ME751
Advanced Computational Multibody Dynamics

October 3, 2016

Dan Negrut
University of Wisconsin-Madison

Quote of the Day

“Everything we hear is an opinion, not a fact. Everything we see is a perspective, not the truth.”

“You have power over your mind - not outside events. Realize this, and you will find strength.”

-- Marcus Aurelius [121 AD – 180 AD]

Before we get started…

- Last Time:
  - Discussed Virtual Displacement and Variation of a Function
  - Covered the ·**r** , ·` case
  - Covered the ·**r**, ·**p** case

- Today:
  - Compute the virtual work associated with a mechanical system made up of nb bodies
  - Work our way towards establishing the EOM that govern the dynamics of a mechanical system

- Reading Assignment:
  - Please read Sections 10.1 and 10.2 of Haug's book: http://sbel.wisc.edu/Courses/ME451/BookHaugVolONE/Chapter10.pdf
    - This will put help things in perspective insofar Kinematic Analysis is concerned

- Miscellaneous
  - Please upload on the ME751 Forum the link to your GitHub repo
  - It'll allow me to see your progress on simEngine3D

Closing Comments, Virtual Displacements [1/2]

- Over the next lectures we'll express the virtual variation of a function that depends on the position and orientation of one or more bodies in the system using one of two sets of virtual displacements: $\delta\mathbf{r}$ and $\delta\bar{\pi}$, or $\delta\mathbf{r}$ and $\delta\mathbf{p}$

- At the end of the day, a virtual displacement of the bodies in the system will lead to a virtual variation of a generic constraint Φ^α, $\alpha \in \{DP1, DP2, D, CD, \mathbf{p}\}$:

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \Phi^\alpha_\mathbf{r}\delta\mathbf{r} + \bar{\Pi}(\Phi^\alpha)\delta\bar{\pi} = \Phi^\alpha_\mathbf{r}\delta\mathbf{r} + \Phi^\alpha_\mathbf{p}\delta\mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \begin{bmatrix} \Phi^\alpha_\mathbf{r} & \bar{\Pi}(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi^\alpha) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$$

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \begin{bmatrix} \Phi^\alpha_\mathbf{r} & \Phi^\alpha_\mathbf{p} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{p} \end{bmatrix} = \Phi^\alpha_\mathbf{q} \cdot \delta\mathbf{q}$$

Closing Comments, Virtual Displacements [2/2]

- Recall that we are supposed to collect *all* m constraints and stick them together in one big $\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \end{bmatrix}$, if we plan to work with virtual displacements expressed in terms of $\delta\mathbf{r}$ and $\delta\bar{\pi}$, or in $\mathbf{\Phi}^F = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \\ \mathbf{\Phi}^{\mathbf{P}}(\mathbf{p}) \end{bmatrix}$, if we plan to work with $\delta\mathbf{r}$ and $\delta\mathbf{p}$.
- Recall that any one of the constraints in $\mathbf{\Phi}$ is one of the four basic GCons that we introduced
- When interested in the variation of $\mathbf{\Phi}$, we simply stack together the variation of each of the GCons that enters in $\mathbf{\Phi}$. The situation for $\mathbf{\Phi}^F$ is similar, except that here you need to account explicitly for the Euler Parameter normalization constraints.
- A virtual displacement of the bodies in the system will lead to a virtual variation $\delta\mathbf{\Phi}$ that depends on the position and orientation of the bodies:

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{r}}\delta\mathbf{r} + \bar{\mathbf{\Pi}}(\mathbf{\Phi})\delta\bar{\pi} \qquad \text{or} \qquad \delta\mathbf{\Phi}^F = \mathbf{\Phi}^F_{\mathbf{r}}\delta\mathbf{r} + \mathbf{\Phi}^F_{\mathbf{p}}\delta\mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta\mathbf{\Phi}(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$$

$$\delta\mathbf{\Phi}^F(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \mathbf{\Phi}^F_{\mathbf{r}} & \mathbf{\Phi}^F_{\mathbf{p}} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{p} \end{bmatrix} = \mathbf{\Phi}^F_{\mathbf{q}} \cdot \delta\mathbf{q}$$

The Concept of Consistent Virtual Displacements

- Framework: assume that your $\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}$ is such that the constraints are satisfied; i.e., $\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0}$. Apply now a virtual displacement $\delta\mathbf{r}_i$ and $\delta\mathbf{p}_i$ to each body i in the system.

