ME751
Advanced Computational Multibody Dynamics

September 14, 2016

Quote of the Day

"My own business always bores me to death; I prefer other people's."

-- Oscar Wilde

Looking Ahead, Friday

- Need to wrap up proposal
- Things look kind of ok. For now, Friday lecture is on
- Friday: last week's assignment is due at 9:30 am
- Friday: time to assign next homework
  - Next assignment has two components
    - Component 1: read the proposal and provide feedback by Saturday evening
      - PDF of proposal to be emailed to you Friday by mid-day
        - Make comments and highlight things that don't make sense (more instructions in follow up email)
    - Component 2: the usual pen and paper stuff, to be posted online

Before we get started…

- Last Time:
  - Finished Calculus review: Implicit Function Theorem
    - Getting a function out of a relation (equation)
    - We'll probably get to use today
  - Introduced the concept of Geometric Vector
    - Definition and five basic operations you can do with Geometric Vectors
    - Combined them into simple operations
    - Clunky to manipulate
  - Introduced reference frames to simplify handling of Geometric Vectors
  - Introduced Algebraic Vectors to manipulate Geometric Vectors represented in Reference Frames
- Today:
  - Changing the RF for representing a Geometric Vector
  - Angular velocity of a rigid body
  - Degree of freedom count, relative to 3D rotation of a rigid body

[Last topic covered on Mo]

Reference Frames: Nomenclature & Notation

- G-RF: Global Reference Frame (the "world" reference frame)
  - This RF is unique
  - This RF is fixed; that is, its location & orientation don't change in time

- L-RF: Local Reference Frame
  - It typically represents a RF that is *rigidly* attached to a moving rigid body
  - Notation issue
    - An algebraic vector represented in an L-RF has either a prime , as in $\mathbf{s}'$, or it has an overbar, like in $\bar{\mathbf{s}}$
    - The book *always* uses a prime, I will almost always use on overbar

Differentiation of Vectors
(pp.315, Haug book)

- Assumption: for the sake of this discussion on vector differentiation, the geometric vectors are assumed to be represented in a G-RF. Therefore:

$$\dot{\vec{\mathbf{i}}} = \dot{\vec{\mathbf{j}}} = \dot{\vec{\mathbf{k}}} = \mathbf{0}$$

How do you know that the derivative of the sum is the sum of the derivatives?

- Due to the assumption above, one has:

$$\begin{aligned} \dot{\vec{\mathbf{a}}} &\equiv \tfrac{d}{dt}\vec{\mathbf{a}}(t) = \tfrac{d}{dt}[a_x(t)\vec{\mathbf{i}} + a_y(t)\vec{\mathbf{j}} + a_z(t)\vec{\mathbf{k}}] \\ &= \left[\tfrac{d}{dt}a_x(t)\right]\vec{\mathbf{i}} + \left[\tfrac{d}{dt}a_y(t)\right]\vec{\mathbf{j}} + \left[\tfrac{d}{dt}a_z(t)\right]\vec{\mathbf{k}} \\ &= \dot{a}_x(t)\vec{\mathbf{i}} + \dot{a}_y(t)\vec{\mathbf{j}} + \dot{a}_z(t)\vec{\mathbf{k}} \end{aligned}$$

- Note: the algebraic representation of the time derivative of $\vec{\mathbf{a}}$ is the time derivative of the algebraic representation of $\vec{\mathbf{a}}$:

$$\dot{\mathbf{a}} \equiv \frac{d}{dt}\mathbf{a}(t) = \left[\frac{d}{dt}a_x(t), \frac{d}{dt}a_y(t), \frac{d}{dt}a_z(t)\right]^T = [\dot{a}_x(t), \dot{a}_y(t), \dot{a}_z(t)]^T$$

Differentiation of Vectors
(pp.315)

- Similarly, by taking one more time derivative, it is easy to see that the second time derivative of a geometric vector has the following algebraic vector representation

$$\ddot{\mathbf{a}} \equiv \frac{d}{dt}(\dot{\mathbf{a}}(t)) = \left[\frac{d^2}{dt^2}a_x(t), \frac{d^2}{dt^2}a_y(t), \frac{d^2}{dt^2}a_z(t)\right]^T = [\ddot{a}_x(t), \ddot{a}_y(t), \ddot{a}_z(t)]^T$$

- Likewise, consider the only operation introduced so far involving two geometric vectors that leads to a real number: the inner product

$$\frac{d}{dt}[\vec{\mathbf{a}}(t) \cdot \vec{\mathbf{b}}(t)] = \frac{d}{dt}[a_x(t)b_x(t) + a_y(t)b_y(t) + a_z(t)b_z(t)] = \frac{d}{dt}[\mathbf{a}^T(t) \cdot \mathbf{b}(t)]$$

Differentiation of Vectors
(pp.315)

- The concluding remark is that as long as we are working in a G-RF, the time derivative of a geometric vector has an algebraic representation that comes in line with our expectations. Specifically:
  - Simply take the time derivative of the components of the algebraic representation

- The time derivative of a basic geometric vector operation has a counterpart in taking time derivative of the appropriate operation involving associated algebraic vectors
  - We just saw this for the inner product
  - Works the same for the time derivative of the sum of two geometric vectors, scaling a geometric vector by a number, etc.

