ME751
Advanced Computational Multibody Dynamics

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Quote of the Day

"If you don't know where you are going, you might wind up someplace else."
-- Yogi Berra

Before we get started…

- Last Time:
  - Discussed partial derivatives. There was a $\mathbf{p} - \boldsymbol{\omega}$ fork aspect we dealt with
  - Discussed computation of $\bar{\mathbf{\Pi}}$
  - Quick remarks on Position Analysis + Newton Raphson
  - Wrapped up Kinematics Analysis

- Today:
  - Start discussing the Dynamics Analysis
  - Discuss about Virtual Displacements and Variation of a Function
    - We're facing the same $\mathbf{p} - \boldsymbol{\omega}$ fork issue

- New homework: assigned today, due next Friday at 9:30 am

Purpose of Chapter 11

- At the end of this chapter you should understand what “dynamics” means and how you should go about carrying out a dynamics analysis

- We’ll learn how to:
  - Formulate the equations that govern the time evolution of a system of bodies in 3D motion
    - These equations are differential equations and they are called the “equations of motion”
    - As many bodies as you wish, connected by any joints we’ve learned about…
  - Compute the reaction forces in any joint connecting any two bodies in the mechanism
  - Account for the effect of external forces in the equations of motion

The Idea, in a Nutshell…

- **Kinematics**
  - As <u>many</u> constraints as generalized coordinates
  - <u>No spare</u> degrees of freedom left
  - Position, velocity, acceleration found as the solution of algebraic problems
  - We do not care whatsoever about forces applied to the system
    - We are told what the motions are; this suffices for the purpose of kinematics

- **Dynamics**
  - You only have a <u>few</u> constraints imposed on the system
  - You have <u>extra</u> degrees of freedom
  - The system evolves in time as a result of external forces applied on it
  - We very much care about forces applied and inertia properties of the components of the mechanism (mass, mass moment of inertia)

A Relevant Question…

- Dynamics **key** question: How can I get the acceleration of each body of the mechanism?
  - Note: If you know the acceleration you can integrate it twice to get velocity and position information for each body
  - In other words, you want to get this quantity:

$$\ddot{\mathbf{q}}_i = \begin{bmatrix} \ddot{\mathbf{r}}_i \\ \ddot{\mathbf{p}}_i \end{bmatrix}$$

  - Alternatively, you can get first

$$\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\bar{\omega}}_i \end{bmatrix}$$

    - Then use the fact that there is a relationship of the type (see previous lecture)

$$\ddot{\mathbf{p}} \longleftrightarrow \dot{\bar{\omega}}$$

Looking Back; Looking Ahead

- Looking back: recall ME240 dynamics (for particle): $\boldsymbol{F} = \boldsymbol{m} \cdot \boldsymbol{a}$
  - Right way to state this: $\boldsymbol{m} \cdot \boldsymbol{a} = \boldsymbol{F}$, which is the “equation of motion” (EOM)
    - Acceleration, which is what we care about, would then simply be $\boldsymbol{a} = \boldsymbol{F}/\boldsymbol{m}$

- Looking ahead (next week):
  - Step 1: we’ll first show that the EOM for a rigid body is

$$
\begin{aligned}
\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} &= \mathbf{0}_{3nb} \qquad \text{Equations of Motion governing translation} \\
\bar{\mathbf{J}}\dot{\bar{\omega}} - \bar{\mathbf{n}} + \tilde{\bar{\omega}}\bar{\mathbf{J}}\bar{\omega} &= \mathbf{0}_{3nb} \qquad \text{Equation of Motion governing rotation}
\end{aligned}
$$

  - Step 2: formulate the equations of motion for a system of bodies interacting through contact, friction, and bilateral constraints

[New Important Topic]
Virtual Displacements

- Rest of the lecture today:
  - Discuss the concept of “virtual displacements”
  - Discuss the concept of “consistent virtual displacements”

- Warm up, for deriving the EOM

Motivation

- Why do we have to talk about "virtual displacements" (VDs)?
  - Because they play a crucial role in evaluating the virtual work

- Why do we care about virtual work?
  - Because it is the crucial ingredient required to formulate the equations of motion (EOM)

- How are the EOM formulated actually?
  - Apply D'Alembert's Principle; then fall back on the Principle of Virtual Work

- The Principle of Virtual Work:
  - Powerful tool used to get EOM in rigid and deformable body dynamics
  - "At equilibrium, the virtual work of forces acting on a system is zero"

