{T}} \rho^i \int\limits_{V^i} \left[\tilde{\bar{\mathbf{u}}}^{i\mathrm{T}} \left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\mathbf{u}}}^{i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i\right] \mathrm{d}V^i$$

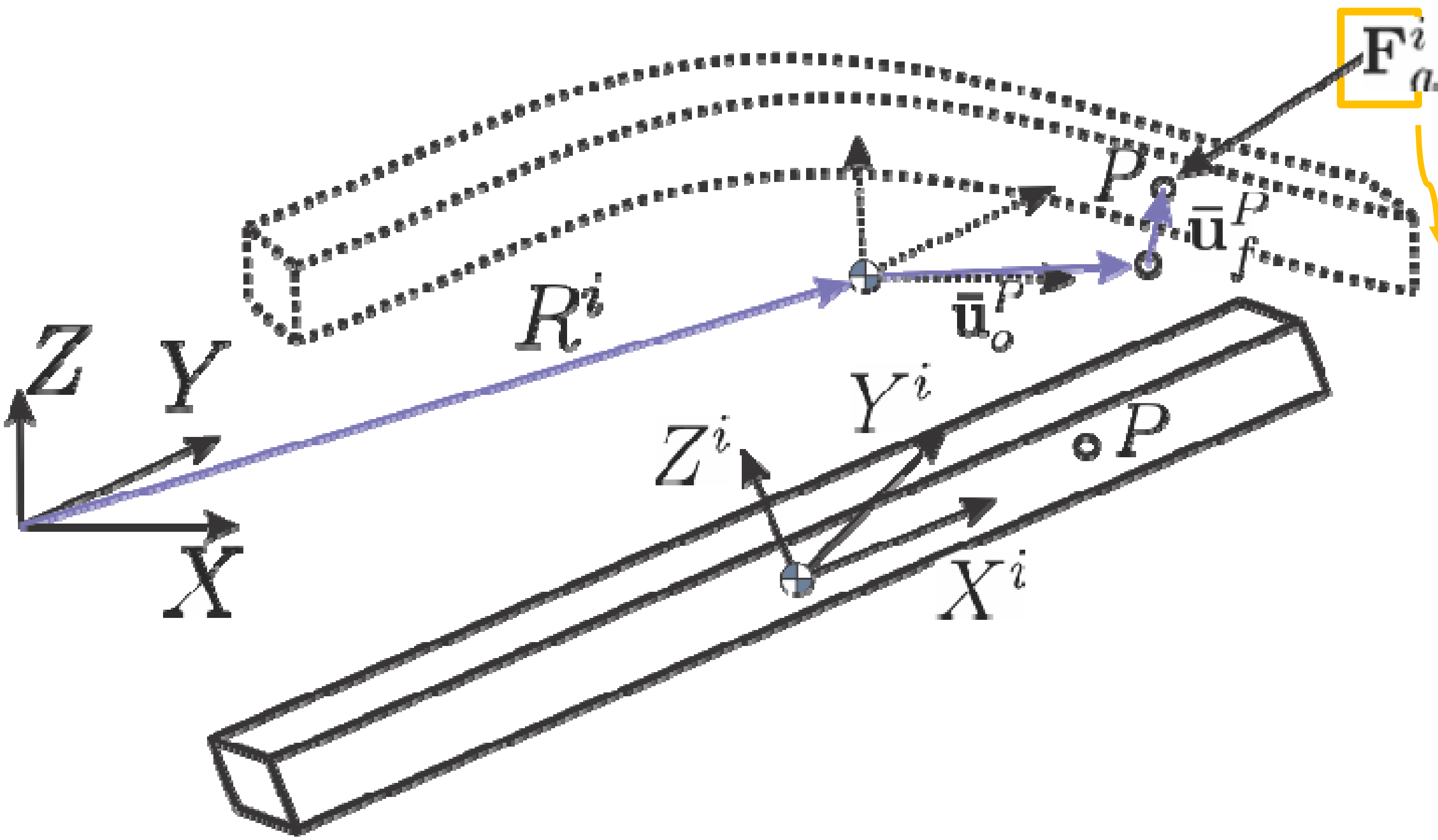
$$\left(\mathbf{Q}_v^i\right)_f = -\rho^i \int\limits_{V^i} \mathbf{S}^{i\mathrm{T}} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i\right] \mathrm{d}V^i$$

These expressions may be obtained by deriving the previous equation or via the Principle of Virtual Work. The Euler parameter identity $\dot{\bar{\mathbf{G}}}^i \dot{\boldsymbol{\theta}}^i = \mathbf{0}$ is used above to simplify the expressions.