- Next logical step: discuss how to take time derivative of operations that involve algebraic vectors

[Already covered in Lecture 1 (for most part)]

Differentiation of Algebraic Vectors: Rules

- Assume that $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{b} \in \mathbb{R}^3$ are all functions of time. Then the following hold (HOMEWORK):

$$\frac{d}{dt}(\mathbf{a}(t) + \mathbf{b}(t)) = \dot{\mathbf{a}} + \dot{\mathbf{b}}$$

$$\frac{d}{dt}(\alpha \mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}}$$

$$\frac{d}{dt}(\mathbf{a}^T \mathbf{b}) = \dot{\mathbf{a}}^T \mathbf{b} + \mathbf{a}^T \dot{\mathbf{b}}$$

$$\frac{d}{dt}(\tilde{\mathbf{a}}\mathbf{b}) = \dot{\tilde{\mathbf{a}}}\mathbf{b} + \tilde{\mathbf{a}}\dot{\mathbf{b}}$$

Take a minute to reflect on this, specifically, on what its geometric counterpart is

$$\dot{\tilde{\mathbf{a}}} = \tilde{\dot{\mathbf{a}}}$$

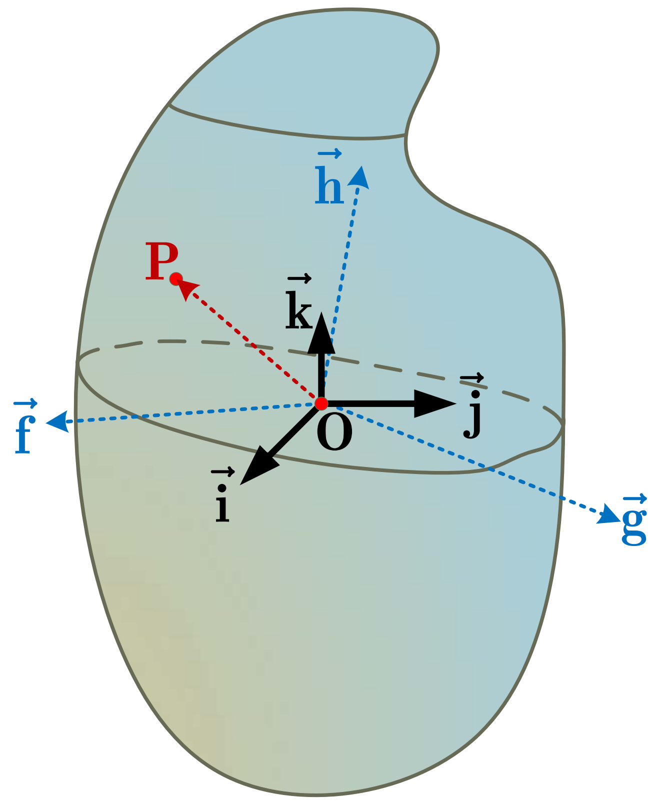
Algebraic Vectors and Reference Frames

- Recall that an algebraic vector was introduced as a representation of a geometric vector in a particular reference frame (RF)

- Question: What if I now want to represent the same geometric vector in a different RF_{new} that is rotated relative to the original RF?
  - We'll run into this transformation over and over again

Problem Setup

- A rigid body is provided and fixed at point O
- G-RF is attached at O
- P is some point of the body
- Geometric vector in red assumes different algebraic representations in the blue L-RF and in the black G-RF
- Question of Interest:
    - What's the relationship between these two representations?



Algebraic Vectors and Reference Frames

- Let $\vec{\mathbf{s}} = \overrightarrow{\mathbf{OP}}$ be a geometric vector (see figure on previous slide)
- In the RF defined by $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$, the geometric vector $\vec{\mathbf{s}}$ is represented as

$$\vec{\mathbf{s}} = s_x \vec{\mathbf{i}} + s_y \vec{\mathbf{j}} + s_z \vec{\mathbf{k}}$$

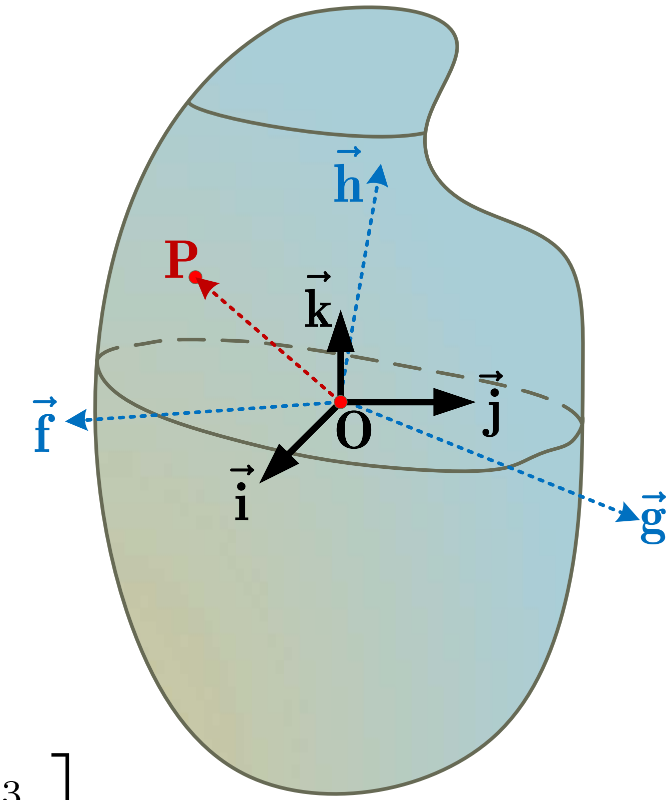
- If I consider a different RF defined by $(\vec{\mathbf{f}}, \vec{\mathbf{g}}, \vec{\mathbf{h}})$, the geometric vector $\vec{\mathbf{s}}$ is represented as

$$\vec{\mathbf{s}} = s_{\bar{x}} \vec{\mathbf{f}} + s_{\bar{y}} \vec{\mathbf{g}} + s_{\bar{z}} \vec{\mathbf{h}}$$