- Question: how should you choose the virtual displacements $\delta\mathbf{r}_i$ and $\delta\mathbf{p}_i$, $i = 1, \ldots, nb$ so that the new configuration is also consistent?

- I am interested in a healthy $\delta\mathbf{q}$:

$$\mathbf{q} \longrightarrow \mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0} \qquad \mathbf{q} + \delta\mathbf{q} \longrightarrow \mathbf{\Phi}^F(\mathbf{q} + \delta\mathbf{q}, t) = \mathbf{\Phi}^F(\mathbf{q}, t) + \delta\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0}$$

- It follows that

$$\delta\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi}^F_{\mathbf{q}}\delta\mathbf{q} = \mathbf{0}$$

- *By definition*, a virtual displacement $\delta\mathbf{q}$ is said to be consistent with the set of constraints present in the system if $\mathbf{\Phi}^F_{\mathbf{q}}\delta\mathbf{q} = \mathbf{0}$ holds

- Note that a similar train of thought can be followed to define a $\begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$ consistent virtual displacement. The condition in that case reads

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \mathbf{0}$$

[Short detour: two ways of posing a virtual rotation]

The $\delta\bar{\pi} \leftrightarrow \delta\mathbf{p}$ Relation

- Recall that a infinitesimal change in orientation, that is, a change from $\mathbf{A} \longrightarrow \mathbf{A} + \delta\mathbf{A}$ can be completely characterized by a vector quantity $\delta\bar{\pi}$
- Recall also that in general an orientation is defined by four Euler Parameters $\mathbf{p}$.
- The key question: suppose you applied a virtual rotation characterized by a vector $\delta\bar{\pi}$. What would be the equivalent virtual change in $\delta\mathbf{p}$ that would lead to the same variation $\delta\mathbf{A}$ in the orientation?
- In other words,

The same · A

$$\mathbf{A} \xrightarrow{\delta\bar{\pi}} \mathbf{A} + \delta\mathbf{A} \qquad \mathbf{A} \xrightarrow{\delta\mathbf{p}=?} \mathbf{A} + \delta\mathbf{A}$$

- I can also pose the question in the opposite direction: if I specified a virtual rotation that is characterized by the change in Euler Parameters $\delta\mathbf{p}$, what would be the value of $\delta\bar{\pi}$ that would lead to the same modification $\delta\mathbf{A}$ of the orientation matrix?

$$\mathbf{A} \xrightarrow{\delta\mathbf{p}} \mathbf{A} + \delta\mathbf{A} \qquad \mathbf{A} \xrightarrow{\delta\bar{\pi}=?} \mathbf{A} + \delta\mathbf{A}$$

The same · A

$\delta \mathbf{p}$ given; $\delta \bar{\pi} = ?$

Useful identity: $\delta \mathbf{A} = \mathbf{E}\,(\delta \mathbf{G})^T$

$$\widetilde{\delta \bar{\pi}} = \mathbf{A}^T \delta \mathbf{A} = 2\mathbf{G}\mathbf{E}^T\mathbf{E}(\delta \mathbf{G})^T$$

$$\Downarrow$$

$$\widetilde{\delta \bar{\pi}} = 2\mathbf{G}(\delta \mathbf{G})^T$$

$$\Downarrow$$

$$\widetilde{\delta \bar{\pi}} = 2(\widetilde{\mathbf{G}\delta \mathbf{p}})$$

$$\Downarrow$$

$$\delta \bar{\pi} = 2\mathbf{G}\delta \mathbf{p}$$

$$\delta \pi = \mathbf{A}\delta \bar{\pi} = 2\mathbf{E}\mathbf{G}^T\mathbf{G}\delta \mathbf{p}$$

$$\Downarrow$$

$$\delta \pi = 2\mathbf{E}\delta \mathbf{p}$$

Draws on pp.344, Haug's book
(Eq. 9.3.32)

$\delta\pi$ given; $\quad \delta\mathbf{p} = ?$

$$\delta\bar{\pi} = 2\mathbf{G}\delta\mathbf{p}$$

$$\mathbf{G}^T\,\delta\bar{\pi} = 2\mathbf{G}^T\mathbf{G}\delta\mathbf{p}$$

$$\delta\mathbf{p} = \frac{1}{2}\mathbf{G}^T\delta\bar{\pi} \quad \longrightarrow \quad \delta\mathbf{p} = \frac{1}{2}\mathbf{E}^T\delta\pi$$

See pp.345, Haug's book

Useful identity: $\mathbf{p}^T(\delta\mathbf{p}) = 0$

Deriving the EOM

The $\mathbf{r} - \bar{\omega}$ EOM Formulation.