Equations of FFR

Applied Forces

External force applied to a deformable body at point P



Can this force make the body rotate? Translate? Deform?

Equations of FFR

Applied Forces The virtual work of a force described in the global frame, $\mathbf{F}_a^i$, applied at P may be expressed as

Generalized force

$$\delta W_a^i = \mathbf{F}_a^i \delta \mathbf{r}_P^i = \boxed{\mathbf{Q}_a^i} \delta \mathbf{q}^i .$$

As a first step, we write the variation of the position vector of a point P in an FFR body i as

$$\delta \mathbf{r}_P^i = \begin{bmatrix} \mathbf{I} & -\mathbf{A}^i \tilde{\bar{\mathbf{u}}}^i \bar{\mathbf{G}}^i & \mathbf{A}^i \mathbf{S}^i \end{bmatrix} \begin{bmatrix} \delta \mathbf{R}^i \\ \delta \boldsymbol{\theta}^i \\ \delta \mathbf{q}_f^i \end{bmatrix} .$$

Combining both equations, one obtains the following generalized force vector

GF rotational coordinates

$$\boxed{\left(\mathbf{Q}_a^i\right)_R} = \mathbf{F}_a^P , \quad \boxed{\left(\mathbf{Q}_a^i\right)_\theta} = -\bar{\mathbf{G}}^{i\mathrm{T}} \tilde{\bar{\mathbf{u}}}^{i\mathrm{T}} \mathbf{A}^{i\mathrm{T}} \mathbf{F}_a^P , \quad \boxed{\left(\mathbf{Q}_a^i\right)_f} = \mathbf{S}^{i\mathrm{T}} \mathbf{A}^{i\mathrm{T}} \mathbf{F}_a^P ,$$

GF translational coordinates

GF flexible coordinates: modal participation factors

where space-dependent vector and matrix, $\bar{\mathbf{u}}^i$ and $\mathbf{S}^i$, respectively, must be particularized at the position of point P.

Equations of FFR

As per discussion with Dan Negrut, the application point of external load was changed to the deformed point location.

Note the following: FFR formulation allows for including the effect of the deformed point location in the generalized rotation coordinates.

$$\delta \mathbf{r}_P^i = \begin{bmatrix} \mathbf{I} & -\mathbf{A}^i \boxed{\tilde{\bar{\mathbf{u}}}^i} \bar{\mathbf{G}}^i & \boxed{\mathbf{A}^i \mathbf{S}^i} \end{bmatrix} \begin{bmatrix} \delta \mathbf{R}^i \\ \delta \boldsymbol{\theta}^i \\ \delta \mathbf{q}_f^i \end{bmatrix} .$$

Can be particularized at the **deformed** location

Matrix of shape functions is always evaluated at the **undeformed** configuration

Note: The loads in FFR do not follow the material, deformed point, as it is based on small deformation finite elements: Shape functions can only be evaluated in the undeformed configuration, so, in terms of deformation it'd be equivalent to think the load is always applied in the undeformed configuration –this is not true for other flexible MBD formulations

Equations of FFR

Equations of Motion Final form:

$$\boxed{\mathbf{M}^i}\ddot{\mathbf{q}}^i + \boxed{\mathbf{D}^i}\dot{\mathbf{q}}^i + \boxed{\mathbf{K}^i}\mathbf{q}^i + \mathbf{C}_{\mathbf{q}^i}^{\mathrm{T}}\lambda = \boxed{\mathbf{Q}_a} + \boxed{\mathbf{Q}_v},$$

