ME751
Advanced Computational Multibody Dynamics

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Quotes of the Day
[from Samuel]

"There will come a time when you believe everything is finished. That will be the beginning."
-- Louis L'Amour, author

"Be yourself, everybody else is already taken."
-- Oscar Wilde

"An approximate answer to the right problem is worth a good deal more than the exact answer to an approximate problem"
-- J. Tukey, statistician

"If we did all the things we are capable of, we would literally astound ourselves."
-- Thomas A. Edison

"Nobody makes a greater mistake than he who did nothing because he could only do a little."
-- Edmund Burke, British statesman and orator

Before we get started…

- Last Time:
  - Accounting for the presence of reaction forces in the expression of the virtual work
  - Obtaining the EOM for a 3D system: the **r-p** formulation

- Today:
  - Loose ends, the $\mathbf{r} - \mathbf{p}$ formulation of the EOM
  - Super briefly talk about the EOM when using Euler Angles
  - Discuss TSDAs and RSDAs
  - Simple example of deriving the EOM for a one body system
  - Inverse Dynamics Analysis
  - Equilibrium Analysis

- Homework
  - Assigned today, due on October 14. Will get you started on the simEngine3D

- Reading:
  - Ed Haug's textbook,11.3 and 11.4

Comment: on the Matrix-Free and Matrix-Form of the EOM

- Matrix-free: dealing with one body at a time, a "low level" perspective

$$\left.\begin{array}{lcl} m_i\ddot{\mathbf{r}}_i + \mathbf{\Phi}_{\mathbf{r}_i}^T\lambda & = & \mathbf{F}_i \\ 4\mathbf{G}_i^T\bar{\mathbf{J}}_i\mathbf{G}_i\ddot{\mathbf{p}}_i + \mathbf{\Phi}_{\mathbf{p}_i}^T\lambda + \mathbf{p}_i\lambda_i^{\mathbf{p}} & = & \hat{\tau}_i \end{array}\right\} \quad \text{for} \quad i = 1, \ldots, nb$$

- The matrix-vector form of the EOM: aggregation of the collection of EOM associated with bodies $i = 1, \ldots, nb$. A more condensed notation for the same thing:

$$\left\{\begin{array}{lcl} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T\lambda & = & \mathbf{F} \\ \mathbf{J}^{\mathbf{P}}\ddot{\mathbf{p}} + \mathbf{\Phi}_{\mathbf{p}}^T\lambda + \mathbf{P}^T\lambda^{\mathbf{P}} & = & \hat{\tau} \end{array}\right.$$

- The matrix-free and matrix-form formulations of the EOM capture the same thing (the second order ODE governing the time evolution of a collection of rigid bodies interconnected through geometric constraints). The matrix-form is useful when it's coupled with the Acceleration Kinematic Constraint Equations to provide the big *linear* system whose solution provides the accelerations and Lagrange multipliers:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 4nb} & \mathbf{0}_{3nb\times nb} & \mathbf{\Phi}_{\mathbf{r}}^T \\ \mathbf{0}_{4nb\times 3nb} & \mathbf{J}^{\mathbf{P}} & \mathbf{P}^T & \mathbf{\Phi}_{\mathbf{p}}^T \\ \mathbf{0}_{nb\times 3nb} & \mathbf{P} & \mathbf{0}_{nb\times nb} & \mathbf{0}_{nb\times nc} \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_{\mathbf{p}} & \mathbf{0}_{nc\times nb} & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \lambda^{\mathbf{P}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\tau} \\ \gamma^{\mathbf{P}} \\ \hat{\gamma} \end{bmatrix}$$

The r-p Formulation of the EOM:
Providing Initial Conditions

- Recall that the claim some slides ago was that the quantities in BLUE were known, while the quantities in RED were unknown and supposed to be computed as the solution of a linear system

- Note that in order to evaluate the blue quantities and pose the problem whose solution provides the red quantities; i.e., the accelerations $\ddot{\mathbf{q}}$ and Lagrange Multipliers λ and $\lambda^{\mathbf{P}}$, you need to have *position* and *velocity* information, $\mathbf{q}$ and $\dot{\mathbf{q}}$, respectively.

