ME451
Kinematics and Dynamics of Machine Systems

Review of Calculus

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- Quote of the day: “Inspiration exists, but it has to find you working.” – Pablo Picasso

Before we get started…

- Last time
  - Time derivatives
  - Partial derivatives (hard, messy, and widely used in ME451)
- Today
  - Wrap up partial derivatives (sensitivity computation)
    - Focus is on chain rule
- HW: assigned last time
  - Due on Th, 9/18, at 9:30 am
  - Problems assigned in class and 2.4.4, 2.5.2, 2.5.7 out of Haug's book
  - Post questions on the forum

Complex Case 1

Scalar Function of Vector Variable

- f is a scalar function of "n" variables: $q_1, \ldots, q_n$

$$f : \mathbb{R}^n \to \mathbb{R}$$

- However, each of these variables q_i in turn depends on a set of "k" other variables $x_1, \ldots, x_k$.

$$\mathbf{q} : \mathbb{R}^k \to \mathbb{R}^n, \quad \mathbf{q} \triangleq \mathbf{q}(\mathbf{x}) = \begin{bmatrix} q_1(x_1, \ldots, x_k) \\ \vdots \\ q_n(x_1, \ldots, x_k) \end{bmatrix}$$

- The composition of f and $\mathbf{q}$ leads to a new function:

$$\phi : \mathbb{R}^k \to \mathbb{R}, \quad \phi(\mathbf{x}) = (f \circ \mathbf{q})(\mathbf{x}) \triangleq f(\mathbf{q}(\mathbf{x}))$$

Chain Rule
Scalar Function of Vector Variable

- Question: how do you compute ϕ_x ?
  - Using our notation:

$$\phi(\mathbf{x}) = (f \circ \mathbf{q})(\mathbf{x}) = f(\mathbf{q}(\mathbf{x})) \quad \Rightarrow \quad \phi_{\mathbf{x}} = ?$$

  - Chain Rule:

$$\phi_{\mathbf{x}} \equiv \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} \equiv f_{\mathbf{q}} \cdot \mathbf{q}_{\mathbf{x}}$$

$$\underbrace{\phi_{\mathbf{x}}}_{\mathbb{R}^{1\times k}} = \underbrace{\underbrace{f_{\mathbf{q}}}_{\mathbb{R}^{1\times n}} \cdot \underbrace{\mathbf{q}_{\mathbf{x}}}_{\mathbb{R}^{n\times k}}}_{\mathbb{R}^{1\times k}}$$

Assignment
[due 09/18]

Assume that $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and consider a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\phi(\mathbf{y}) = 3y_1^2 + \sin y_2$. Assume further that $\mathbf{y}$ depends on a variable $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as follows:

$$\mathbf{y} \triangleq \mathbf{y}(\mathbf{x}) \equiv \begin{bmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log_{10} x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}$$

It follows that ϕ depends on $\mathbf{x}$, implicitly through $\mathbf{y}$. Apply the *chain rule of differentiation* to find the derivative of ϕ with respect to $\mathbf{x}$, that is:

$$\phi_{\mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \end{bmatrix} = ?$$

What is the dimension of the Jacobian (sensitivy matrix) $\phi_{\mathbf{x}}$?

Complex Case 2

Vector Function of Vector Variable

- $\mathbf{F}$ is a vector function of several variables: $q_1, \ldots, q_n$

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$$

- However, each of these variables q_i depends in turn on a set of k other variables $x_1, \ldots, x_k$.

$$\mathbf{q} : \mathbb{R}^k \to \mathbb{R}^n\,, \quad \mathbf{q} \triangleq \mathbf{q}(\mathbf{x}) = \begin{bmatrix} q_1(x_1, \ldots, x_k) \\ \vdots \\ q_n(x_1, \ldots, x_k) \end{bmatrix}$$

- The composition of $\mathbf{F}$ and $\mathbf{q}$ leads to a new function:

$$\mathbf{\Phi} : \mathbb{R}^k \to \mathbb{R}^m\,, \quad \mathbf{\Phi}(\mathbf{x}) = (\mathbf{F} \circ \mathbf{q})(\mathbf{x}) \triangleq \mathbf{F}(\mathbf{q}(\mathbf{x}))$$