- The QUESTION: how are (s_x, s_y, s_z) and $(s_{\bar{x}}, s_{\bar{y}}, s_{\bar{z}})$ related?
- NOTE: The vectors $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$ define the global ('world') RF, and therefore

$$\dot{\vec{\mathbf{i}}} = \dot{\vec{\mathbf{j}}} = \dot{\vec{\mathbf{k}}} = \mathbf{0}$$

Relationship Between L-RF Vectors and G-RF Vectors



$$\overrightarrow{\mathbf{f}} = a_{11}\overrightarrow{\mathbf{i}} + a_{21}\overrightarrow{\mathbf{j}} + a_{31}\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{g}} = a_{12}\overrightarrow{\mathbf{i}} + a_{22}\overrightarrow{\mathbf{j}} + a_{32}\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{h}} = a_{13}\overrightarrow{\mathbf{i}} + a_{23}\overrightarrow{\mathbf{j}} + a_{33}\overrightarrow{\mathbf{k}}$$

$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$a_{11} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{f}})$$
$$a_{12} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{g}})$$
$$a_{13} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{h}})$$

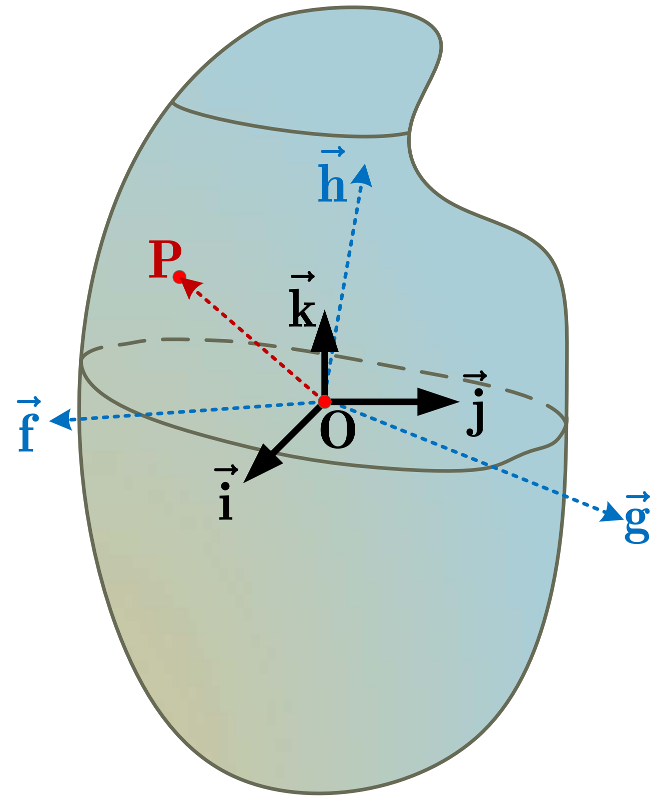
$$a_{21} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{f}})$$
$$a_{22} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{g}})$$
$$a_{23} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{h}})$$

$$a_{31} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{f}})$$
$$a_{32} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{g}})$$
$$a_{33} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{h}})$$

There is a good reason the values a_{ij} above are called "direction cosines".

Relationship Between L-RF and G-RF Representations

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} s_{\bar{x}} \\ s_{\bar{y}} \\ s_{\bar{z}} \end{bmatrix}$$



$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

This is important
(see pp. 321)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\mathbf{f} \quad \mathbf{g} \quad \mathbf{h}]$$

$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

Algebraic Vectors and Reference Frames

- Representing the same geometric vector in a different RF leads to the important concept of Rotation Matrix **A**:
  - Getting the new coordinates, that is, representation of the **same** geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix **A**:

$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

- NOTE 1: what is changed is the RF used to represent the vector
  - We are talking about the *same* geometric vector, in a different RF

- NOTE 2: rotation matrix **A** sometimes called "orientation matrix"

On the Orthonormality of A

- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are mutually orthogonal
- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are are unit vectors

- Therefore, the following hold:

$$\mathbf{f}^T\mathbf{f} = \mathbf{g}^T\mathbf{g} = \mathbf{h}^T\mathbf{h} = 1$$

$$\mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{h} = \mathbf{h}^T\mathbf{f} = 0$$

- Consequently, the rotation matrix **A** is orthonormal:

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}_{3\times 3}$$

The Transformation Matrix A: Further Comments

- The nine entries of matrix **A** are called direction cosines
  - The first column are the direction cosines of **f**, the second contains the direction cosines of **g**, etc.

- Found the representation in G-RF given the one in an L-RF
  - Found L-RF $ G-RF

- Since **A** is orthonormal, easy to find the transformation in the opposite direction: G-RF $ L-RF