- In other words, in order to compute $\ddot{\mathbf{q}}$, λ and $\lambda^{\mathbf{P}}$, you need to have $\mathbf{q}$ and $\dot{\mathbf{q}}$. There is no surprise here, since the EOM represent a set of second order ODEs. You will see next week that when you are dealing with a second order ODE, the initial conditions (ICs) provided are the value of $\mathbf{q}$ and $\dot{\mathbf{q}}$, which therefore allows one to compute $\ddot{\mathbf{q}}$, λ, and $\lambda^{\mathbf{P}}$

The r-p Formulation of the EOM:
The RHS of the Acceleration Kin. Constr. Eq.

- Please keep in mind the following: γ used to denote the RHS of the acceleration equation in the $\mathbf{r}$–$\bar{\omega}$ formulation.
- The quantity $\hat{\gamma}$ is used to denote the RHS of the acceleration equation in the $\mathbf{r}$–$\mathbf{p}$ formulation.
- Using the notation $\mathbf{q} \equiv \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}$, we have

$$\hat{\gamma} = -(\mathbf{\Phi_q \dot{q}})_{\mathbf{q}} \dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t} \dot{\mathbf{q}} - \mathbf{\Phi}_{tt}$$

- Note that we have not provided yet $\hat{\gamma}$ for the four basic GCons
  - Basic approach to computing $\hat{\gamma}^{\alpha}$, where $\alpha \in \{DP1, DP2, D, CD\}$: take two time derivatives of Φ^{α}; set the result equal to zero; move to the right side all quantities that do not depend on $\ddot{\mathbf{r}}$ and $\ddot{\mathbf{p}}$. The quantity on the RHS is your $\hat{\gamma}^{\alpha}$, the right side of the kinematic acceleration constraint equation

The r-p Formulation of the EOM:
Getting the RHS of the Acceleration Equation

- First, recall that we introduced a matrix $\mathbf{B}$ as follows:

$$\frac{\partial[\mathbf{A}(\mathbf{p})\cdot\bar{\mathbf{s}}]}{\partial\mathbf{p}} \equiv \mathbf{B}(\mathbf{p},\bar{\mathbf{s}})$$

- Some helpful identities:

$$\dot{\mathbf{B}}(\mathbf{p},\bar{\mathbf{s}}) = \mathbf{B}(\dot{\mathbf{p}},\bar{\mathbf{s}}) \quad \longrightarrow \quad \text{due to the linearity of the } \mathbf{B}(\mathbf{p},\bar{\mathbf{s}}) \text{ matrix in relation to the variable } \mathbf{p}$$

$$\frac{d[\mathbf{B}(\mathbf{p},\bar{\mathbf{s}})\dot{\mathbf{p}}]}{d\,t} = \mathbf{B}(\dot{\mathbf{p}},\bar{\mathbf{s}})\dot{\mathbf{p}} + \mathbf{B}(\mathbf{p},\bar{\mathbf{s}})\ddot{\mathbf{p}}$$

$$\mathbf{a}_i = \mathbf{A}_i\bar{\mathbf{a}}_i \quad \Rightarrow \quad \dot{\mathbf{a}}_i = \mathbf{B}(\mathbf{p}_i,\bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i \quad \Rightarrow \quad \ddot{\mathbf{a}}_i = \mathbf{B}(\dot{\mathbf{p}}_i,\bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i + \mathbf{B}(\mathbf{p}_i,\bar{\mathbf{a}}_i)\ddot{\mathbf{p}}_i$$

$$\dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \dot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

$$\ddot{\mathbf{d}}_{ij} = \ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\bar{\mathbf{s}}_j^Q)\ddot{\mathbf{p}}_j + \mathbf{B}(\dot{\mathbf{p}}_j,\bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P)\ddot{\mathbf{p}}_i - \mathbf{B}(\dot{\mathbf{p}}_i,\bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

The r-p Formulation of the EOM:
Getting the RHS of the Acceleration Equation

HOMEWORK:

- For the four basic GCons, prove that $\hat{\gamma}$ assumes the values indicated below:

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = 0$$

$$\hat{\gamma}^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = -\mathbf{a}_i^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{a}}_j)\dot{\mathbf{p}}_j - \mathbf{a}_j^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i - 2\dot{\mathbf{a}}_i^T \dot{\mathbf{a}}_j + \ddot{f}(t)$$

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

$$\hat{\gamma}^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = -\mathbf{a}_i^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + \mathbf{a}_i^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - \mathbf{d}_{ij}^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i - 2\dot{\mathbf{a}}_i^T \dot{\mathbf{d}}_{ij} + \ddot{f}(t)$$

$$\Phi^{D}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P)^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

$$\hat{\gamma}^{D}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = -2\mathbf{d}_{ij}^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + 2\mathbf{d}_{ij}^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - 2\dot{\mathbf{d}}_{ij}^T \dot{\mathbf{d}}_{ij} + \ddot{f}(t)$$