Motivation

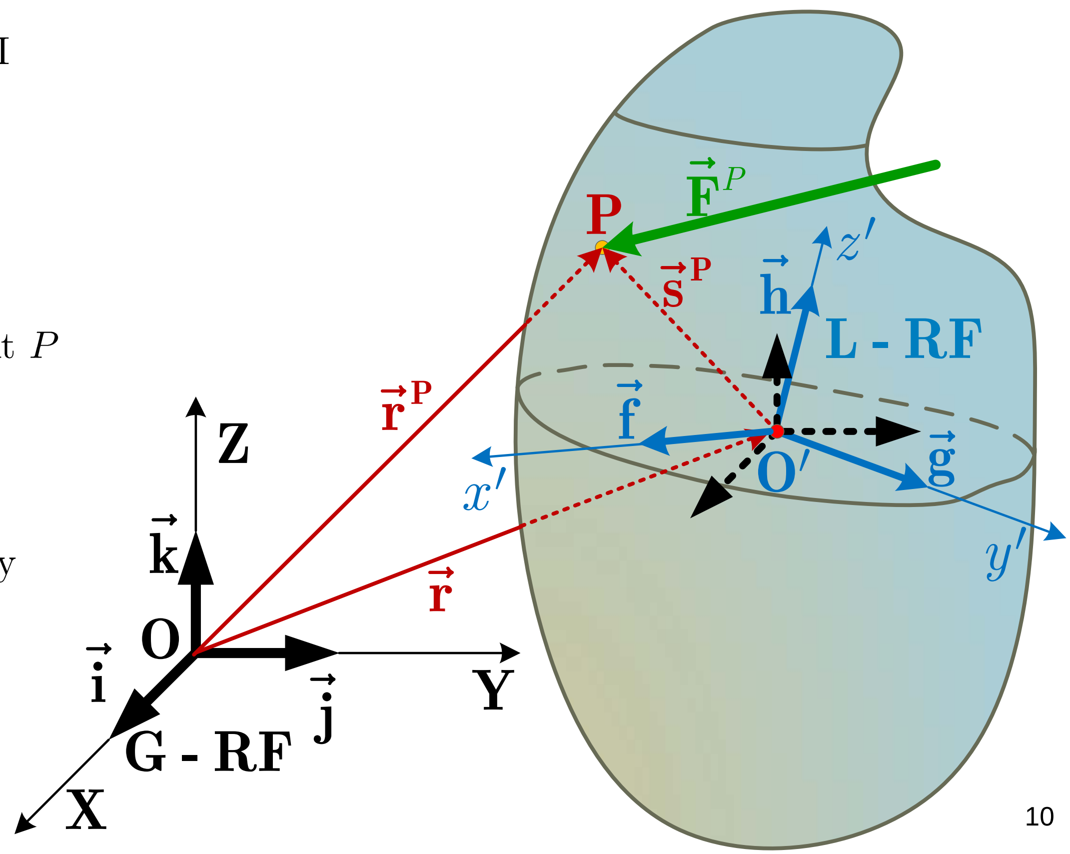
[Cntd.]

- Imagine that a force $\vec{\mathbf{F}}^P$ acts on the rigid body at point P. The work done by this force is

$$\delta W^{F_P} = \delta \mathbf{r}^P \cdot \mathbf{F}^P$$

- Here $\delta \mathbf{r}^P$ represents a very small displacement of the point P. Causes for this displacement:

- The principle of virtual work requires that I should be in a position to consider *any* small displacement of point P
- This generic displacement is called *virtual displacement*
- The size of the virtual displacement of point P is infinitesimally small
- The fact that a body i is connected to other bodies through joints intuitively suggests that a virtual displacement of body i is related to a virtual displacement of body j if bodies i and j are connected through some type of constraint (joint)



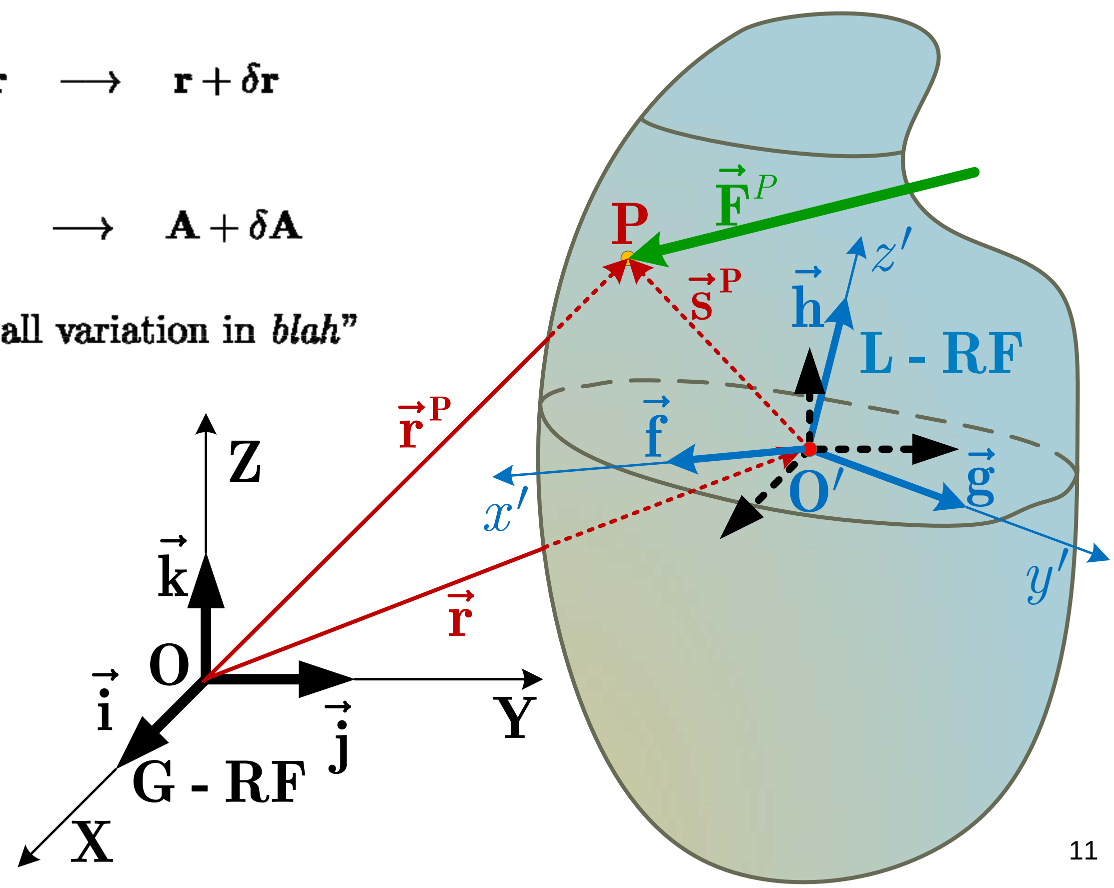
Virtual Translation + Virtual Rotation

- Important observation: since the body is rigid, the small displacement of point P is fully described in terms of a small translation $\delta\mathbf{r}$ and a small change of orientation $\delta\mathbf{A}$ of the L-RF
- Specifically, assume that the change in the L-RF position and orientation are as follows

$$\mathbf{r} \quad \longrightarrow \quad \mathbf{r} + \delta\mathbf{r}$$

$$\mathbf{A} \quad \longrightarrow \quad \mathbf{A} + \delta\mathbf{A}$$

- Read the construct "δ of *blah*" as "a small variation in *blah*"