Spelling out submatrices and vectors...

$$\boxed{\begin{bmatrix} \mathbf{m}_{RR} & \mathbf{m}_{R\theta} & \mathbf{m}_{Rf} \\ & \mathbf{m}_{\theta\theta} & \mathbf{m}_{\theta f} \\ sym. & & \mathbf{m}_{ff} \end{bmatrix}^i} \begin{bmatrix} \ddot{\mathbf{R}} \\ \ddot{\boldsymbol{\theta}} \\ \ddot{\mathbf{q}}_f \end{bmatrix}^i + \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{D}_{ff} \end{bmatrix}^i} \begin{bmatrix} \dot{\mathbf{R}} \\ \dot{\boldsymbol{\theta}} \\ \dot{\mathbf{q}}_f \end{bmatrix}^i + \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{K}_{ff} \end{bmatrix}^i} \begin{bmatrix} \mathbf{R} \\ \boldsymbol{\theta} \\ \mathbf{q}_f \end{bmatrix}^i +$$

$$\begin{bmatrix} \mathbf{C}_{\mathbf{R}^i}^{\mathrm{T}} \\ \mathbf{C}_{\boldsymbol{\theta}^i}^{\mathrm{T}} \\ \mathbf{C}_{\mathbf{q}_f^i}^{\mathrm{T}} \end{bmatrix} \lambda = \boxed{\begin{bmatrix} (\mathbf{Q}_a)_{\mathbf{R}^i} \\ (\mathbf{Q}_a)_{\boldsymbol{\theta}^i} \\ (\mathbf{Q}_a)_{\mathbf{q}_f^i} \end{bmatrix}} + \boxed{\begin{bmatrix} (\mathbf{Q}_v)_{\mathbf{R}^i} \\ (\mathbf{Q}_v)_{\boldsymbol{\theta}^i} \\ (\mathbf{Q}_v)_{\mathbf{q}_f^i} \end{bmatrix}}.$$

How do these matrices look like?

The new terms $\boxed{\mathbf{K}_{ff} \text{ and } \mathbf{D}_{ff}}$ refer to the stiffness and damping matrices of the structure. These two matrices may be obtained by modeling the structure as a linear system using, for instance, the finite element method.

3. Inertia Shape Integrals

- FFR mass matrix is highly nonlinear

Rewriting the following terms...

$$(\mathbf{Q}_v^i)_R = -\mathbf{A}^i \rho^i \int_{V^i} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i \right] \mathrm{d}V^i$$

$$(\mathbf{Q}_v^i)_\theta = \bar{\mathbf{G}}^{i\mathrm{T}} \rho^i \int_{V^i} \left[\tilde{\bar{\mathbf{u}}}^{i\mathrm{T}} \left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\mathbf{u}}}^{i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i \right] \mathrm{d}V^i$$

$$(\mathbf{Q}_v^i)_f = -\rho^i \int_{V^i} \mathbf{S}^{i\mathrm{T}} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i \right] \mathrm{d}V^i$$

$$\boxed{\begin{aligned}
(\mathbf{Q}_v^i)_R &= -\mathbf{A}^i \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{S}}_t^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \bar{\mathbf{S}}^i \dot{\mathbf{q}}_f^i \right] \\
(\mathbf{Q}_v^i)_\theta &= -2\dot{\bar{\mathbf{G}}}^{i\mathrm{T}} \bar{\mathbf{I}}_{\theta\theta}^i \bar{\boldsymbol{\omega}}_i - 2\dot{\bar{\mathbf{G}}}^{i\mathrm{T}} \bar{\mathbf{I}}_{\theta f}^i \dot{\mathbf{q}}_f^i + \bar{\mathbf{G}}^{iT} \dot{\bar{\mathbf{I}}}_{\theta\theta}^i \bar{\boldsymbol{\omega}}_i \\
(\mathbf{Q}_v^i)_f &= -\rho^i \int_{V^i} \mathbf{S}^{i\mathrm{T}} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\mathbf{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \mathbf{S}^i \dot{\mathbf{q}}_f^i \right] \mathrm{d}V^i, \textit{where...}
\end{aligned}}$$