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = \mathbf{c}^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

$$\hat{\gamma}^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - \mathbf{c}^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + \ddot{f}(t)$$

Comments on getting the EOM using Euler Angles ϵ

Recall What Done to Get $\mathbf{r} - \mathbf{p}$ EOM
[1/3]

- Starting point: the expression of the virtual work, single out the virtual work done by the reaction torques:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} [\, \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a\right) - \delta \bar{\pi}_i^T \bar{\mathbf{\Pi}}_i^T (\mathbf{\Phi}) \lambda \,] = 0\end{aligned}$$

- Note that since now we account for the expression of the reaction force/torque, the condition above holds for arbitrary $\delta \mathbf{r}_i$–$\delta \bar{\pi}_i$

- The basic idea is simple: wherever you see $\delta \bar{\pi}_i$ in the expression of the virtual work of few slides ago, replace it with $2\mathbf{G}_i \delta \mathbf{p}_i$; wherever you see $\dot{\bar{\omega}}_i$, replace it with $2\mathbf{G}_i \ddot{\mathbf{p}}_i$. Additionally, when dealing with the virtual work of the reaction torques, recall the identity in red on the previous slide. We end up with the following:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} [\, \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \delta \mathbf{p}_i^T 2\mathbf{G}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i 2\mathbf{G}_i \ddot{\mathbf{p}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a\right) - \delta \mathbf{p}_i^T \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda \,] = 0\end{aligned}$$

- In the expression above, the variations in Euler Parameters are arbitrary as long as they are healthy; i.e.,

$$\mathbf{p}_i^T \delta \mathbf{p}_i = 0 \qquad i = 1, \ldots, nb \qquad \text{i.e.,} \qquad \mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}} \delta \mathbf{p} = \mathbf{0}_{nb}$$

The Formulation of the EOM Using Euler Angles
[2/3]

BONUS HOMEWORK PROBLEM

- Follow the approach used to get the EOM in the $\mathbf{r}$–$\mathbf{p}$ to produce the EOM for the $\mathbf{r}$–ϵ formulation, where $\epsilon_i = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}_i$ denotes the set of three Euler Angles used to capture the orientation of body i in 3D space. Specifically, start from the expression of the virtual work to obtain the following second order ODEs:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T \lambda &= \mathbf{F} \\ \mathbf{J}^{\epsilon}\ddot{\epsilon} + \mathbf{\Phi}_{\epsilon}^T \lambda &= \breve{\tau} \end{aligned}$$

- Given that $\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{nb} \end{bmatrix}_{3nb}$, you will have to obtain the ODEs above and the expression of $\mathbf{J}^{\epsilon}$ and $\breve{\tau}$

- If you take upon this challenge, you'll have to recall that early on in the semester when we discussed about Euler Angles we showed that there is a [almost everywhere] nonsingular matrix $\mathbf{B}(\epsilon)$ so that (we'll denote by $\mathbf{D} = \mathbf{A}^T\mathbf{B}$)

$$\omega = \mathbf{B}\dot{\epsilon} \qquad\qquad \bar{\omega} = \mathbf{D}\dot{\epsilon}$$

The Formulation of the EOM Using Euler Angles [3/3]

- Note that the EOMs on the previous slide is what ADAMS uses
- Just like before, the ODEs above are augmented with the acceleration kinematic constraint equation $\mathbf{\Phi_r}\ddot{\mathbf{r}} + \mathbf{\Phi}_\epsilon \ddot{\epsilon} = \breve{\gamma}$ to obtain:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb \times 3nb} & \mathbf{\Phi}_\mathbf{r}^T \\ \mathbf{0}_{3nb \times 3nb} & \mathbf{J}^\epsilon & \mathbf{\Phi}_\epsilon^T \\ \mathbf{\Phi_r} & \mathbf{\Phi}_\epsilon & \mathbf{0}_{nc \times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\epsilon} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \breve{\tau} \\ \breve{\gamma} \end{bmatrix}$$

- You would have to determine now for the four basic GCons the expression of $\breve{\gamma}$
- Not going to pursue this since
  - Euler Angles have a nasty singularity that you can run into relatively easily
  - The ingredients that enter the EOMs and kinematic acceleration constraint equations, such as $\mathbf{J}^\epsilon$, $\mathbf{\Phi}_\epsilon$, etc., are very cumbersome and computationally expensive to evaluate
- Note: I uploaded on the class website a PDF that describes in detail the formulation of the EOM and their numerical solution
  - See link 'ADAMS/Solver Primer' (2004 vintage document)