Virtual Displacement of Point P

- Original position of P:

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\bar{\mathbf{s}}^P$$

- Position of P after the small change in the position and orientation of the rigid body:

$$\mathbf{r}^P + \delta\mathbf{r}^P = (\mathbf{r} + \delta\mathbf{r}) + (\mathbf{A} + \delta\mathbf{A})\bar{\mathbf{s}}^P$$

- Net change in position of point P:

$$\delta\mathbf{r}^P = \underbrace{(\mathbf{r}^P + \delta\mathbf{r}^P)}_{\text{Location, after Virtual Displacement}} - \underbrace{\mathbf{r}^P}_{\text{Location, Original}} = \delta\mathbf{r} + \delta\mathbf{A}\ \bar{\mathbf{s}}^P$$

- Quick remarks:
  - Dimensions: $\delta\mathbf{r}$ is 3×1, and $\delta\mathbf{A}$ is 3×3
  - The change in orientation, $\delta\mathbf{A}$, is not quite random. This is because the new matrix $\mathbf{A}+\delta\mathbf{A}$, which corresponds to the new orientation after the rigid body is nudged, should represent an actual orientation, that is, it must satisfy the orthonormality condition

$$(\mathbf{A} + \delta\mathbf{A})^T(\mathbf{A} + \delta\mathbf{A}) = \mathbf{I}_3$$

Comments on Change in Orientation, $\delta\mathbf{A}$

- First, keep in mind that the changes in position and orientation are **small**.
- Translation, in mathematical lingo: products of two changes in position and/or orientation are ignored

$$\delta\mathbf{r}^T\delta\mathbf{r} \approx 0 \qquad (\delta\mathbf{A})^T\delta\mathbf{A} \approx \mathbf{0}_{3\times3} \qquad \delta\mathbf{A}(\delta\mathbf{A})^T \approx \mathbf{0}_{3\times3}$$

- Key Result: There is a **vector** that is the generator of the matrix $\mathbf{A}\delta\mathbf{A}$. This vector is called **virtual rotation**: $\delta\bar{\pi}$
- Proof:

$$(\mathbf{A}+\delta\mathbf{A})^T(\mathbf{A}+\delta\mathbf{A}) = \mathbf{I}_3 \Rightarrow \mathbf{A}^T\mathbf{A} + \mathbf{A}^T\delta\mathbf{A} + (\delta\mathbf{A})^T\mathbf{A} + \underbrace{(\delta\mathbf{A})^T\delta\mathbf{A}}_{\mathbf{0}} = \mathbf{I}_3$$

$$\Rightarrow \qquad \mathbf{A}^T\delta\mathbf{A} + (\delta\mathbf{A})^T\mathbf{A} = \mathbf{0}_{3\times3} \qquad \Rightarrow \qquad \mathbf{A}^T\delta\mathbf{A} = -(\delta\mathbf{A})^T\mathbf{A} = -[\mathbf{A}^T\delta\mathbf{A}]^T$$

- Thus, the matrix $\mathbf{A}^T\delta\mathbf{A}$ is skew symmetric. As such, there should be a vector, denote it $\delta\bar{\pi}$, so that

$$\widetilde{\delta\bar{\pi}} = \mathbf{A}^T\delta\mathbf{A}$$

- The vector $\delta\bar{\pi}$ is called the virtual rotation vector, and therefore the change in the orientation matrix $\delta\mathbf{A}$ can be expressed in terms of the virtual rotation vector as

$$\delta\mathbf{A} = \mathbf{A}\widetilde{\delta\bar{\pi}}$$

The Invariance Property of · `

- Note that when representing the virtual rotation vector in the G-RF, one gets

$$\widetilde{\delta\pi} = \mathbf{A}\widetilde{\delta\bar{\pi}}\mathbf{A}^T \qquad \Rightarrow \qquad \widetilde{\delta\pi} = (\delta\mathbf{A})\mathbf{A}^T \qquad \Rightarrow \qquad \delta\mathbf{A} = \widetilde{\delta\pi}\mathbf{A}$$

- The virtual rotation vector $\delta\pi$ was implicitly defined by the identity $\widetilde{\delta\pi} = (\delta\mathbf{A})\mathbf{A}^T$. This somewhat suggests that $\delta\pi$ is related to the matrix $\mathbf{A}$. What follows proves that this is not the case, instead, $\delta\pi$ is an attribute of the rigid body the L-RF is attached to.

- First, assume that there are two different virtual rotation vectors: $\delta\pi_1$, which goes along with L-RF$_1$, and $\delta\pi_2$, which goes along with L-RF$_2$, where the two L-RFs are rigidly attached to the same body

- Then, since $\mathbf{A}_2 = \mathbf{A}_1\mathbf{C}$, we have $\delta\mathbf{A}_2 = \delta\mathbf{A}_1\mathbf{C}$, which implies that

$$\widetilde{\delta\pi}_2\mathbf{A}_2 = \widetilde{\delta\pi}_1\mathbf{A}_1\mathbf{C}$$

- Since $\mathbf{A}_2 = \mathbf{A}_1\mathbf{C}$, we get that

$$\widetilde{\delta\pi}_2 = \widetilde{\delta\pi}_1 \qquad \Rightarrow \qquad \delta\pi_2 = \delta\pi_1$$

- In other words, the virtual rotation is an attribute of the body, not of the L-RF rigidly attached to it.

