ME751
Advanced Computational Multibody Dynamics

Section 9.2

September 12, 2016

"Age is an issue of mind over matter. If you don't mind, it doesn't matter." Mark Twain

Quote of the day

"Age is an issue of mind over matter. If you don't mind, it doesn't matter."
-- Mark Twain

Before we get started…

- Last Time:
  - Wrapped up Linear Algebra Review
  - Started Calculus Review
    - Important concept: “canonical form” of a quantity for computing its partial derivative

- Today:
  - Finish Calculus Review
  - Geometric Vectors
    - Definition and five basic operations you can do with G. Vectors
    - Introduced reference frames to simplify handling of G. Vectors
  - Introduce Algebraic Vectors (the algebraic counterpart of Geometric Vectors)
  - Understand what it takes to change a RF

- HW assigned on Friday, available at class website
  - Due on September 16 at 9:30 am
  - Available at class website, http://sbel.wisc.edu/Courses/ME751/2016/
  - Solutions emailed to you (selected from solutions you provide)

- Quick remark: no class next Monday (possibly Friday too – I’ll let you know)

Useful Formulas

- A couple of useful formulas (derived based on concepts introduced in the previous lecture)

$$\frac{\partial(\mathbf{g}^T\mathbf{p})}{\partial\mathbf{q}} = \mathbf{g}^T\mathbf{p}_\mathbf{q} + \mathbf{p}^T\mathbf{g}_\mathbf{q}$$

$$\frac{\partial}{\partial\mathbf{q}}(\mathbf{C}\mathbf{q}) = \mathbf{C}$$

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{x}^T\mathbf{C}\mathbf{y}) = \mathbf{y}^T\mathbf{C}^T$$

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{C}\mathbf{q}) = \dot{\mathbf{p}}^T\mathbf{C}\mathbf{q} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$

Assumptions:
$\mathbf{g} = \mathbf{g}(\mathbf{q})$
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix
$\mathbf{y}$ doesn't depend on $\mathbf{x}$

The dimensions of the vectors and matrix above such that all the operations listed can be carried out.

Example

- Derive the last equality on previous slide
- Can you expand that equation further?

Assumptions:
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{C}\mathbf{q}) = \dot{\mathbf{p}}^T\mathbf{C}\mathbf{q} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$

The Implicit Function Theorem (IFT)

- The Implicit Function Theorem provides the guarantee that a relation can locally be turned into a function
  - What do I mean by 'relation'?
    - Here is an example:

$$x^2 + y^2 - 8 = 0 \tag{1}$$

  - What do I mean by 'function'?
    - Here is the function that goes with the above example:

$$y(x) = \sqrt{8 - x^2}$$

  - What do I mean by 'locally'?
    - The meaning of 'locally' is the fact that if I take $x = 3$, relation (1) ceases to define a function anymore. When $x = 2$, in a *neighborhood* of this value things are good, but there is no guarantee that you can make a *global* assumption about the nature of the function $y(x)$ that comes out of a relation.

The Implicit Function Theorem

- There is one more important thing to be considered in relation to the "locality" aspect
  - Note that both x=2,y=2 and x=2, y=-2 verify relation (1) in previous slide
  - However, x=2,y=2 forces the relation to lead to this function:

$$y(x) = \sqrt{8 - x^2}$$

  - Yet x=2, y=-2 forces the relation to lead to this different function,

$$y(x) = -\sqrt{8 - x^2}$$

- Conclusion: the values x_0, y_0 around which you seek the function that comes out of a relation play a role in defining the expression of that function

The Implicit Function Theorem:
Setting the Stage, Formally

- Let $\mathbf{u} : \mathbb{R}^{n+m} \to \mathbb{R}^m$, and for convenience we will use two letters to denote the entries of any element of $\mathbb{R}^{n+m}$ such as in $(x_1, \ldots, x_n, y_1, \ldots, y_m)^T = (\mathbf{x}, \mathbf{y}) \equiv \mathbf{z}$.

- Note that by setting $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{0}_m$ we procure the *relation* we referenced a couple of slides ago.

- Assume we have a point $(a_1, \ldots, a_n, b_1, \ldots, b_m)^T \in \mathbb{R}^{n+m}$ that satisfy our relation; i.e., $\mathbf{u}(\mathbf{a}, \mathbf{b}) = \mathbf{0}_m$.

- What we want to accomplish here is to find, in the neighborhood of the point $\mathbf{x} = \mathbf{a}$, the *function* of several slides ago that is induced by the *relation* $\mathbf{u}(\mathbf{x}, \mathbf{y}) = \mathbf{0}_m$. We will call this function $\mathbf{v}(\mathbf{x})$, and note that $\mathbf{v} : \mathcal{N}_{\mathbf{a}} \to \mathbb{R}^m$, where $\mathcal{N}_{\mathbf{a}}$ is a neighborhood of the point (open set of) $\mathbf{x} = \mathbf{a}$.

- Note that if this is indeed the function induced by the relation, then we must have that $\mathbf{u}(\mathbf{x}, \mathbf{v}(\mathbf{x})) = \mathbf{0}_m$; that is, $\mathbf{y} = \mathbf{v}(\mathbf{x})$.