One slide overview

EOM in $\mathbf{r}-\omega$ and $\mathbf{r}-\mathbf{p}$ and $\mathbf{r}-\epsilon$

- Quick overview, Newton-Euler form of the equations of motion:
  - The $\mathbf{r}$–$\bar{\omega}$ formulation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^{T}\lambda &= \mathbf{F} \\ \bar{\mathbf{J}}\dot{\bar{\omega}} + \bar{\mathbf{\Pi}}^{T}(\mathbf{\Phi})\lambda &= \tau \end{aligned}$$

  - The $\mathbf{r}$–$\mathbf{p}$ formulation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^{T}\lambda &= \mathbf{F} \\ \mathbf{J}^{\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{\Phi}_{\mathbf{p}}^{T}\lambda + \mathbf{P}^{T}\lambda^{\mathbf{p}} &= \hat{\tau} \end{aligned}$$

  - The $\mathbf{r}$–ϵ formulation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^{T}\lambda &= \mathbf{F} \\ \mathbf{J}^{\epsilon}\ddot{\epsilon} + \mathbf{\Phi}_{\epsilon}^{T}\lambda &= \breve{\tau} \end{aligned}$$

Dynamics in the Absence of Constraints

- If there are no constraints, we will be left with the following problems to solve:
- The $\mathbf{r}$–$\bar{\omega}$ formulation:

$$\mathbf{M}\ddot{\mathbf{r}} = \mathbf{F}$$

$$\bar{\mathbf{J}}\dot{\bar{\omega}} = \tau$$

  - This represents a collection of differential equations (ODEs). Note that it's not clear in this from how to get the orientation of the body, this only provides a way to compute through integration the angular velocity $\bar{\omega}$

- The $\mathbf{r}$–$\mathbf{p}$ formulation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} &= \mathbf{F} \\ \mathbf{J^p}\ddot{\mathbf{p}} + \mathbf{P}^T\lambda^{\mathbf{p}} &= \hat{\tau} \end{aligned}$$

$$\mathbf{P}\ddot{\mathbf{p}} = \gamma^{\mathbf{p}}$$

  - This represents a collection of differential and algebraic equations (DAEs). Tricky to solve.

- The $\mathbf{r}$–ϵ formulation:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} &= \mathbf{F} \\ \mathbf{J}^{\epsilon}\ddot{\epsilon} &= \breve{\tau} \end{aligned}$$

  - This represents a collection of second order ODEs. Straightforward to solve yet plagued by singularities.

Comments,
Not Wanting to Deal W/ Constraints

- The tricky part: coming up with the set of forces that each body experiences. Recall that we assume that forces are known quantities; i.e., they are an input to the dynamics problem that tells you how the bodies move in response to the forces applied on them

- Previous slide contains the set of equations that most video gaming engines will solve. Usually, creative ways are used to eliminate GCons

- Examples of typical forces in video gaming
  - Contact and friction forces: a shell hits a Doom character (deal with this next week)
  - User provided: playing tennis in Wii

- In engineering applications you'll typically have GCons present in a mechanism (suspension of a vehicle, hinges on a door, etc.)

- Examples of typical forces in engineering applications:
  - Springs, dampers, actuators: translational or rotational (deal with this next)
  - Forces coming out of tires: tough to deal with, particularly when terrain deforms under tire

Discussion on Applied Forces and Torques

Virtual Work:
Contribution of concentrated forces/torques

- From slides 19 & 20 of October 3, the virtual work produced by the active forces and torques is

$$\sum_{U\in\mathcal{U}_i}\left[\delta\mathbf{r}_i^T+\delta\bar{\pi}_i^T\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\right]\cdot\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\delta\bar{\pi}_i^T\cdot\bar{\mathbf{n}}_V^a$$

$$=\quad\delta\mathbf{r}_i^T\cdot\sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a+\delta\bar{\pi}_i^T\cdot\left[\sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a\right]$$

$$=\quad\delta\mathbf{r}_i^T\mathbf{F}_i^a+\delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^a$$

- Notation used:
  - Total active force acting on body i:

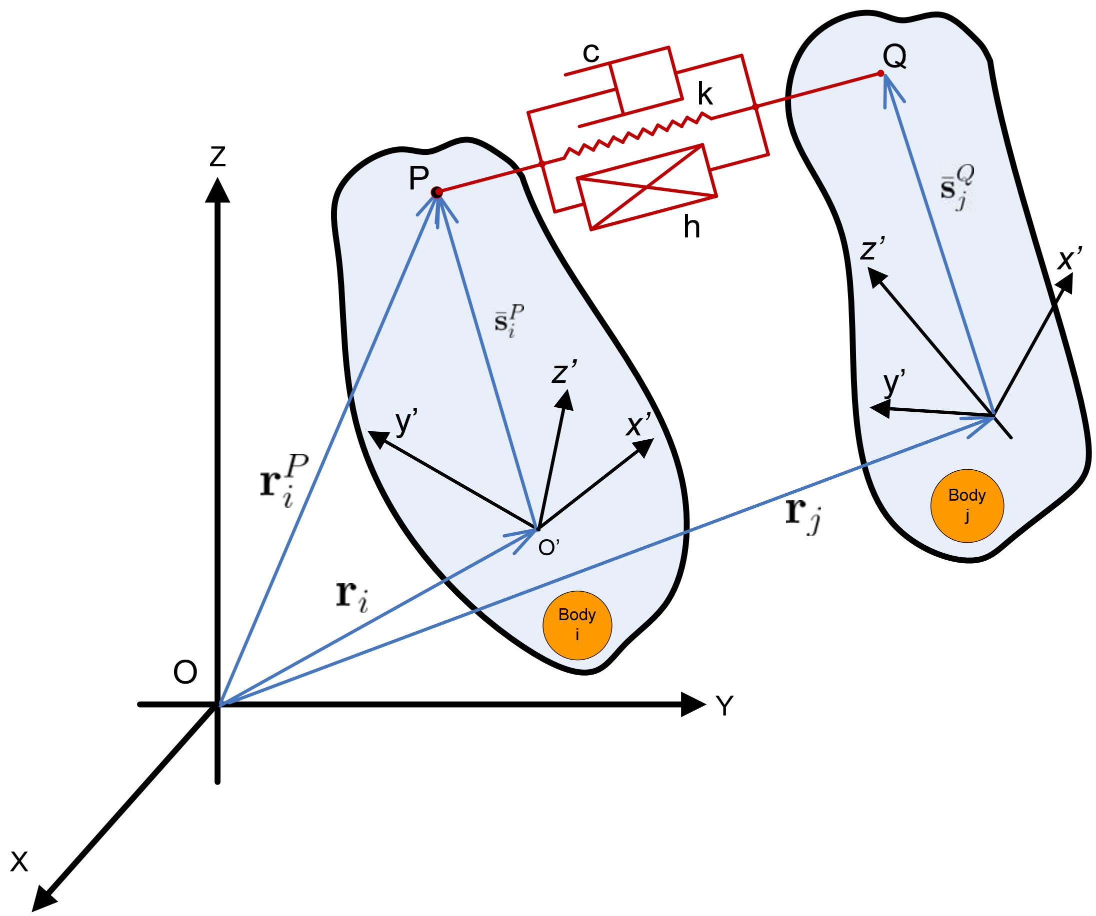
$$\mathbf{F}_i^a=\sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a$$

  - Total active torque acting on body i:

$$\bar{\mathbf{n}}_i^a=\sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a$$

Concentrated Forces: TSDA
(Translational Spring-Damper-Actuator) – pp.445

- Setup: You have a translational spring-damper-actuator acting between point P_i on body i, and P_j on body j



- Translational spring, stiffness *k*
  - Zero stress length (given): l_0
- Translational damper, coefficient c
- Actuator (hydraulic, electric, etc.) – symbol used "h"

Concentrated Forces: TSDA

- Brief description: a force element that acts between point P on body i and point Q on body j. Consequently, each body will regard this as a point (concentrated) force.