Putting Things in Perspective
[Nomenclature issues]

- Virtual Translation: an infinitesimal translation $\delta\mathbf{r}$ of the L-RF. Performed with the time held fixed.

- Virtual Change in Orientation: an infinitesimal change in the orientation of the body captured in a change $\delta\mathbf{A}$ of the orientation matrix $\mathbf{A}$ associated with the L-RF. The virtual change $\delta\mathbf{A}$ in orientation is performed with the time held fixed.

- Virtual Rotation: a vector quantity $\delta\bar{\pi}$ that is the generator of $\mathbf{A}^T\delta\mathbf{A}$. In other words, $\widetilde{\delta\bar{\pi}} = \mathbf{A}^T\delta\mathbf{A}$, from where $\delta\mathbf{A} = \mathbf{A}\widetilde{\delta\bar{\pi}}$.

- Virtual Displacement: the combination of a virtual translation and a virtual rotation.

- Virtual Variation of a function (expression): change in the value of a function that depends on the location and orientation of a body in the system as a result of a virtual displacement applied to that body

Variational Calculus

- We have a function (or expression) that depends on the location and orientation of the bodies in a mechanical system

- Examples of such expressions:

$$\begin{aligned}
\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) &= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) \\
\mathbf{d}_{ij} &= \mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P \\
\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= \mathbf{c}^T \mathbf{d}_{ij} - f(t)
\end{aligned}$$

- The fundamental question that we want to answer today: what is the variation in the value of the function when the location and orientation of the bodies in the system slightly change as a result of applying a virtual displacement?

- The answer to this question is the subject of the calculus of variations

Formulas, Calculus of Variations

Rule 1 Variation of a constant quantity $\mathbf{c}$ (applies to scalars c or matrices $\mathbf{C}$ as well):

$$\delta(\mathbf{c}) = \mathbf{0}$$

- Example use: calculate the variation of $\mathbf{c}^T \mathbf{d}_{ij}$

Rule 2 Variation of a sum of two vectors:

$$\delta(\mathbf{u} + \mathbf{v}) = \delta\mathbf{u} + \delta\mathbf{v}$$

- Example use: calculate the variation of $\mathbf{r}_i + \mathbf{A}_i \bar{\mathbf{s}}_i^P$

Rule 3 Variation of the product of two matrices:

$$\delta(\mathbf{U}\mathbf{V}) = (\delta\mathbf{U})\mathbf{V} + \mathbf{U}(\delta\mathbf{V})$$

- Example use: calculate the variation of $\mathbf{A}(\mathbf{p}) = \mathbf{E}\mathbf{G}^T$

Rule 4 Variation of the product of matrix times a vector:

$$\delta(\mathbf{U}\mathbf{v}) = (\delta\mathbf{U})\mathbf{v} + \mathbf{U}(\delta\mathbf{v})$$

- Example use: calculate the variation of $\mathbf{G}\mathbf{p}$

Rule 5 Variation of the product of two vectors:

$$\delta(\mathbf{u}^T \mathbf{v}) = \mathbf{v}^T(\delta\mathbf{u}) + \mathbf{u}^T(\delta\mathbf{v})$$

- Example use: calculate the variation of $\mathbf{a}_i^T \mathbf{a}_j$

Not difficult to prove.
We'll skip though.

Virtual Variation, Basic GCons: Φ^{DP1}

- Recall that

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = \mathbf{a}_i{}^T \mathbf{a}_j - f(t) = 0$$

- Assume that body i experiences a virtual displacement characterized by $\begin{bmatrix} \delta \mathbf{r}_i \\ \delta \bar{\pi}_i \end{bmatrix}$, and the body j experiences a virtual displacement characterized by $\begin{bmatrix} \delta \mathbf{r}_j \\ \delta \bar{\pi}_j \end{bmatrix}$. Therefore, $\mathbf{A}_i \longrightarrow \mathbf{A}_i + \delta \mathbf{A}_i$, and $\mathbf{A}_j \longrightarrow \mathbf{A}_j + \delta \mathbf{A}_j$.

- This variation in the attitude of bodies i and j will lead to a variation in the value of Φ^{DP1}. Specifically, $\bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j \longrightarrow \bar{\mathbf{a}}_i^T (\mathbf{A}_i + \delta \mathbf{A}_i)^T (\mathbf{A}_j + \delta \mathbf{A}_j) \bar{\mathbf{a}}_j$.

- Therefore,

$$\begin{aligned}
\delta \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) &= \bar{\mathbf{a}}_i^T (\mathbf{A}_i + \delta \mathbf{A}_i)^T (\mathbf{A}_j + \delta \mathbf{A}_j) \bar{\mathbf{a}}_j - \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j \\
&= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \delta \mathbf{A}_j \bar{\mathbf{a}}_j + \bar{\mathbf{a}}_i^T (\delta \mathbf{A}_i)^T \mathbf{A}_j \bar{\mathbf{a}}_j \\
&= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \widetilde{\delta \bar{\pi}}_j \bar{\mathbf{a}}_j + \bar{\mathbf{a}}_j^T \mathbf{A}_j^T \mathbf{A}_i \widetilde{\delta \bar{\pi}}_i \bar{\mathbf{a}}_i \\
&= -\bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \tilde{\bar{\mathbf{a}}}_j \delta \bar{\pi}_j - \bar{\mathbf{a}}_j^T \mathbf{A}_j^T \mathbf{A}_i \tilde{\bar{\mathbf{a}}}_i \delta \bar{\pi}_i
\end{aligned}$$

- Compare to $\dot{\Phi}^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t))$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{DP1}$ and $\delta \Phi^{DP1}$.