$$\bar{\mathbf{s}} = \mathbf{A}^T \mathbf{s}$$

Summarizing the Key Points

- Linking two algebraic vector representations of the same geometric vector

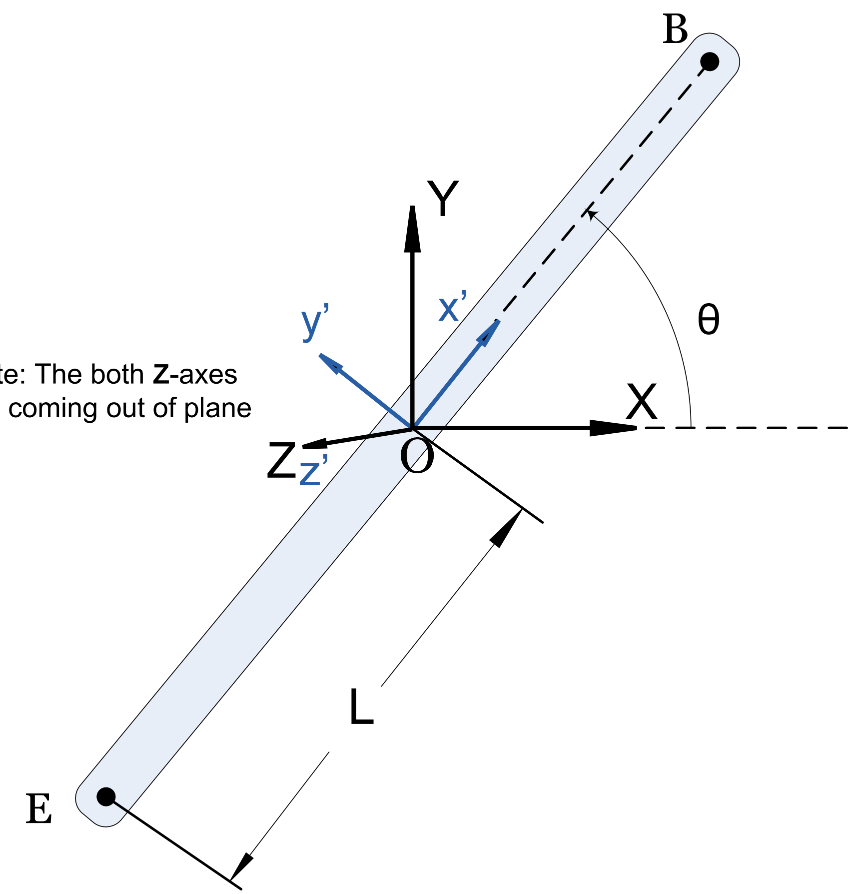
$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

- Sometimes called a change of base or reference frame
- **A**’s columns made up of the representation of **f**, **g**, and **h** in the new RF
  - The algebraic vectors **f**, **g**, and **h** define the “old”, “local”, “initial” RF, that is, that reference frame in which where $\bar{\mathbf{s}}$ is expressed

[AO]

Example: Assembling Matrix A

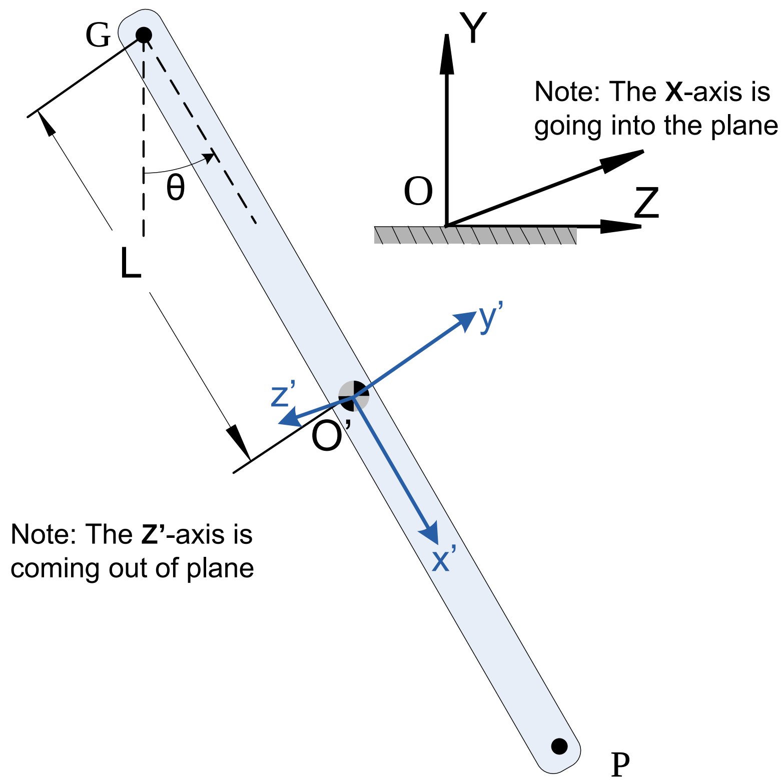
Note: The both **Z**-axes are coming out of plane



- Express the geometric vector $\overrightarrow{OB}$ in the local reference frame OX'Y'.
- Express the same geometric vector in the global reference frame OXY
- Do the same for the geometric vector $\overrightarrow{OE}$

[HOMEWORK]

Assembling A



- Express the geometric vector $\overrightarrow{O'P}$ in the local reference frame O'X'Y'Z'.
- Express the same geometric vector in the global reference frame OXYZ
- Do the same for the geometric vector $\overrightarrow{O'G}$

- Note that the plane (**O'X'Y'**) is parallel to the (**OYZ**) plane
- Note that **O** and **O'** should have been coincident; avoided to do that to prevent clutter of the figure (you should work under this assumption though)

[two slide topic] RF Change: The Outer Product and Cross Product Matrix

- Problem Setup:
  - We saw how to switch between L-RF and G-RF when it comes to the algebraic representation of a geometric vector
    - Boils down to multiplication by the rotation matrix **A**

  - Recall that associated with each algebraic vector there is a cross product matrix

- **Question**: How do you have to change the *cross product matrix* when you go from a L-RF to the G-RF ?