- Using the notation introduced on slide 19 of October 3 when we discussed about active forces/torques we have that $P \in \mathcal{U}_i$, where $\mathcal{U}_i$ is the collection of all points on body i where there are concentrated forces applied to the body. Our force $\mathbf{F}_P^a$ will contribute to the body i resultant force $\mathbf{F}_i^a$ **and** resultant torque $\bar{\mathbf{n}}_i^a$:

$$\boxed{\mathbf{F}_i^a = \sum_{U \in \mathcal{U}_i} \mathbf{F}_U^a} \qquad \text{and} \qquad \bar{\mathbf{n}}_i^a = \boxed{\sum_{U \in \mathcal{U}_i} \tilde{\bar{\mathbf{s}}}_i^U \mathbf{A}_i^T \mathbf{F}_U^a} + \sum_{V \in \mathcal{V}_i} \bar{\mathbf{n}}_V^a$$

  - The same discussion holds for body j, there we'll have the force $\mathbf{F}_Q^a$ that acts on body j and reflects in the overall resultant force and also makes a contribution to the expression of the torque

- In order to indicate that this is not your average concentrated force $\mathbf{F}_P^a$ applied at point P but rather a special kind, we'll denote this particular force by $\mathbf{F}_P^{TSDA}$ to indicate that it is the force produced by a Translational Spring-Damper-Actuator element.

Concentrated Forces: TSDA

- Attributes of the TSDA:
  - The body i, location of point P on body i
  - The body j, location of point Q on body j
  - The value k of the spring stiffness, along with the zero tension length of the spring, l_0
  - The value of the damping coefficient c
  - The expression of the actuation force h, which depends on the distance l_{ij} between points P and Q, its rate of change $\dot{l}_{ij}$, and time t. You typically encounter actuators in mechatronics systems, when you want to control the motion of a system through an actuator (penumatic, hydraulic, electric, etc.). We assume a scalar function is provided to us to capture the magnitude of the force produced by the actuator in the form

$$h(l_{ij}, \dot{l}_{ij}, t) = \ldots\ldots$$

- Expression of the TSDA force:

$$f^{TSDA}(i, \bar{\mathbf{s}}^P, j, \bar{\mathbf{s}}^Q, k, l_0, c, h, t) = k(l_{ij} - l_0) + c\dot{l}_{ij} + h(l_{ij}, \dot{l}_{ij}, t)$$

Concentrated Forces: TSDA

- Nomenclature:

$$\mathbf{d}_{ij} = \mathbf{r}_j^Q - \mathbf{r}_i^P = \mathbf{r}_j + \mathbf{A}_i\bar{\mathbf{s}}^Q - \mathbf{r}_i - \mathbf{A}_j\bar{\mathbf{s}}^P$$

$$l_{ij} = ||\mathbf{d}_{ij}|| \qquad \mathbf{e}_{ij} = \frac{\mathbf{d}_{ij}}{l_{ij}}$$

- In this case, the force of interest is

$$\mathbf{F}_P^{TSDA} = f^{TSDA} \cdot \mathbf{e}_{ij}$$

- Quick remarks
  - Note that you'll run into trouble if you ever end up with $l_{ij} = 0$ (look at the definition of $\mathbf{e}_{ij}$, l_{ij} shows in the denominator). In other words, make sure the point P on body i and point Q on body j never assume the same location. If you end up with this particular situation, see Haug's book pp. 446 for a discussion of how to handle this scenario
  - IMPORTANT: Note that the discussion above was associated with body i. Body j would see *exactly* the same value for f^{TSDA}, yet when it comes to the direction along which the force acts, it'll act along $\mathbf{e}_{ji}$, where

$$\mathbf{e}_{ji} = \frac{\mathbf{d}_{ji}}{l_{ij}} = \frac{-\mathbf{d}_{ij}}{l_{ij}} = -\mathbf{e}_{ij}$$