Chain Rule
Vector Function of Vector Variable

- Question: how do you compute $\Phi_{\mathbf{x}}$?
  - Using our notation:

$$\mathbf{\Phi}(\mathbf{x}) = (\mathbf{F} \circ \mathbf{q})(\mathbf{x}) = \mathbf{F}(\mathbf{q}(\mathbf{x})) \quad \Rightarrow \quad \mathbf{\Phi}_{\mathbf{x}} = ?$$

  - Chain Rule:

$$\mathbf{\Phi}_{\mathbf{x}} \equiv \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} \equiv \mathbf{F}_{\mathbf{q}} \cdot \mathbf{q}_{\mathbf{x}}$$

$$\underbrace{\mathbf{\Phi}_{\mathbf{x}}}_{\mathbb{R}^{m \times k}} = \underbrace{\underbrace{\mathbf{F}_{\mathbf{q}}}_{\mathbb{R}^{m \times n}} \cdot \underbrace{\mathbf{q}_{\mathbf{x}}}_{\mathbb{R}^{n \times k}}}_{\mathbb{R}^{m \times k}}$$

[handout]

Example

Assume that $\mathbf{B} \in \mathbb{R}^{m \times n}$ is a matrix that doesn't depend on $\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$. Show that:

$$\frac{\partial (\mathbf{Bx})}{\partial \mathbf{x}} = \mathbf{B}$$

Assignment

[due 09/18]

Assume that $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and consider a function $\mathbf{\Phi} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $\mathbf{\Phi}(\mathbf{y}) = \begin{bmatrix} 2y_1 + y_2^2 \\ y_1 y_2 \end{bmatrix}$. Assume further that $\mathbf{y}$ depends on a variable $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as follows:

$$\mathbf{y} \triangleq \mathbf{y}(\mathbf{x}) \equiv \begin{bmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log_{10} x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}$$

It follows that $\mathbf{\Phi}$ depends on $\mathbf{x}$, implicitly through $\mathbf{y}$. Apply the *chain rule of differentiation* to find the derivative of $\mathbf{\Phi}$ with respect to $\mathbf{x}$, that is:

$$\mathbf{\Phi}_{\mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial \mathbf{\Phi}}{\partial x_1} & \frac{\partial \mathbf{\Phi}}{\partial x_2} & \frac{\partial \mathbf{\Phi}}{\partial x_3} \end{bmatrix} = ?$$

What is the dimension of the result $\mathbf{\Phi}_{\mathbf{x}}$?

Complex Case 3

Vector Function of Vector Variables

- $\mathbf{F}$ is a vector function of 2 vector variables $\mathbf{q}$ and $\mathbf{p}$:

$$\mathbf{F} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m}$$

- Both $\mathbf{q}$ and $\mathbf{p}$ in turn depend on a set of k other variables $\mathbf{x} = [x_1, \cdots, x_k]^T$:

$$\mathbf{q} : \mathbb{R}^{k} \to \mathbb{R}^{n_1} \quad \mathbf{q} \triangleq \mathbf{q}(x_1, \ldots, x_k)$$

$$\mathbf{p} : \mathbb{R}^{k} \to \mathbb{R}^{n_2} \quad \mathbf{p} \triangleq \mathbf{p}(x_1, \ldots, x_k)$$

- A new function $\Phi(\mathbf{x})$ is defined as:

$$\mathbf{\Phi} : \mathbb{R}^{k} \to \mathbb{R}^{m}, \quad \mathbf{\Phi}(\mathbf{x}) \triangleq \mathbf{F}\left(\mathbf{q}(\mathbf{x}), \mathbf{p}(\mathbf{x})\right)$$

- Example: a force (which is a vector quantity), depends on the generalized positions and velocities

Chain Rule

Vector Function of Vector Variables

- Question: how do you compute $\mathbf{\Phi_x}$?

  - Using our notation:

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{F}\left(\mathbf{q}(\mathbf{x}), \mathbf{p}(\mathbf{x})\right) \quad \Rightarrow \quad \mathbf{\Phi_x} = ?$$

  - Chain Rule:

$$\mathbf{\Phi_x} \equiv \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \mathbf{x}} \equiv \mathbf{F_q} \cdot \mathbf{q_x} + \mathbf{F_p} \cdot \mathbf{p_x}$$

$$\underbrace{\mathbf{\Phi_x}}_{\mathbb{R}^{m \times k}} = \underbrace{\underbrace{\mathbf{F_q}}_{\mathbb{R}^{m \times n_1}} \cdot \underbrace{\mathbf{q_x}}_{\mathbb{R}^{n_1 \times k}} + \underbrace{\mathbf{F_p}}_{\mathbb{R}^{m \times n_2}} \cdot \underbrace{\mathbf{p_x}}_{\mathbb{R}^{n_2 \times k}}}_{\mathbb{R}^{m \times k}}$$