$$\tilde{\mathbf{s}} \overset{?}{\rightleftarrows} \bar{\tilde{\mathbf{s}}}$$

[two slide topic] RF Change: The Outer Product and Cross Product Matrix

- The geometric vector representation: I have two geometric vectors, $\vec{\mathbf{s}}, \vec{\mathbf{v}}$ and care about their outer product,

$$\vec{\mathbf{c}} = \vec{\mathbf{s}} \times \vec{\mathbf{v}}$$

Representation of $\vec{\mathbf{c}}$ in a G-RF

$$\mathbf{c} = \tilde{\mathbf{s}} \cdot \mathbf{v}$$

Representation of $\vec{\mathbf{c}}$ in a L-RF

$$\overline{\mathbf{c}} = \tilde{\overline{\mathbf{s}}} \cdot \overline{\mathbf{v}}$$

$$\mathbf{c} = \mathbf{A}\overline{\mathbf{c}}$$

$$\Downarrow$$

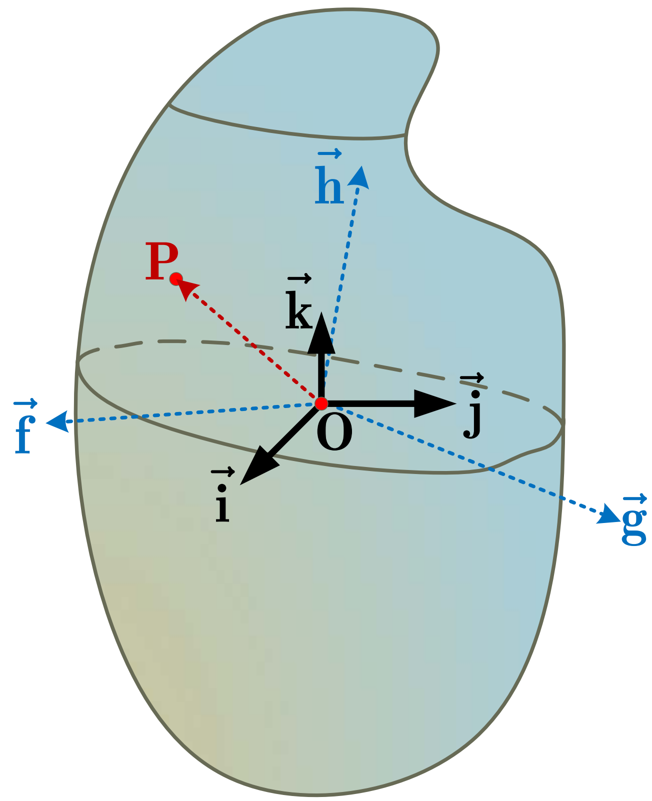
$$\tilde{\mathbf{s}} = \widetilde{(\mathbf{A}\bar{\mathbf{s}})} = \mathbf{A}\tilde{\bar{\mathbf{s}}}\mathbf{A}^T$$

$$\tilde{\bar{\mathbf{s}}} = \widetilde{(\mathbf{A}^T\mathbf{s})} = \mathbf{A}^T\tilde{\mathbf{s}}\mathbf{A}$$

[new very important topic]

Angular Velocity: Intro

- The motivating question: How does the orientation matrix **A** change in time?
- Matrix **A** changes whenever the representation of **f**, **g**, or **h** in the G-RF changes
- Example: Assume blue RF is attached to the body (the L-RF) and the black is the G-RF, fixed to ground
  - A ball joint (spherical joint) present between the body and ground at point **O**



Angular Velocity: Intro

- Note that if **f**, **g**, and **h** change, then a_{11}, a_{21},…, a_{33} change
  - In other words, **A**=**A**(t)

- Recall how the orientation matrix **A** was defined:

$$\overrightarrow{\mathbf{f}}(t) = a_{11}(t)\overrightarrow{\mathbf{i}} + a_{21}(t)\overrightarrow{\mathbf{j}} + a_{31}(t)\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{g}}(t) = a_{12}(t)\overrightarrow{\mathbf{i}} + a_{22}(t)\overrightarrow{\mathbf{j}} + a_{32}(t)\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{h}}(t) = a_{13}(t)\overrightarrow{\mathbf{i}} + a_{23}(t)\overrightarrow{\mathbf{j}} + a_{33}(t)\overrightarrow{\mathbf{k}}$$

Note that $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ do not depend on time (G-RF is fixed).

$$\mathbf{f}(t) = \begin{bmatrix} a_{11}(t) \\ a_{21}(t) \\ a_{31}(t) \end{bmatrix} \qquad \mathbf{g}(t) = \begin{bmatrix} a_{12}(t) \\ a_{22}(t) \\ a_{32}(t) \end{bmatrix} \qquad \mathbf{h}(t) = \begin{bmatrix} a_{13}(t) \\ a_{23}(t) \\ a_{33}(t) \end{bmatrix}$$

$$\mathbf{A}(t) = [\mathbf{f}(t) \quad \mathbf{g}(t) \quad \mathbf{h}(t)] = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

Angular Velocity: Getting There…

- Recall that $\mathbf{AA}^T = \mathbf{I}_3$. Take time derivative to get:

$$\dot{\mathbf{A}}\mathbf{A}^T + \mathbf{A}\dot{\mathbf{A}}^T = \mathbf{0}_{3\times 3} \qquad \Rightarrow \qquad \dot{\mathbf{A}}\mathbf{A}^T = -\mathbf{A}\dot{\mathbf{A}}^T$$

- Notice the following:
  - The matrix $\dot{\mathbf{A}}\mathbf{A}^T$ is a 3×3 matrix
  - The matrix $\dot{\mathbf{A}}\mathbf{A}^T$ is skew-symmetric

- CONCLUSION: there must be a vector, ω, whose cross product matrix is equal to the 3×3 skew symmetric matrix $\dot{\mathbf{A}}\mathbf{A}^T$:

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T$$

- This vector ω is called the angular velocity of the L-RF with respect to the G-RF.

[Short Detour]:

Two L-RF Attached to Same Body

- Problem Setup:
  - You have one rigid body and two different L-RF rigidly attached to that body
    - Rigidly attached means that their relative orientation never change
    - Rigidly attached to the body = “welded” to the body， they move as the body moves
  - Call the local references frames and orientation matrices L-RF$_1$, $\mathbf{A}_1$, and L-RF$_2$, $\mathbf{A}_2$, respectively
- Question: what is the relationship between $\mathbf{A}_1$ and $\mathbf{A}_2$?