Road Map

- Introduce the forces acting on one body present in a mechanical system
  - Distributed
  - Concentrated
- Express the virtual work produced by each of these forces acting on *one* body
- Evaluate the virtual work for the entire mechanical system
- Apply principle of virtual work and obtain the EOM

- Next time:
  - Eliminate the reaction forces from the expression of the virtual work
  - Obtain the constrained EOM (Newton-Euler form)
  - Express the reaction (constraint) forces given the Lagrange multipliers

Types of Forces & Torques

- Distributed over the volume of a body (color red)
  - Inertia forces
  - Forces induced by external fields (gravity, electro-magnetic, etc.)
    - We'll call them "mass-distributed"
  - "Internal interaction" forces
- Concentrated at a point (color blue)
  - Reaction (or constraint) forces and torques
  - Action (or applied, or external) forces and torques

On Principles

- Principle of Virtual Work
  - Applies to a collection of particles
  - States that at equilibrium, the virtual work of the forces acting on the collection of particle is zero

- D'Alembert's Principle
  - For a collection of particles moving around you can still fall back on the Principle of Virtual Work when you also include in the set of forces acting on each particle i its inertia force

$$\sum_i \delta \mathbf{r}_i^T \cdot (\mathbf{F}_i - m_i \mathbf{a}_i) = 0$$

  - Note: we are talking here about a collection of *particles*
    - Consequently, we'll have to regard each rigid body as a collection of particles that are rigidly connected to each other and that together make up the body

Dealing with Inertia Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$

- Expression of the force:

$$-\ddot{\mathbf{r}}_i^P \; dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot [-\ddot{\mathbf{r}}_i^P \; dm_i(P)]$$

- Comments:

  – The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P)$$

Dealing with Mass-Distributed Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. A force per unit mass, $\mathbf{f}_i(P)$, is assumed to act at point P.

- Expression of the force:

$$\mathbf{f}_i(P)\, dm_i(P)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P)$$

  - This type of force is rarely seen in classical multibody dynamics. Exception: the force due to the gravitational field, which leads to the weight of the body. In this case $\mathbf{f}_i(P) = \mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration of magnitude $g \approx 9.81 \frac{m}{s^2}$ (in Madison, WI).

Dealing with "Internal Interaction" Forces

- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. We also consider an *arbitrary* point R on body i. The focus is on the **internal force** acting between the mass elements $dm_i(P)$ and $dm_i(R)$.

- The expression of this type of force acting at point P is obtained by considering the contribution of each point R of the body:

$$\int_{m_i} \mathbf{f}_i(P, R)\, dm_i(R)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \int_{m_i} \mathbf{f}_i(P, R)\, dm_i(R)$$

- Comments:
  - The total virtual work produced by this type of force when acting at all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \left[\int_{m_i} \mathbf{f}_i(P, R)\, dm_i(R) \right] dm_i(P) = \int_{m_i} \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P, R)\, dm_i(R)\, dm_i(P)$$

  - The assumption that we make is that the force $\mathbf{f}_i(P, R)$ acts along the line connecting points P and R. In other words, $\mathbf{f}_i(P, R) dm_i(R) = k(\mathbf{r}_i^P - \mathbf{r}_i^R)$, where k is a scalar that might depend on the points P and R.

Dealing with Constraint Reaction Forces

- Framework: We assume that a set of constraints acts on body i. These constraints most often lead to the presence of reaction forces. We will assume that the constraints on body i are producing reaction (constraint) forces acting at a collection of points generically denoted by $\mathcal{Q}_i$.

- Expression of this type of force acting at point $Q \in \mathcal{Q}_i$:

$$\mathbf{F}_Q^r$$

- Virtual work produced by this set of forces:

$$\sum_{Q \in \mathcal{Q}_i} [\delta \mathbf{r}_i^Q]^T \cdot \mathbf{F}_Q^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction forces $\mathbf{F}_Q^r$ for $Q \in \mathcal{Q}_i$
  - A separate discussion will follow on the meaning of the points $\mathcal{Q}_i$

Dealing with Constraint Reaction Torques

- Framework: We assume that a set of constraints acts on body i. These constraints most often lead to the presence of reaction torques. We will assume that the constraints on body i are producing reaction (constraint) torques acting at a collection of points generically denoted by $\mathcal{Z}_i$.