Example

Assume that $\mathbf{q} = \mathbf{q}(\mathbf{x}) \in \mathbb{R}^n$ and $\mathbf{p} = \mathbf{p}(\mathbf{x}) \in \mathbb{R}^n$. Show that:

$$\frac{\partial\left(\mathbf{q}^T\mathbf{p}\right)}{\partial\mathbf{x}} = \mathbf{q}^T\mathbf{p}_{\mathbf{x}} + \mathbf{p}^T\mathbf{q}_{\mathbf{x}}$$

Complex Case 4

Time Derivatives

- In the previous slides we talked about functions f depending on q, where q in turn depends on another variable x

- The most common scenario in ME451 is when the variable x is actually time, t
  - You have a function that depends on the generalized coordinates **q**, and in turn the generalized coordinates are functions of time (they change in time, since we are talking about kinematics/dynamics here…)
  - Case 1: scalar function that depends on an array of *m* time-dependent generalized coordinates:

$$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi \triangleq \phi(\mathbf{q}(t))$$

  - Case 2: vector function (of dimension n) that depends on an array of *m* time-dependent generalized coordinates:

$$\mathbf{\Phi} : \mathbb{R} \to \mathbb{R}^n, \quad \mathbf{\Phi} \triangleq \mathbf{\Phi}(\mathbf{q}(t))$$

Chain Rule
Time Derivatives

- Question: what are the time derivatives of Φ and $\mathbf{\Phi}$?

- Applying the chain rule of differentiation, the results in both cases can be written formally in the exact same way, except the dimension of the result will be different

  - Case 1: scalar function

$$\dot{\phi} \triangleq \frac{d\phi}{dt} = \frac{d\phi\left(\mathbf{q}(t)\right)}{dt} = \frac{\partial \phi}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \phi_{\mathbf{q}}\dot{\mathbf{q}}\,, \quad \dot{\phi} \in \mathbb{R}$$

  - Case 2: vector function

$$\dot{\mathbf{\Phi}} \triangleq \frac{d\mathbf{\Phi}}{dt} = \frac{d\mathbf{\Phi}\left(\mathbf{q}(\mathbf{t})\right)}{dt} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}\,, \quad \dot{\mathbf{\Phi}} \in \mathbb{R}^n$$

Example
Time Derivatives

Assume $\mathbf{q} \in \mathbb{R}^3$ is an array of generalized coordinates:

$$\mathbf{q} = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

- Find the time derivative of the scalar function $\phi(\mathbf{q}(t)) = 3x(t) + 2L\sin\theta(t)$
- Find the time derivative of the vector function

$$\mathbf{\Phi} = \begin{bmatrix} 3x(t) + 2L\sin\theta(t) \\ y(t) - 2L\cos\theta(t) \end{bmatrix}$$

Summary of Useful Formulas

$$\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{g}^T\mathbf{h}\right) = \mathbf{g}^T\mathbf{h}_{\mathbf{q}} + \mathbf{h}^T\mathbf{g}_{\mathbf{q}}$$

$$\frac{\partial}{\partial \mathbf{q}}\left(\mathbf{B}\mathbf{q}\right) = \mathbf{B}$$

$$\frac{\partial}{\partial \mathbf{p}}\left(\mathbf{p}^T\mathbf{B}\mathbf{q}\right) = \mathbf{q}^T\mathbf{B}^T$$

$$\frac{d}{dt}\left(\mathbf{p}^T\mathbf{B}\mathbf{q}\right) = \mathbf{q}^T\mathbf{B}^T\dot{\mathbf{p}} + \mathbf{p}^T\mathbf{B}\dot{\mathbf{q}}$$

Assumptions:

- $\mathbf{g} = \mathbf{g}(\mathbf{q})\,, \quad \mathbf{h} = \mathbf{h}(\mathbf{q})$
- $\mathbf{B}$ is a constant matrix
- $\mathbf{q}$ does not depend on $\mathbf{p}$
- Vector and matrix dimensions are such that all operations are possible.