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{f}_1 & \mathbf{g}_1 & \mathbf{h}_1 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} \mathbf{f}_2 & \mathbf{g}_2 & \mathbf{h}_2 \end{bmatrix}$$

[Short Detour, Cntd.]:

Two L-RF Attached to Same Body

- The important observation: since both L-RF$_1$ and L-RF$_2$ are "welded" to the rigid body, their relative attitude (orientation) doesn't change in time

- Equivalent way of saying this:

$$\mathbf{f}_2(t) = c_{11}\mathbf{f}_1(t) + c_{21}\mathbf{g}_1(t) + c_{31}\mathbf{h}_1$$
$$\mathbf{g}_2(t) = c_{12}\mathbf{f}_1(t) + c_{22}\mathbf{g}_1(t) + c_{32}\mathbf{h}_1$$
$$\mathbf{h}_2(t) = c_{13}\mathbf{f}_1(t) + c_{23}\mathbf{g}_1(t) + c_{33}\mathbf{h}_1$$

$$\Downarrow$$

$$\begin{bmatrix} \mathbf{f}_2(t) & \mathbf{g}_2(t) & \mathbf{h}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(t) & \mathbf{g}_1(t) & \mathbf{h}_1(t) \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{A}_2(t) = \mathbf{A}_1(t) \cdot \mathbf{C}$$

- The important point: **C** is a constant matrix (since L-RF$_1$ and L-RF$_2$ are "welded" to the rigid body)

The Invariance Property of $

- Recall that we saw that the angular velocity was implicitely defined by the identity $\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T$. This somewhat suggests that ω is related to the matrix $\mathbf{A}$. What follows proves that this is not the case, instead, ω is an attribute of the rigid body the L-RF is attached to.

- First, assume that there are two different angular velocities: ω_1, which goes along with L-RF$_1$, and ω_2, which goes along with L-RF$_2$, where the two L-RFs are rigidly attached to the same body

- Then, since $\mathbf{A}_2 = \mathbf{A}_1\mathbf{C}$, we have $\dot{\mathbf{A}}_2 = \dot{\mathbf{A}}_1\mathbf{C}$, which implies that

$$\tilde{\omega}_2\mathbf{A}_2 = \tilde{\omega}_1\mathbf{A}_1\mathbf{C}$$

- Since $\mathbf{A}_2 = \mathbf{A}_1\mathbf{C}$, we get that

$$\tilde{\omega}_2 = \tilde{\omega}_1$$

- In other words, the angular velocity is an attribute of the body, not of the L-RF rigidly attached to it.

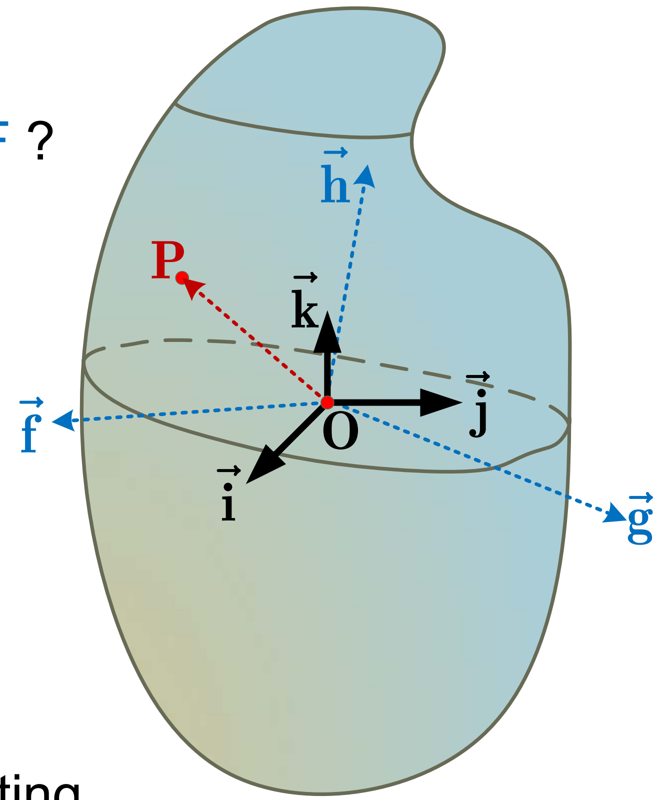
Angular Velocity: On Its Representation in the L-RF

- Assume you have a L-RF attached a body
- Assume that the angular velocity is $
- Question: what is its representation in the L-RF ?

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T \qquad \text{and} \qquad \tilde{\bar{\omega}} = \mathbf{A}^T\tilde{\omega}\mathbf{A}$$

$$\Downarrow$$

$$\tilde{\bar{\omega}} = \mathbf{A}^T\dot{\mathbf{A}}$$



- Therefore, we have that

$$\boxed{\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T \qquad \text{and} \qquad \tilde{\bar{\omega}} = \mathbf{A}^T\dot{\mathbf{A}}}$$

- Note that this also yields two ways of representing the time derivative of **A**:

$$\boxed{\dot{\mathbf{A}} = \tilde{\omega}\mathbf{A} \qquad \text{and} \qquad \dot{\mathbf{A}} = \mathbf{A}\tilde{\bar{\omega}}}$$

The Second Time Derivative of A

- Straight forward application of the definition of the first time derivative of **A** combined with the chain rule of differentiation