[Short Detour]:

On the Variation of $\mathbf{d}_{ij}$, that is, $\delta\mathbf{d}_{ij}$

- Recall that

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P = \mathbf{r}_j + \mathbf{s}_j^Q - \mathbf{r}_i - \mathbf{s}_i^P$$

- Assume that body i experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_i \\ \delta\bar{\pi}_i \end{bmatrix}$, and the body j experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_j \\ \delta\bar{\pi}_j \end{bmatrix}$. Therefore, $\mathbf{r}_i \longrightarrow \mathbf{r}_i + \delta\mathbf{r}_i$ and $\mathbf{A}_i \longrightarrow \mathbf{A}_i + \delta\mathbf{A}_i$. Likewise, $\mathbf{r}_j \longrightarrow \mathbf{r}_j + \delta\mathbf{r}_j$ and $\mathbf{A}_j \longrightarrow \mathbf{A}_j + \delta\mathbf{A}_j$.
- This variation in the attitude of bodies i and j will lead to a variation in the value of $\mathbf{d}_{ij}$. Specifically, $\mathbf{d}_{ij} \longrightarrow \mathbf{d}_{ij} + \delta\mathbf{d}_{ij}$. In other words,

$$\mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P \longrightarrow \mathbf{r}_j + \delta\mathbf{r}_j + (\mathbf{A}_j + \delta\mathbf{A}_j)\bar{\mathbf{s}}_j^Q - [\mathbf{r}_i + \delta\mathbf{r}_i + (\mathbf{A}_i + \delta\mathbf{A}_i)\bar{\mathbf{s}}_i^P]$$

- Therefore,

$$\begin{aligned}
\delta\mathbf{d}_{ij} &= (\mathbf{d}_{ij} + \delta\mathbf{d}_{ij}) - \mathbf{d}_{ij} \\
&= \delta\mathbf{r}_j + \delta\mathbf{A}_j\bar{\mathbf{s}}_j^Q - \delta\mathbf{r}_i - \delta\mathbf{A}_i\bar{\mathbf{s}}_i^P \\
&= \delta\mathbf{r}_j + \mathbf{A}_j\widetilde{\delta\bar{\pi}}_j\bar{\mathbf{s}}_j^Q - \delta\mathbf{r}_i - \mathbf{A}_i\widetilde{\delta\bar{\pi}}_i\bar{\mathbf{s}}_i^P \\
&= \delta\mathbf{r}_j - \mathbf{A}_j\tilde{\bar{\mathbf{s}}}_j^Q\delta\bar{\pi}_j - \delta\mathbf{r}_i + \mathbf{A}_i\tilde{\bar{\mathbf{s}}}_i^P\delta\bar{\pi}_i
\end{aligned}$$

- Compare to $\dot{\mathbf{d}}_{ij}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\mathbf{d}}_{ij}$ and $\delta\mathbf{d}_{ij}$.

Virtual Variation, Basic GCons: Φ^{DP2}

- Recall that

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = {\mathbf{a}_i}^T \mathbf{d}_{ij} - f(t) = 0$$

- Assume that body i experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_i \\ \delta\bar{\pi}_i \end{bmatrix}$, and the body j experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_j \\ \delta\bar{\pi}_j \end{bmatrix}$. Therefore, $\mathbf{r}_i \longrightarrow \mathbf{r}_i + \delta\mathbf{r}_i$ and $\mathbf{A}_i \longrightarrow \mathbf{A}_i + \delta\mathbf{A}_i$. Likewise, $\mathbf{r}_j \longrightarrow \mathbf{r}_j + \delta\mathbf{r}_j$ and $\mathbf{A}_j \longrightarrow \mathbf{A}_j + \delta\mathbf{A}_j$.

- This variation in the attitude of bodies i and j will lead to a variation in the value of Φ^{DP2}. Specifically, $\Phi^{DP2} \longrightarrow \Phi^{DP2} + \delta\Phi^{DP2}$.

- We have that (see Rule 5, Rule 2)

$$\begin{aligned}
\delta\Phi^{DP2} &= \mathbf{a}_i^T \delta\mathbf{d}_{ij} + \mathbf{d}_{ij}^T \delta\mathbf{a}_i \\
&= \mathbf{a}_i^T \left[\delta\mathbf{r}_j - \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta\bar{\pi}_j - \delta\mathbf{r}_i + \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta\bar{\pi}_i\right] - \mathbf{d}_{ij}^T \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta\bar{\pi}_i \\
&= \mathbf{a}_i^T \delta\mathbf{r}_j - \mathbf{a}_i^T \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta\bar{\pi}_j - \mathbf{a}_i^T \delta\mathbf{r}_i + \left[\left(\mathbf{a}_i^T \mathbf{A}_i - \mathbf{d}_{ij}^T \mathbf{A}_i\right) \tilde{\bar{\mathbf{s}}}_i^P\right] \delta\bar{\pi}_i
\end{aligned}$$

- Compare to $\dot{\Phi}^{DP2}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{DP2}$ and $\delta\Phi^{DP2}$.