- Expression of this type of torque acting at point $Z \in \mathcal{Z}_i$ when represented in the L-RF:

$$\bar{\mathbf{n}}_Z^r$$

- Virtual work produced by these reaction torques:

$$\sum_{Z \in \mathcal{Z}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_Z^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction torques $\bar{\mathbf{n}}_Z^r$ for $Z \in \mathcal{Z}_i$
  - Note that since we are talking about *rigid* bodies, we have the same virtual rotation $\delta \bar{\pi}_i$ no matter what point $Z \in \mathcal{Z}_i$ of the rigid body we are dealing with

Dealing with Active Forces

- Framework: We assume that a set of active forces acts on body i. These active forces are acting at a collection of points generically denoted by $\mathcal{U}_i$.
- Expression of this type of force acting at point $U \in \mathcal{U}_i$:

$$\mathbf{F}_U^a$$

- Virtual work produced by this set of forces:

$$\sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}_U^a$$

Dealing with Active Torques

- Framework: We assume that a set of active torques acts on body i. We will assume that these active torques are acting at a collection of points generically denoted by $\mathcal{V}_i$.
- Expression of this type of torque acting at point $V \in \mathcal{V}_i$, expressed in the L-RF$_i$:

$$\bar{\mathbf{n}}_V^a$$

- Virtual work produced by this set of torques:

$$\sum_{V \in \mathcal{V}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_V^a$$

- Comments: Note that since we are talking about *rigid* bodies, we have the same virtual rotation $\delta \bar{\pi}_i$ no matter which of the torques acting on the rigid body we are dealing with

Short Detour: On the Choice of L-RF$_i$

- We will choose the L-RF of each body so that it is a centroidal reference frame. In other words, the L-RF$_i$ is located at the center of mass of body i, for $i \in \{1, \ldots, nb\}$
- For a centroidal reference frame, by definition,

$$\int_{m_i} \bar{\mathbf{s}}^P \, dm_i(P) = \mathbf{0}_3$$

- The definition of the mass moment of inertia tensor:

$$\bar{\mathbf{J}} = \int_{m_i} -\tilde{\bar{\mathbf{s}}}^P \, \tilde{\bar{\mathbf{s}}}^P \, dm_i(P) = \begin{bmatrix} \bar{J}_{xx} & \bar{J}_{xy} & \bar{J}_{xz} \\ \bar{J}_{yx} & \bar{J}_{yy} & \bar{J}_{yz} \\ \bar{J}_{zx} & \bar{J}_{zy} & \bar{J}_{zz} \end{bmatrix}$$

Short Detour: On the Choice of L-RF$_i$

- The constant matrix $\bar{\mathbf{J}}$ represents an attribute that depends on the shape (geometry) of the body and the distribution of mass within that geometry
- Recall that a careful choice of the orientation of L-RF leads to this matrix $\bar{\mathbf{J}}$ being diagonal. When L-RF is chosen like that, it becomes a principal reference frame
- To conclude, to keep things simple yet without any loss of generality in terms of formulating the EOM, we will assume that for each body i we selected the L-RF$_i$ so that it is a centroidal and principal RF
- NOTE: When can't you assume to have a centroidal and principal RF? When (a) the body if flexible, or (b) when you are solving an optimization problem and the geometry (shape) of the body changes in response to the very optimization process

Putting Things in Perspective

- What we have accomplished so far
  - Expressed the virtual work done by each of the forces acting on *one* body "i"
  - Decided to work with centroidal and principal LRFs

- Coming up next
  - Evaluate the virtual work associated with a mechanism; i.e., several bodies
    - We're dealing w/ a mechanical system made up of nb bodies
  - Clean up the complicated expression of the system-level virtual work
    - Express the virtual work as a function of the virtual displacements of the nb bodies

The Expression of the Virtual Work

- Principle of Virtual Work, applied for a collection of rigid bodies interconnected through an arbitrary collection of constraints:

$$\delta W = \sum_{i=1}^{nb} \Bigg[\int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \, dm_i(P) + \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P) \, dm_i(P)$$

$$+ \int_{m_i} \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P,R) \, dm_i(R) \, dm_i(P) + \sum_{Q \in \mathcal{Q}_i} [\delta \mathbf{r}_i^Q]^T \cdot \mathbf{F}_Q^r + \sum_{Z \in \mathcal{Z}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_Z^r$$

$$+ \sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}_U^a + \sum_{V \in \mathcal{V}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_V^a \Bigg] = 0$$