- Using the angular velocity and its derivative expressed in the G-RF:

$$\ddot{\mathbf{A}} = \dot{\tilde{\omega}}\mathbf{A} + \tilde{\omega}\dot{\mathbf{A}} = \tilde{\dot{\omega}}\mathbf{A} + \tilde{\omega}\tilde{\omega}\mathbf{A} = (\tilde{\dot{\omega}} + \tilde{\omega}\tilde{\omega})\mathbf{A}$$

- Using the angular velocity and its derivative expressed in the L-RF:

$$\ddot{\mathbf{A}} = \mathbf{A}\dot{\tilde{\bar{\omega}}} + \dot{\mathbf{A}}\tilde{\bar{\omega}} = \mathbf{A}\tilde{\dot{\bar{\omega}}} + \mathbf{A}\tilde{\bar{\omega}}\tilde{\bar{\omega}} = \mathbf{A}(\tilde{\dot{\bar{\omega}}} + \tilde{\bar{\omega}}\tilde{\bar{\omega}})$$

Degrees of Freedom Count, Orientation

- The rotation matrix $\mathbf{A}$ has nine direction cosines
- Recall the story of the birth of the $\mathbf{A}$ matrix: we started with a L-FR attached to a rigid body. Having a L-RF means that we have a triplet $\overrightarrow{\mathbf{f}}$, $\overrightarrow{\mathbf{g}}$, and $\overrightarrow{\mathbf{h}}$. Having the triplet allowed us to generate the matrix $\mathbf{A}$ (since we had its columns)
- Note the following: for each orientation (attitude) of the rigid body, we have a certain set of entries in the $\mathbf{A}$ matrix. Likewise, for each set of entries in matrix $\mathbf{A}$, we have an attitude of the rigid body
- Moral of the story: as soon as we have a set of entries in $\mathbf{A}$; i.e., a set of direction cosines, we basically determined the attitude of the rigid body relative to the G-RF

[Cntd.]

Degrees of Freedom Count, Orientation

- QUESTION: How many of the nine directions cosines can we specify?
  - In order words, how many free parameters do we have in conjunction with the rotation matrix **A**?

- ANSWER: We have three of them
  - We're going to show that we can choose three of the direction cosines and express the other six as a function of the three chosen ones
  - Why can I pull off the "express six as a function of three" trick?
    - Because of these six conditions that the direction cosines satisfy:

$$\mathbf{f}^T\mathbf{f} = \mathbf{g}^T\mathbf{g} = \mathbf{h}^T\mathbf{h} = 1 \quad \& \quad \mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{h} = \mathbf{h}^T\mathbf{f} = 0$$

  - How are we going to prove the "six function of three" trick?
    - Use the Implicit Function Theorem

[Cntd.] [pp.324]

Degrees of Freedom Count, Orientation

- Consider the *relation* $\mathbf{u}(a_{11}, a_{12}, \ldots, a_{33}) = \mathbf{0}_6$, where $\mathbf{u}$ is defined as:

$$\mathbf{u}(a_{11}, a_{12}, \ldots, a_{33}) = \begin{bmatrix} \frac{1}{2}\mathbf{f}^T\mathbf{f} - \frac{1}{2} \\ \frac{1}{2}\mathbf{g}^T\mathbf{g} - \frac{1}{2} \\ \frac{1}{2}\mathbf{h}^T\mathbf{h} - \frac{1}{2} \\ \mathbf{f}^T\mathbf{g} \\ \mathbf{g}^T\mathbf{h} \\ \mathbf{h}^T\mathbf{f} \end{bmatrix}$$

- I have six scalar relations that capture the interplay between nine direction cosines

- I'm going to apply the Implicit Function Theorem to show that I can always express six of them as a function $\mathbf{v}$ that depends on the other three of them

[Cntd.] [pp.324]

Degrees of Freedom Count, Orientation

- First, introduce the following notation to simplify the math:

$$\mathbf{q} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{33} \end{bmatrix}$$

- Second, I'll compute the Jacobian $\mathbf{u_q}$, since I'm about to use the Implicit Function Theorem:

$$\mathbf{u_q} = \begin{bmatrix} \mathbf{f}^T & 0 & 0 \\ 0 & \mathbf{g}^T & 0 \\ 0 & 0 & \mathbf{h}^T \\ \mathbf{g}^T & \mathbf{f}^T & 0 \\ 0 & \mathbf{h}^T & \mathbf{g}^T \\ \mathbf{h}^T & 0 & \mathbf{f}^T \end{bmatrix}_{6\times 9}$$

- Third, the Jacobian $\mathbf{u_q}$ has full row rank. You can prove this by showing that the only linear combination of the rows that can produce a zero vector requires that the coefficients be all zero. Specifically, I will prove that $\mathbf{u_q}^T\mathbf{x} = \mathbf{0}_9 \Rightarrow x_1 = x_2 = \ldots = x_6 = 0$.

[Cntd.] [pp.324]

Degrees of Freedom Count, Orientation

- Fourth, note that we just proved that the rank of $rank(\mathbf{u_q}) = 6$. This means that its column rank it's also six, which means that out of the nine columns of $\mathbf{u_q}$ I can choose six that are linearly independent. I'm going to group these six columns together to make up a matrix that I'm going to call $\mathbf{V}$. For this 6×6 matrix, I have that $det(\mathbf{V}) \neq 0$.

- Fifth, get the remaining three (out of nine) columns, and organize them as matrix $\mathbf{U}$. At this point, we have the following: a *relation* $\mathbf{u}(\mathbf{q}) = \mathbf{0}_6$, and the Jacobian $\mathbf{u_q} = [\mathbf{U} \;\; \mathbf{V}]$, with $det(\mathbf{V}) \neq 0$. According to the Implicit Function Theorem, the six direction cosines that are associated with the columns that were used to make up $\mathbf{V}$ can be expressed as a function that depends on the other three direction cosines.