Virtual Variation, Basic GCons: Φ^D

- Recall that the GCon-CD assumes the expression

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = 0$$

- Assume that body i experiences a virtual displacement characterized by $\begin{bmatrix} \delta \mathbf{r}_i \\ \delta \bar{\pi}_i \end{bmatrix}$, and the body j experiences a virtual displacement characterized by $\begin{bmatrix} \delta \mathbf{r}_j \\ \delta \bar{\pi}_j \end{bmatrix}$. Therefore, $\mathbf{r}_i \longrightarrow \mathbf{r}_i + \delta \mathbf{r}_i$ and $\mathbf{A}_i \longrightarrow \mathbf{A}_i + \delta \mathbf{A}_i$. Likewise, $\mathbf{r}_j \longrightarrow \mathbf{r}_j + \delta \mathbf{r}_j$ and $\mathbf{A}_j \longrightarrow \mathbf{A}_j + \delta \mathbf{A}_j$.
- This variation in the attitude of bodies i and j will lead to a variation in the value of Φ^D. Specifically, $\Phi^D \longrightarrow \Phi^D + \delta \Phi^D$.
- We have that (see Rule 2, Rule 5)

$$\begin{aligned}
\delta \Phi^D &= \mathbf{d}_{ij}^T (\delta \mathbf{d}_{ij}) + (\delta \mathbf{d}_{ij}^T) \mathbf{d}_{ij} \\
&= 2 \mathbf{d}_{ij}^T \delta \mathbf{d}_{ij} \\
&= 2 \mathbf{d}_{ij}^T \left[\delta \mathbf{r}_j - \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta \bar{\pi}_j - \delta \mathbf{r}_i + \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta \bar{\pi}_i \right] \\
&= 2 \mathbf{d}_{ij}^T \delta \mathbf{r}_j - 2 \mathbf{d}_{ij}^T \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta \bar{\pi}_j - 2 \mathbf{d}_{ij}^T \delta \mathbf{r}_i + 2 \mathbf{d}_{ij}^T \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta \bar{\pi}_i
\end{aligned}$$

- Compare to $\dot{\Phi}^D$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^D$ and $\delta \Phi^D$.

Virtual Variation, Basic GCons: $\mathbf{\Phi}^{CD}$

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = 0$$

- Assume that body i experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_i \\ \delta\bar{\pi}_i \end{bmatrix}$, and the body j experiences a virtual displacement characterized by $\begin{bmatrix} \delta\mathbf{r}_j \\ \delta\bar{\pi}_j \end{bmatrix}$. Therefore, $\mathbf{r}_i \longrightarrow \mathbf{r}_i + \delta\mathbf{r}_i$ and $\mathbf{A}_i \longrightarrow \mathbf{A}_i + \delta\mathbf{A}_i$. Likewise, $\mathbf{r}_j \longrightarrow \mathbf{r}_j + \delta\mathbf{r}_j$ and $\mathbf{A}_j \longrightarrow \mathbf{A}_j + \delta\mathbf{A}_j$.

- This variation in the attitude of bodies i and j will lead to a variation in the value of Φ^{CD}. Specifically, $\Phi^{CD} \longrightarrow \Phi^{CD} + \delta\Phi^{CD}$.

- We have that (see Rule 1, Rule 5)

$$\begin{aligned}
\delta\Phi^{CD} &= \mathbf{c}^T \delta\mathbf{d}_{ij} \\
&= \mathbf{c}^T \left[\delta\mathbf{r}_j - \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta\bar{\pi}_j - \delta\mathbf{r}_i + \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta\bar{\pi}_i \right] \\
&= \mathbf{c}^T \delta\mathbf{r}_j - \mathbf{c}^T \mathbf{A}_j \tilde{\bar{\mathbf{s}}}_j^Q \delta\bar{\pi}_j - \mathbf{c}^T \delta\mathbf{r}_i + \mathbf{c}^T \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta\bar{\pi}_i
\end{aligned}$$

- Compare to $\dot{\Phi}^{CD}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{CD}$ and $\delta\Phi^{CD}$.

Virtual Variation, Basic GCons: Putting It All Together

- Gather now all the virtual translations and rotations in two big vectors:

$$\delta \mathbf{r} = \begin{bmatrix} \delta \mathbf{r}_1 \\ \cdots \\ \delta \mathbf{r}_{nb} \end{bmatrix}_{3\,nb} \qquad \text{and} \qquad \delta \bar{\pi} = \begin{bmatrix} \delta \bar{\pi}_1 \\ \cdots \\ \delta \bar{\pi}_{nb} \end{bmatrix}_{3\,nb}$$

- We want to express the variation of a basic constraint Φ^α, where $\alpha \in \{DP1, DP2, D, CD\}$, in terms of $\delta \mathbf{r}$ and $\delta \bar{\pi}$.
- The key observation is that $\delta \Phi^\alpha$ assumes the form

Related to variations in position $\delta \mathbf{r}$

Related to variations in orientation $\delta \bar{\pi}$

$$\delta \Phi^\alpha = \left[\mathbf{0}_{1\times3} \ldots \mathbf{0}_{1\times3} \;\; \Phi^\alpha_{\mathbf{r}_i} \;\; \mathbf{0}_{1\times3} \ldots \mathbf{0}_{1\times3} \;\; \Phi^\alpha_{\mathbf{r}_j} \;\; \mathbf{0}_{1\times3} \ldots \ldots \mathbf{0}_{1\times3} \;\; \bar{\mathbf{\Pi}}_i \;\; \mathbf{0}_{1\times3} \ldots \mathbf{0}_{1\times3} \;\; \bar{\mathbf{\Pi}}_j \;\; \mathbf{0}_{1\times3} \ldots \right] \cdot \begin{bmatrix} \delta \mathbf{r}_1 \\ \cdots \\ \delta \mathbf{r}_{nb} \\ \\ \delta \bar{\pi}_1 \\ \cdots \\ \delta \bar{\pi}_{nb} \end{bmatrix}$$

Body 1, **Transl.** — Body i, **Transl.** — Body j, **Transl.** — Body i, **Rotation** — Body j, **Rotation**