[Short Detour]:

Virtual Translation of a Point & Acceleration of a Point

- Recall the expression of a virtual translation of a point P on body i as a result of a virtual displacement $\delta \mathbf{r}_i$ and $\delta \bar{\pi}_i$ of the L-RF$_i$

$$\delta \mathbf{r}_i^P = \delta \mathbf{r}_i - \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta \bar{\pi}_i$$

$$\Downarrow$$

$$\left[\delta \mathbf{r}_i^P\right]^T = \delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T$$

- We'll need the acceleration of an arbitrary point P, obtained as:

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \bar{\mathbf{s}}_i^P$$

$$\Downarrow$$

$$\dot{\mathbf{r}}_i^P = \dot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P$$

$$\Downarrow$$

$$\ddot{\mathbf{r}}_i^P = \ddot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P + \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \bar{\mathbf{s}}_i^P$$

Virtual Work: Contribution of the Reaction Forces/Torques

- Virtual work produced by the reaction forces and torques is

$$
\begin{aligned}
& \sum_{Q \in \mathcal{Q}_i} \left[\delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^Q \mathbf{A}_i^T \right] \cdot \mathbf{F}_Q^r + \sum_{Z \in \mathcal{Z}_i} \delta \bar{\pi}_i^T \cdot \bar{\mathbf{n}}_Z^r \\
= \; & \delta \mathbf{r}_i^T \cdot \sum_{Q \in \mathcal{Q}_i} \mathbf{F}_Q^r + \delta \bar{\pi}_i^T \cdot \left[\sum_{Q \in \mathcal{Q}_i} \tilde{\bar{\mathbf{s}}}_i^Q \mathbf{A}_i^T \mathbf{F}_Q^r + \sum_{Z \in \mathcal{Z}_i} \bar{\mathbf{n}}_Z^r \right] \\
= \; & \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^r
\end{aligned}
$$

- Notation used:
  - Total reaction force acting on body i:

$$
\mathbf{F}_i^r = \sum_{Q \in \mathcal{Q}_i} \mathbf{F}_Q^r
$$

  - Total reaction torque acting on body i:

$$
\bar{\mathbf{n}}_i^r = \sum_{Q \in \mathcal{Q}_i} \tilde{\bar{\mathbf{s}}}_i^Q \mathbf{A}_i^T \mathbf{F}_Q^r + \sum_{Z \in \mathcal{Z}_i} \bar{\mathbf{n}}_Z^r
$$

Virtual Work: Contribution of the Active Forces/Torques

- Virtual work produced by the active forces and torques is

$$\sum_{U\in\mathcal{U}_i}\left[\delta\mathbf{r}_i^T+\delta\bar{\pi}_i^T\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\right]\cdot\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\delta\bar{\pi}_i^T\cdot\bar{\mathbf{n}}_V^a$$

$$=\delta\mathbf{r}_i^T\cdot\sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a+\delta\bar{\pi}_i^T\cdot\left[\sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a\right]$$

$$=\delta\mathbf{r}_i^T\mathbf{F}_i^a+\delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^a$$

- Notation used:
  - Total active force acting on body i:

$$\mathbf{F}_i^a=\sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a$$

  - Total active torque acting on body i:

$$\bar{\mathbf{n}}_i^a=\sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a+\sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a$$

Virtual Work: Contribution of Internal Forces

- Based on discussion at pp. 418 of Haug's book (see Eqs. 11.1.4, 11.1.5), the virtual work of the internal forces in a rigid body is zero:

$$\int_{m_i}\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P,R)\, dm_i(R)\, dm_i(P) = 0$$

- This goes back to the fact that (a) $\mathbf{f}_i(P,R) dm_i(R) = k(\mathbf{r}_i^P - \mathbf{r}_i^R)$, where k is a scalar that might depend on the points P and R, an assumption made a couple of slides ago, and (b) the body is rigid, from where $(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = \text{const.}$

$$(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = \text{const.} \qquad \Rightarrow \qquad (\delta\mathbf{r}^P - \delta\mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = 0$$