Inertia Shape Integrals

$$\left(\mathbf{Q}_v^i\right)_R = -\mathbf{A}^i\left[\left(\tilde{\bar{\omega}}_i\right)^2\bar{\mathbf{S}}_t^i + 2\tilde{\bar{\omega}}_i\bar{\mathbf{S}}^i\dot{\mathbf{q}}_f^i\right]$$

$$\left(\mathbf{Q}_v^i\right)_\theta = -2\dot{\bar{\mathbf{G}}}^{i\mathrm{T}}\bar{\mathbf{I}}_{\theta\theta}^i\bar{\omega}_i - 2\dot{\bar{\mathbf{G}}}^{i\mathrm{T}}\bar{\mathbf{I}}_{\theta f}^i\dot{\mathbf{q}}_f^i + \bar{\mathbf{G}}^{iT}\dot{\bar{\mathbf{I}}}_{\theta\theta}^i\bar{\omega}_i$$

$$\left(\mathbf{Q}_v^i\right)_f = -\rho^i\int_{V^i}\mathbf{S}^{i\mathrm{T}}\left[\left(\tilde{\bar{\omega}}_i\right)^2\bar{\mathbf{u}}^i + 2\tilde{\bar{\omega}}_i\mathbf{S}^i\dot{\mathbf{q}}_f^i\right]\mathrm{d}V^i, \textit{where...}$$

$$\bar{\mathbf{S}}_t^i = \int_{V^i}\rho^i\bar{\mathbf{u}}^i\mathrm{d}V^i$$

$$\bar{\mathbf{S}}^i = \int_{V^i}\rho^i\mathbf{S}^i\mathrm{d}V^i$$

$$\bar{\mathbf{I}}_{\theta\theta}^i = \int_{V^i}\rho^i\tilde{\bar{\mathbf{u}}}^{iT}\tilde{\bar{\mathbf{u}}}^i\mathrm{d}V^i \triangleq \text{Inertia tensor of } \boxed{\text{deformable}} \text{ body}$$

$$\bar{\mathbf{I}}_{\theta f}^i = \int_{V^i}\rho^i\tilde{\bar{\mathbf{u}}}^i\mathbf{S}^i\mathrm{d}V^i$$

Inertia Shape Integrals

The terms of the nonlinear FFR mass matrix can be written in terms of inertia shape integrals...

$$\mathbf{m}^i_{R\theta} = -\int_{V^i} \mathbf{A}^i \tilde{\bar{\mathbf{u}}}^i \bar{\mathbf{G}}^i \mathrm{d}V^i = -\mathbf{A}^i \boxed{\left[\int_{V^i} \tilde{\bar{\mathbf{u}}}^i \mathrm{d}V^i\right]} \bar{\mathbf{G}}^i = -\mathbf{A}^i \tilde{\bar{\mathbf{S}}}^i_t \bar{\mathbf{G}}^i$$

$$\mathbf{m}^i_{Rf} = \mathbf{A}^i \boxed{\int_{V^i} \rho^i \mathbf{S}^i \mathrm{d}V^i} = \mathbf{A}^i \bar{\mathbf{S}}^i$$

$$\mathbf{m}^i_{\theta f} = \bar{\mathbf{G}}^{iT} \boxed{\int_{V^i} \rho^i \tilde{\bar{\mathbf{u}}}^i \mathbf{S}^i \mathrm{d}V^i} = \bar{\mathbf{G}}^{iT} \bar{\mathbf{I}}^i_{\theta f}$$

$$\mathbf{m}^i_{ff} = \boxed{\int_{V^i} \rho^i \mathbf{S}^{iT} \mathbf{S}^i \mathrm{d}V^i} = \bar{\mathbf{S}}^i_{11} + \bar{\mathbf{S}}^i_{22} + \bar{\mathbf{S}}^i_{33}$$