Virtual Variation, Basic GCons: Putting It All Together

- Using the notation:

$$\delta \mathbf{r} = \begin{bmatrix} \delta \mathbf{r}_1 \\ \cdots \\ \delta \mathbf{r}_{nb} \end{bmatrix}_{3\,nb} \qquad \text{and} \qquad \delta \bar{\pi} = \begin{bmatrix} \delta \bar{\pi}_1 \\ \cdots \\ \delta \bar{\pi}_{nb} \end{bmatrix}_{3\,nb}$$

- We express the variation of a basic constraint Φ^α, where $\alpha \in \{DP1, DP2, D, CD\}$, in terms of $\delta \mathbf{r}$ and $\delta \bar{\pi}$ as

$$\delta \Phi^\alpha = \begin{bmatrix} \Phi_{\mathbf{r}} & \bar{\Pi}(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix} = \bar{\mathbf{R}} \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix}$$

- Equivalently,

$$\delta \Phi^\alpha = \begin{bmatrix} \Phi_{\mathbf{r}} & \Pi(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \pi \end{bmatrix} = \mathbf{R} \begin{bmatrix} \delta \mathbf{r} \\ \delta \pi \end{bmatrix}$$

- Recall that by definition (see previous lecture), $\bar{\Pi}(\Phi^\alpha)$ is the coefficient matrix that multiplies $\bar{\omega}$ in the time derivative $\dot{\Phi}^\alpha$.

End, Variations in a Function due to Virtual Displacements $\delta\mathbf{r}$ and $\delta\bar{\pi}$

Begin, Variations in a Function due to Virtual Displacements $\delta\mathbf{r}$ and $\delta\mathbf{p}$

The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta \mathbf{p}$ Virtual Rotation

- Framework: assume you have a vector quantity that depends on $\mathbf{p}$. Assume that the value of $\mathbf{p}$ changes to $\mathbf{p} + \delta\mathbf{p}$. What is the variation in the quantity that depends on $\mathbf{p}$ due to the said change?

- Specifically, assume the vector quantity of interest is $\mathbf{u}$, and $\mathbf{u}$ depends on $\mathbf{p}$ and possibly time t:

$$\mathbf{u} = \mathbf{u}(\mathbf{p}, t)$$

- I am interested at a fixed time t in the $\delta\mathbf{u}$ below given $\mathbf{p}$, $\delta\mathbf{p}$, and the expression of $\mathbf{u}(\mathbf{p})$:

$$\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p}, t) \qquad \mathbf{p} + \delta\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p} + \delta\mathbf{p}, t) = \mathbf{u}(\mathbf{p}, t) + \delta\mathbf{u}$$

$$\delta\mathbf{u} = ?$$

The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta\mathbf{p}$ Virtual Rotation

[Cntd.]:

- The answer to question of interest, $\delta\mathbf{u}(\mathbf{p}) =?$, is obtained using a Taylor series expansion:

$$\begin{aligned}\mathbf{u}(\mathbf{p}+\delta\mathbf{p},t) &= \mathbf{u}(\mathbf{p},t) + \mathbf{u}_{\mathbf{p}}\delta\mathbf{p} + \ldots \\ &\approx \mathbf{u}(\mathbf{p},t) + \mathbf{u}_{\mathbf{p}}\delta\mathbf{p}\end{aligned}$$

- Then

$$\delta\mathbf{u}(\mathbf{p}) = \mathbf{u}(\mathbf{p}+\delta\mathbf{p},t) - \mathbf{u}(\mathbf{p},t) = \mathbf{u}_{\mathbf{p}}\delta\mathbf{p}$$

- In the argument above, we rely on the fact that the virtual rotations, that is, the perturbations $\delta\mathbf{p}$, are small and therefore higher order terms that contain entries of $\delta\mathbf{p}$, that is, δe_0, δe_1, δe_2, or δe_3, can be safely approximated to be zero.

- Important observation: note that the time does note play a role in figuring out what the variation in $\mathbf{u}$ is. In other words, looking into the variation of $\mathbf{u}$ is an exercise that is carried out at a certain time t, and time is held fixed.

- Note that the same argument applies if u is a scalar function that depends on $\mathbf{p}$. In that case,

$$\delta u(\mathbf{p},t) = u_{\mathbf{p}}\delta\mathbf{p}$$

Exercise

- Calculate the variation of the function $\mathbf{u}(\mathbf{p}) = \mathbf{A}(\mathbf{p})\bar{\mathbf{s}}$ due to a variation $\delta\mathbf{p}$ in the Euler Parameters. The vector $\bar{\mathbf{s}}$ does not depend on $\mathbf{p}$.

Exercise

- Calculate the variation of the function $u(\mathbf{p}) = \mathbf{p}^T\mathbf{p} - 1$ due to a variation $\delta\mathbf{p}$ in the Euler Parameters

Quick Question

- Note that when interested in variations as induced by virtual rotations of the $\delta\mathbf{p}$ flavor (as opposed to the $\delta\bar{\pi}$ flavor), it is very straightforward to produce the quantity of interest:

$$\delta\mathbf{u}(\mathbf{p}) = \mathbf{u}_{\mathbf{p}}\delta\mathbf{p}$$

- Why did not we take the same approach for the $\delta\bar{\pi}$?
  - We couldn't do this direct approach for the the same reason we couldn't find a set of three variables whose time derivative is the angular velocity $\bar{\omega}$
  - Specifically, there is no concept of partial derivative $\mathbf{u}_{\bar{\pi}}$ to work with, and therefore we have to resort to the process that in the end expresses the variation $\delta\mathbf{u}$ or the time derivative $\dot{\mathbf{u}}$ using $\bar{\mathbf{\Pi}}(\mathbf{u})$ and $\delta\bar{\pi}$, or $\bar{\mathbf{\Pi}}(\mathbf{u})$ and $\bar{\omega}$, respectively