- Part of next assignment

Virtual Work:
Contribution of Mass-Distributed Forces

- Move on to the mass distributed internal force:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\, dm_i(P) = \int_{m_i} [\delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T] \cdot \mathbf{f}_i(P)\, dm_i(P)$$

$$= \delta \mathbf{r}_i^T \int_{m_i} \mathbf{f}_i(P)\, dm_i(P) + \delta \bar{\pi}_i^T \int_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T \mathbf{f}_i(P)\, dm_i(P)$$

$$\equiv \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \bar{\pi}_i^T \cdot \bar{\mathbf{n}}_i^m$$

- Notation used:

$$\mathbf{F}_i^m = \int_{m_i} \mathbf{f}_i(P)\, dm_i(P) \qquad \bar{\mathbf{n}}_i^m = \int_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T \mathbf{f}_i(P)\, dm_i(P)$$

- Superscript m indicates that this is mass distributed force (force per unit mass)
- If the $\mathbf{f}_i(P) = \mathbf{g}$; i.e., we only have the unit of mass subject to the gravitational force, then

$$\mathbf{F}_i^m = m_i \mathbf{g} \qquad \bar{\mathbf{n}}_i^m = \mathbf{0}_3$$

Virtual Work: Contribution of Inertia Forces

- Virtual work of the inertia force turns out to be more challenging:

$$\int\limits_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \, dm_i(P) = \int\limits_{m_i} [\delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T] \cdot \left[\ddot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P + \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \bar{\mathbf{s}}_i^P \right] \, dm_i(P)$$

$$= \quad \delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i \int\limits_{m_i} \, dm_i(P) + \delta \mathbf{r}_i^T \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \, dm_i(P) + \delta \mathbf{r}_i^T \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \int\limits_{m_i} \bar{\mathbf{s}}_i^P \, dm_i(P)$$

$$+ \quad \delta \bar{\pi}_i^T \int\limits_{m_i} \bar{\mathbf{s}}_i^P \, dm_i(P) \mathbf{A}_i \ddot{\mathbf{r}}_i + \delta \bar{\pi}_i^T \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P \, dm_i(P) - \delta \bar{\pi}_i^T \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\mathbf{s}}}_i^P \, dm_i(P) \dot{\bar{\omega}}_i$$

$$= \quad \delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i + \delta \bar{\pi}_i^T \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i + \delta \bar{\pi}_i^T \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i$$

- We used the fact that $\bar{\mathbf{J}} = \int\limits_{m_i} -\tilde{\bar{\mathbf{s}}}^P \, \tilde{\bar{\mathbf{s}}}^P \, dm_i(P)$ and the following identity (see Haug's book, bottom of pp.420)

$$\int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P \, dm_i(P) = \tilde{\bar{\omega}}_i \left[- \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\mathbf{s}}}_i^P \, dm_i(P) \right] \bar{\omega}_i = \tilde{\bar{\omega}}_i \bar{\mathbf{J}} \bar{\omega}_i$$

Virtual Work: Putting Things in Perspective

- At this point, the expression of the virtual work assumes the form:

$$
\begin{aligned}
\delta W &= \sum_{i=1}^{nb} \left[-\delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i - \delta \bar{\pi}_i^T \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \delta \bar{\pi}_i^T \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \bar{\pi}_i^T \cdot \bar{\mathbf{n}}_i^m \right. \\
&+ \left. \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^r \ \right] = 0
\end{aligned}
$$

- Alternatively,

$$
\begin{aligned}
\delta W &= \sum_{i=1}^{nb} \left[\ \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) \right. \\
&+ \left. \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r \right) \ \right] = 0
\end{aligned} \tag{1}
$$

Virtual Work: Putting Things in Perspective

- Since Eq.(1) on previous slide should hold for *any* set of virtual displacements $(\delta \mathbf{r}_1, \delta \bar{\pi}_1)$, $(\delta \mathbf{r}_2, \delta \bar{\pi}_2)$,. . ., $(\delta \mathbf{r}_{nb}, \delta \bar{\pi}_{nb})$, then we necessarily have that for $i = 1, \ldots, nb$:

$$\begin{aligned} -\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r &= \mathbf{0}_3 \\ -\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r &= \mathbf{0}_3 \end{aligned}$$

- Equivalently,

$$\begin{aligned} m_i \ddot{\mathbf{r}}_i &= \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \\ \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i &= \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r - \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i \end{aligned}$$

- The set of equations above represent the EOM for the system of nb bodies.