Inertia Shape Integrals

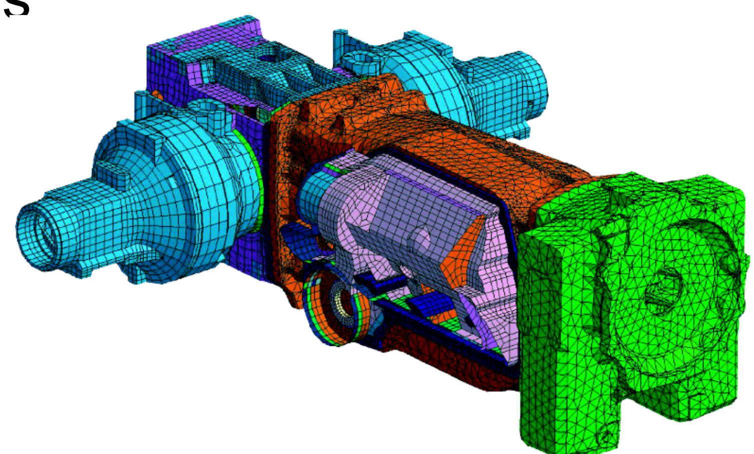
But... Why are these inertia shape integrals noteworthy?

Inertia shape integrals ...

- only depend on *selected, known* shapes
- are integrated over the volume of the solid –simplifications available for 1D, 2D solids
- are **constant** over the simulation!
- only need to be obtained once at the beginning of the simulation

4. Need for Model Order Reduction

- So far we assumed we knew a set of modes that represent the body's deformation
- In practice, FE analysis is used to have complex models of structures/flexible bodies
- FEA models may have hundreds of thousands or millions of degrees of freedom
- Each finite element describe local shapes
- At low and middle frequencies, structures vibrate describing 'smooth' shapes



FEM Model of a rear suspension including CVT gear

Source: http://ecs.magna.com/Finite-Element-Analysis.3153.0.html?&L=1

Need for Model Order Reduction

- MOR is included in commercial/research software at a preprocessing stage
- Before the dynamic simulation we need to *choose* how to reduce our system
- The goodness of the reduction will determine the accuracy of our results

Subspaces for MOR

FFR model has a large number of degrees of freedom $\mathbf{q}_f \in \mathbb{R}^{N\times 1}$ which are approximated in a subspace $\mathbf{V}_s \in \mathbb{R}^{N\times n}$ by a reduced flexible coordinate vector $\tilde{\mathbf{q}}_f \in \mathbb{R}^{n\times 1}$

$$\mathbf{q}_f \approx \mathbf{V}_s \tilde{\mathbf{q}}_f$$

To obtain a unique solution the residual should be orthogonal on another subspace $\mathbf{W}_s \in \mathbb{R}^{N\times n}$

For a linear system (finite element model), the reduced system looks like...

$$\boxed{\begin{aligned} &\tilde{\mathbf{M}}_f \ddot{\tilde{\mathbf{q}}}_f + \tilde{\mathbf{D}}_f \dot{\tilde{\mathbf{q}}}_f + \tilde{\mathbf{K}} \tilde{\mathbf{q}}_f = \tilde{\mathbf{B}}_f \tilde{\mathbf{u}}_f, \\ &\tilde{\mathbf{y}} = \tilde{\mathbf{C}}_f \tilde{\mathbf{q}}_f, \end{aligned}}$$

How do these equations equivalent to the full-order ones? (not reduced)

where

$$\tilde{\mathbf{M}}_f = \mathbf{W}^T \mathbf{M}_f \mathbf{V},\ \tilde{\mathbf{D}}_f = \mathbf{W}^T \mathbf{D}_f \mathbf{V},\ \tilde{\mathbf{K}}_f = \mathbf{W}^T \mathbf{K}_f \mathbf{V} \in \mathbb{R}^{n\times n}$$

$$\tilde{\mathbf{B}}_f = \mathbf{W}^T \mathbf{B}_f \in \mathbb{R}^{n\times p}, \tilde{\mathbf{C}}_f = \mathbf{W}^T \mathbf{C}_f \in \mathbb{R}^{r\times n}$$