Virtual Variation, Basic GCons: Φ^{DP1} [The $\delta\mathbf{p}$ Flavor]

- Recall that

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = \mathbf{a}_i{}^T \mathbf{a}_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_i} = \mathbf{0}_{1\times 3} \qquad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} = \mathbf{a}_j^T \mathbf{B}\left(\mathbf{p}_i, \bar{\mathbf{a}}_i\right)$$

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_j} = \mathbf{0}_{1\times 3} \qquad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} = \mathbf{a}_i^T \mathbf{B}\left(\mathbf{p}_j, \bar{\mathbf{a}}_j\right)$$

- Putting it all together, $\delta\Phi^{DP1} = \Phi^{DP1}_{\mathbf{q}} \delta\mathbf{q}$, where,

$$\Phi^{DP1}_{\mathbf{q}} = \left[\mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \ldots \mathbf{0}_{1\times 4} \ \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} \ \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4} \ \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} \ \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4}\right]$$

Partials with respect to $\mathbf{r}$ ← | → Partials with respect to $\mathbf{p}$

$$= \left[\mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \ldots \mathbf{0}_{1\times 4} \ \mathbf{a}_j^T \mathbf{B}\left(\mathbf{p}_i, \bar{\mathbf{a}}_i\right) \ \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4} \ \mathbf{a}_i^T \mathbf{B}\left(\mathbf{p}_j, \bar{\mathbf{a}}_j\right) \ \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4}\right]$$

Body 1, **r** | Body i, **r** | Body j, **r** | Body i-1, **p** | Body i, **p** | Body i+1, **p** | Body j-1, **p** | Body j, **p** | Body j+1, **p** | Body nb, **p**

[Short Detour]:

Computing $\delta \mathbf{d}_{ij}$

- Recall that

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P = \mathbf{r}_j + \mathbf{s}_j^Q - \mathbf{r}_i - \mathbf{s}_i^P$$

- Recall also that

$$\begin{aligned} [\mathbf{d}_{ij}]_{\mathbf{q}_i,\mathbf{q}_j} &= [-\mathbf{I}_3 \quad -(\mathbf{s}_i^P)_{\mathbf{p}_i} \quad \mathbf{I}_3 \quad (\mathbf{s}_j^Q)_{\mathbf{p}_j}] \\ &= [-\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)] \end{aligned}$$

- It follows that

$$\delta \mathbf{d}_{ij} = [-\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)] \cdot \begin{bmatrix} \delta \mathbf{r}_i \\ \delta \mathbf{p}_i \\ \delta \mathbf{r}_j \\ \delta \mathbf{p}_j \end{bmatrix} = [\mathbf{d}_{ij}]_{\mathbf{q}} \cdot \delta \mathbf{q}$$

Virtual Variation, Basic GCons: Φ^{DP2} [The $\delta\mathbf{p}$ Flavor]

- Recall that

$$\Phi^{DP2}(i,\bar{\mathbf{a}}_i,\bar{\mathbf{s}}_i^P,j,\bar{\mathbf{s}}_j^Q,f(t)) = \bar{\mathbf{a}}_i^T\mathbf{A}_i^T\mathbf{d}_{ij} - f(t) = \mathbf{a}_i{}^T\mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{DP2}_{\mathbf{q}_i,\mathbf{q}_j}(\mathbf{a}_i,\mathbf{d}_{ij}) = \begin{bmatrix} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P) & \mathbf{a}_i^T & \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\bar{\mathbf{s}}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta\Phi^{DP2} = \begin{bmatrix} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{s}}_i^P) & \mathbf{a}_i^T & \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\bar{\mathbf{s}}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^{DP2}_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Basic GCons: Φ^D [The $\delta \mathbf{p}$ Flavor]

- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) \quad = \quad \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = 0$$

- It also that

$$\Phi^D_{\mathbf{q}_i, \mathbf{q}_j} = \left[\begin{array}{cccc} -2\mathbf{d}_{ij}^T & -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & 2\mathbf{d}_{ij}^T & 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{array} \right]$$

- It follows that

$$\delta\Phi^D = \left[\begin{array}{cccc} -2\mathbf{d}_{ij}^T & -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & 2\mathbf{d}_{ij}^T & 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{array} \right] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^D_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Basic GCons: Φ^{CD} [The $\delta\mathbf{p}$ Flavor]

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{CD}_{\mathbf{q}_i,\mathbf{q}_j} = \begin{bmatrix} -\mathbf{c}^T & -\mathbf{c}^T\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{c}^T & \mathbf{c}^T\mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta\Phi^{CD} = \begin{bmatrix} -\mathbf{c}^T & -\mathbf{c}^T\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{c}^T & \mathbf{c}^T\mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^{CD}_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Euler Parameter Normalization Constraint: $\Phi^{\mathbf{p}}$

- Recall that the Euler Parameter normalization constraint assumes the expression

$$\Phi_i^{\mathbf{p}} = \mathbf{p}_i^T \mathbf{p}_i - 1 = 0$$

- Recall also that

$$(\Phi_i^{\mathbf{p}})_{\mathbf{q}_i} = \begin{bmatrix} \mathbf{0}_{1\times 3} & 2\mathbf{p}_i^T \end{bmatrix}$$

- It follows that

$$\delta\Phi_i^{\mathbf{p}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & 2\mathbf{p}_i^T \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \end{bmatrix} = (\Phi_i^{\mathbf{p}})_{\mathbf{q}} \cdot \delta\mathbf{q}$$