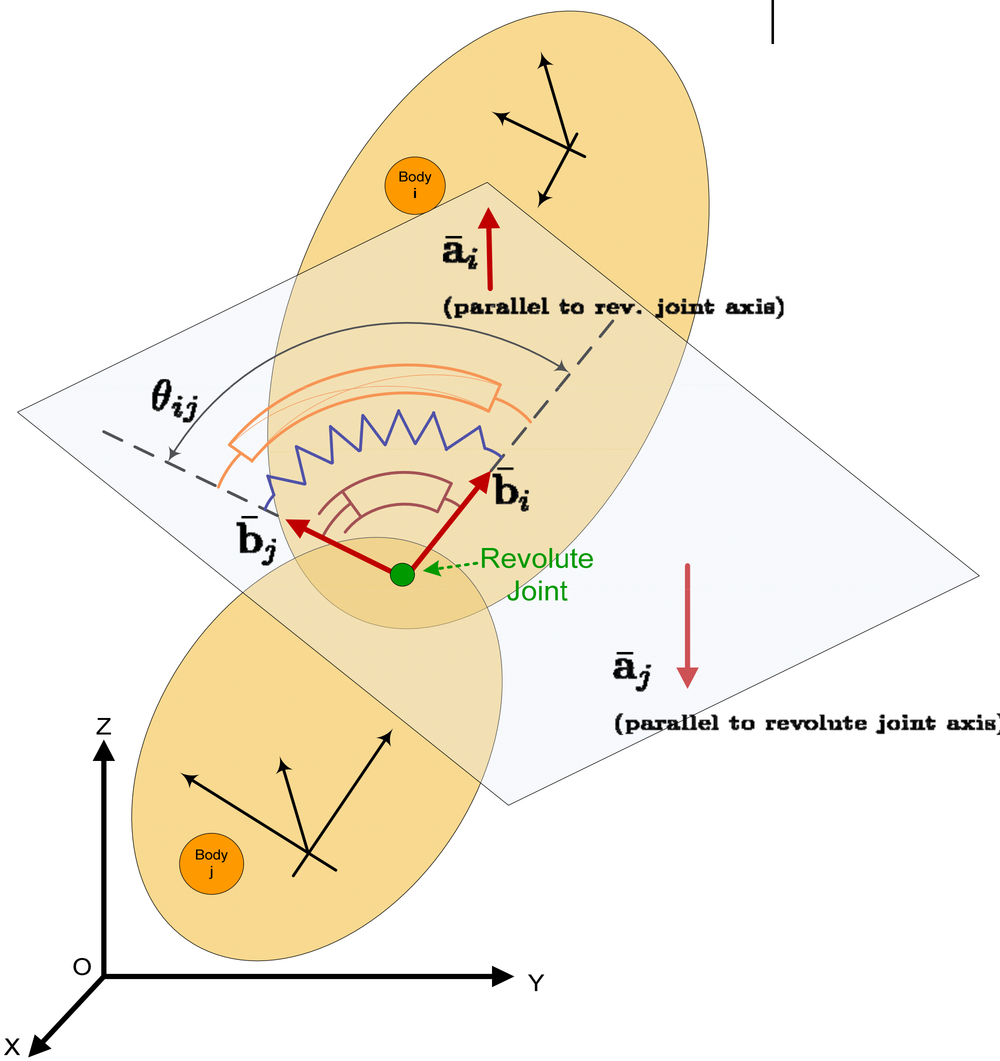
$$\mathbf{F}_Q^{TSDA} = f^{TSDA} \cdot \mathbf{e}_{ji}$$

  - Note therefore that $\mathbf{F}_P^{TSDA} = -\mathbf{F}_Q^{TSDA}$

Concentrated Torques: RSDA

(Rotational Spring-Damper-Actuator) – pp.448

- Setup: You have a rotational spring-damper-actuator acting between two lines, each line rigidly attached to one of the bodies (dashed lines in figure)



- Rotational spring, stiffness k
- Rotational damper, coefficient c
- Actuator (hydraulic, electric, etc.) – symbol used "h"

Virtual Work:
Contribution of the active forces/torques

- Brief description: a torque element that acts between body i and body j. Called RSDA, Rotational Spring-Damper-Actuator. Very similar to TSDA, except it leads to a torque rather than a force

- In the vast majority of cases although not always, an RSDA is attached to revolute (most often) or cylindrical joints. As such, there will be an axis around which the torque acts. This axis is obtained based on the attributes of the revolute or cylindrical joint. See discussion on the attributes of the RSDA

- Following the discussion for the TSDA, we will only focus on the torque $\mathbf{n}_V^a$ as 'felt' by body i, with the understanding the body j would experience $-\mathbf{n}_V^a$.

- To explicitly indicate its origin/nature, we'll use the notation $\mathbf{n}_{\bar{\mathbf{a}}_i}^{RSDA}$. Here, $\bar{\mathbf{a}}_i$ represents the axis about which the torque is applied when acting on body i

- Note that back when we formulated the EOM we worked with torques expressed in the local reference frame; i.e., $\mathbf{n}_V^a$ is used in the EOM as $\bar{\mathbf{n}}_V^a$

- Using the notation introduced on slide 20 of October 3 this torque will end up in $\bar{\mathbf{n}}_i^a$, which accumulates all torques acting on body i:

$$\bar{\mathbf{n}}_i^a = \sum_{U \in \mathcal{U}_i} \tilde{\bar{\mathbf{s}}}_i^U \mathbf{A}_i^T \mathbf{F}_U^a + \boxed{\sum_{V \in \mathcal{V}_i} \bar{\mathbf{n}}_V^a}$$