- **IMPORTANT CONCLUSION:** You need three variable to express all the other six entires of the orienation matrix $\mathbf{A}$. In other words, once you specifiy the value of the three direction cosines, you actually define the orientation of the rigid body that the L-RF associated with $\mathbf{A}$ is attached to.

- **NOTE:** I do not know which three out of the nine direction cosines can be used to express the other six as a function of. In fact, chances are that as $\mathbf{A}$ changes in time, every once in a while you'd have to change the three direction cosines, since you always have to have a $\mathbf{V}$ that is nonsingular.

Remarks,
Orientation Degrees of Freedom

- QUESTION:
  - Why do we obsess about how many degrees of freedom do we have?

- ANSWER:
  - We need to know how many generalized coordinates we'll have to include in our set of unknowns when solving for the time evolution of a dynamic system
  - In this context, we arrived to the conclusion that three direction cosines are independent and need to be accounted for. The other six can be immediately computed once the value of the three independent direction cosines becomes available.
    - As seen soon, in Kinematics and Dynamics we solve for three, and recover the other six direction cosines

Remarks,

Orientation Degrees of Freedom [Cntd.]

- QUESTION:
  - Do I really have to choose three direction cosines and include them in the set of generalized coordinates used to understand the time evolution of the mechanical system?

- ANSWER:
  - No, this is not common
  - What is important is the number of generalized coordinates that are needed
  - Specifically:
    - I can choose three other quantities, call them „, ˘, ‡, that I decide to adopt as my three rotation generalized coordinates as long as there is a ONE-TO-ONE mapping between these three generalized coordinates and *the special set of three out of the nine* direction cosines of the rotation matrix **A**

Remarks,
Orientation Degrees of Freedom [Cntd.]

- Here are a couple of possible scenarios:
  - I indeed choose three generalized coordinates: this is what Euler did, when he chose the Euler Angles to define the entries of **A** and thus capture the orientation of a L-RF with respect to the G-RF
  - I can choose a set of quaternion, or Euler parameters. There is four of them: e_0, e_1, e_2, and e_3, but they are related through a normalization condition:

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

    - Quaternions: born on Monday, October 16, 1843 in one of Sir William Rowan Hamilton's moments of inspiration
  - If you want to be extreme, chose all of the nine direction cosines as generalized coordinates but also added to the equations of motion the following set of six algebraic constraints:

$$\mathbf{f}^T\mathbf{f} = \mathbf{g}^T\mathbf{g} = \mathbf{h}^T\mathbf{h} = 1 \quad \& \quad \mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{h} = \mathbf{h}^T\mathbf{f} = 0$$

[Short Detour:]

Hopping from RF to RF

- The discussion framework:
  - Recall that when going from one L-RF_2 to a different L-RF_1, there is a transformation matrix that multiplies the representation of a geometric vector in L-RF_2 to get the representation of the geometric vector in L-RF_1 :

$$\bar{\mathbf{s}}_1 = \mathbf{A}_{12} \cdot \bar{\mathbf{s}}_2$$

- Question: What happens if you want to go from L-RF_3 to L-RF_2 and then eventually to the representation in L-RF_1 ?

- Why are we curious?
  - Comes into play when dealing with Euler Angles

[End Detour:]

Hopping from RF to RF

- Going from L-RF$_3$ to L-RF$_2$ to L-RF$_1$:

$$\bar{\mathbf{s}}_2 = \mathbf{A}_{23} \cdot \bar{\mathbf{s}}_3 \quad \oplus \quad \bar{\mathbf{s}}_1 = \mathbf{A}_{12} \cdot \bar{\mathbf{s}}_2 \quad \Rightarrow \quad \bar{\mathbf{s}}_1 = \mathbf{A}_{12}\mathbf{A}_{23} \cdot \bar{\mathbf{s}}_3$$

- The basic idea is clear, you keep multiplying rotation matrices like that to hope from RF to RF until you arrive to your final destination

- However, how would you actually go about computing $\mathbf{A}_{ij}$ if you have $\mathbf{A}_i$ and $\mathbf{A}_j$ (that is the two rotation matrices from RF$_i$ to RF$_j$, respectively, into the G-RF)?
  - This means that you hop from L-RF$_j$ to L-RF$_i$
  - Keep in mind the invariant here, that is, the geometric vector $\vec{s}$ whose representation you are playing with:

$$\left.\begin{array}{l} \mathbf{s} = \mathbf{A}_i \bar{\mathbf{s}}_i \\ \mathbf{s} = \mathbf{A}_j \bar{\mathbf{s}}_j \end{array}\right\} \quad \Rightarrow \quad \mathbf{A}_i \bar{\mathbf{s}}_i = \mathbf{A}_j \bar{\mathbf{s}}_j \quad \Rightarrow \quad \bar{\mathbf{s}}_i = \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{s}}_j \quad \Rightarrow \quad \boxed{\mathbf{A}_{ij} = \mathbf{A}_i^T \mathbf{A}_j}$$

A Matter of Notation

- On the previous slide we used the notation $\mathbf{A}_i$ for the matrix that gives the transformation from L-RF$_i$ to the G-RF
- A more consistent notation would have been $\mathbf{A}_{0i}$, to indicate that the destination is "0"; i.e., the G-RF, and the source is L-RF$_i$
- This notation convention; i.e., having two subscripts, would help put things in perspective when we discussed the matrices $\mathbf{A}_{ij} = \mathbf{A}_i^T \mathbf{A}_j$ on the previous slide: L-RF$_i$ is the destination, L-RF$_j$ is the source
- NOTE: Because of convenience, we'll continue to use $\mathbf{A}_i$ instead of $\mathbf{A}_{0i}$