Concentrated Torques: RSDA

- Attributes of the RSDA:
  - The body i, the axis $\bar{\mathbf{a}}_i$ (a unit vector) about which the torque is applied on body i, and the unit vector $\bar{\mathbf{b}}_i$, perpendicular to $\bar{\mathbf{a}}_i$, used to measure the relative rotational angle θ_{ij}
  - The body j, the axis $\bar{\mathbf{a}}_j$ (a unit vector) about which the torque is applied on body j, and the unit vector $\bar{\mathbf{b}}_j$, perpendicular to $\bar{\mathbf{a}}_j$, used to measure the relative rotational angle θ_{ij}
  - The value k of the spring stiffness, along with the zero tension angle of the spring, θ_0
  - The value of the damping coefficient c
  - The expression of the actuation torque h, which depends on the relative angle θ_{ij}, its rate of change $\dot{\theta}_{ij}$, and time t. You typically encounter actuators in mechatronics systems, when you want to control the motion of a system through an actuator (torque coming from an electric motor of an engine). We assume a scalar function is provided to us to capture the magnitude of the torque produced by the actuator in the form

$$h(\theta_{ij}, \dot{\theta}_{ij}, t) = \ldots\ldots$$

- Expression of the RSDA torque:

$$n^{RSDA}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{a}}_j, \bar{\mathbf{b}}_j, k, \theta_0, c, h, t) = k(\theta_{ij} - \theta_0) + c\dot{\theta}_{ij} + h(\theta_{ij}, \dot{\theta}_{ij}, t)$$

Concentrated Torques: RSDA

- When computing $\bar{\mathbf{n}}_{\bar{\mathbf{a}}_i}^{RSDA}$, the angle θ_{ij} is measured from $\mathbf{b}_i$ to $\mathbf{b}_j$ (see figure).
  - The $\mathbf{b}_i$ and $\mathbf{b}_j$ vectors are represented in the global reference frame so that we measure θ_{ij} in a consistent fashion
  - Note that θ_{ij} and θ_{ji} are measured differently. The discussion here is focused on what body i experiences. Body j would automatically experience the opposite torque
- For the RSDA, the torque acting on body i is computed as

$$\bar{\mathbf{n}}_{\bar{\mathbf{a}}_i}^{RSDA} = n^{RSDA} \cdot \bar{\mathbf{a}}_i$$

- Quick remarks
  - When defining the attributes of the RSDA we mentioned that $\bar{\mathbf{a}}_i$ defines the axis of the revolute/cylindrical joint and as seen above it is also used in the definition of the torque. Yet we have not specified a sense that goes along with that direction. By definition the sense is so that an extension of the rotational spring leads to the torque as expressed in the equation above
  - The discussion covers the body i. For body j, a similar train of thought leads to the conclusion that

$$\bar{\mathbf{n}}_{\bar{\mathbf{a}}_j}^{RSDA} = n^{RSDA} \cdot \bar{\mathbf{a}}_j$$

  - Note that $\mathbf{a}_i = -\mathbf{a}_j$ since

$$\mathbf{n}_{\bar{\mathbf{a}}_i}^{RSDA} = -\mathbf{n}_{\bar{\mathbf{a}}_j}^{RSDA} \quad \Rightarrow \quad \mathbf{A}_i\, \bar{\mathbf{n}}_{\bar{\mathbf{a}}_i}^{RSDA} = -\mathbf{A}_j\, \bar{\mathbf{n}}_{\bar{\mathbf{a}}_j}^{RSDA} \quad \Rightarrow \quad \mathbf{A}_i\, n^{RSDA} \cdot \bar{\mathbf{a}}_i = -\mathbf{A}_j\, n^{RSDA} \cdot \bar{\mathbf{a}}_j$$
$$\Rightarrow \quad \mathbf{A}_i \bar{\mathbf{a}}_i = -\mathbf{A}_j \bar{\mathbf{a}}_j \quad \Rightarrow \quad \mathbf{a}_i = -\mathbf{a}_j$$

[AO]
Example: EOM for a Dangling Cube

- See handout, available also online
- Units are all SI
- Cube of mass 6, length of edge is 2
- Hanging from a corner at point P
- A force applied at opposite corner, at point Q
- Moving under gravity $\mathbf{g}$

- What's the work order?
  - Formulate the EOM using the $\mathbf{r} - \boldsymbol{\omega}$ formulation
  - Get the linear system whose solution provides accelerations and Lagr. multiplier