The Implicit Function Theorem:
Bringing in the Jacobian

- We will need the Jacobian of the relation we were provided. Specifically, consdier the partial derivative of $\mathbf{u}$:

$$\frac{\partial \mathbf{u}}{\partial \mathbf{z}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1}(\mathbf{a},\mathbf{b}) & \cdots & \frac{\partial u_1}{\partial x_n}(\mathbf{a},\mathbf{b}) & \frac{\partial u_1}{\partial y_1}(\mathbf{a},\mathbf{b}) & \cdots & \frac{\partial u_1}{\partial y_m}(\mathbf{a},\mathbf{b}) \\ \dots & \ddots & \dots & \dots & \ddots & \dots \\ \frac{\partial u_m}{\partial x_1}(\mathbf{a},\mathbf{b}) & \cdots & \frac{\partial u_m}{\partial x_n}(\mathbf{a},\mathbf{b}) & \frac{\partial u_m}{\partial y_1}(\mathbf{a},\mathbf{b}) & \cdots & \frac{\partial u_m}{\partial y_m}(\mathbf{a},\mathbf{b}) \end{bmatrix} = \begin{bmatrix} \mathbf{u_x}(\mathbf{a},\mathbf{b}) & \mathbf{u_y}(\mathbf{a},\mathbf{b}) \end{bmatrix}$$

- Note that this partial derivative was evaluated at the 'good' point $(\mathbf{a},\mathbf{b})$; i.e., $\mathbf{a}$, and $\mathbf{b}$, satisfy the relation $\mathbf{u}(\mathbf{a},\mathbf{b}) = \mathbf{0}_m$.

- We concentrate on the $m \times m$ matrix $\mathbf{u_y}(\mathbf{a},\mathbf{b})$. Specifically, the determinant of this submatrix will be assumed to nonzero; i.e., $\mathbf{u_y}(\mathbf{a},\mathbf{b})$ is nonsingular.

The Implicit Function Theorem:

The Actual Formal Thing

Theorem: Assume that the function $\mathbf{u}(\mathbf{x}, \mathbf{y})$ introduced two slides ago is continuously differentiale, and assume that at the 'good' point $(\mathbf{a}, \mathbf{b})$, we have that $\mathbf{u}(\mathbf{a}, \mathbf{b}) = \mathbf{0}_m$ and $det[\mathbf{u_y}(\mathbf{a}, \mathbf{b})] \neq 0$. Then there exists

- an open set $\mathcal{N}_\mathbf{a}$ (neighborhood of $\mathbf{a}$)
- an open set $\mathcal{N}_\mathbf{b}$ (neighborhood of $\mathbf{b}$)
- a *unique* continously differentiable function $\mathbf{v} : \mathcal{N}_\mathbf{a} \to \mathcal{N}_\mathbf{b}$

such that,

$$\mathbf{u}(\mathbf{x}, \mathbf{v}(\mathbf{x})) = \mathbf{0}_m \quad \text{for any} \quad \mathbf{x} \in \mathcal{N}_\mathbf{a}$$

- Important observation regarding differentiability of **v(x)**:

It can be proved that whenever we have the additional hypothesis that $\mathbf{u}$ is continuously differentiable up to k times inside $\mathcal{N}_\mathbf{a} \times \mathcal{N}_\mathbf{b}$, then the same holds true for the induced function $\mathbf{v}$ inside $\mathcal{N}_\mathbf{a}$. For instance, if $k = 1$, we have that for any $\mathbf{x} \in \mathcal{N}_\mathbf{a}$

$$\frac{\partial \mathbf{v}}{\partial x_j}(\mathbf{x}) = -\left[\mathbf{u_y}(\mathbf{x}, \mathbf{v}(\mathbf{x}))\right]^{-1} \cdot \frac{\partial \mathbf{u}}{\partial x_j}(\mathbf{x})$$

The Implicit Function Theorem:

Revisiting the Motivating Example

- Recall our orginal example, where the *relation* $u(x, y) = 0$ was provided using $u(x, y) = x^2 + y^2 - 8$. Note that in this case $n = m = 1$.

- The Jacobian of interest is simple in this case: $\mathbf{J} = [2x \quad 2y]$

- The subjacobian of interest is $u_y(x, y) = 2y$ (in fact for this simple case, u_y doesn't even depend on x).

- Note that for any value $y \neq 0$, one has that $det(u_y) \neq 0$, which is what we need to get a unique function $v(x)$ defined in the neighborhood of any point a, where (a, b) is chosen such that is satifies $u(a, b) = 0$. For instance, $(1, \sqrt{7})$ qualifies as such a point.

- It's interesting to take a close look to see what happens when actually $det(u_y) = 0$. One such a point, but not the only one, in our case would be $(2\sqrt{2}, 0)$. In this case one can easily notice that there is a loss of uniquness since both $v(x) = \sqrt{8 - x^2}$ and $v(x) = -\sqrt{8 - x^2}$ are equally good candidates.

The Implicit Function Theorem:
Final Remarks

- Note that the expression of the function v(x) is unknown
  - We are only told that locally such a function exists
  - We'll have to use other means to figure out the value of v given a value of x

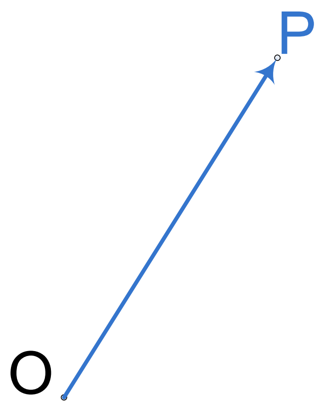
- The Implicit Function Theorem: important theorem of Applied Math

- Make friends with Implicit Function Theorem

End: Review of Calculus
Begin: 3D Kinematics of a Rigid Body

Geometric Vectors

- What is a “Geometric Vector” [GV]?
  - Element of the 3D Euclidian vector space
  - A quantity that has three attributes:
    - A direction (given by the blue line)
    - A sense (from O to P)
    - A magnitude, ||OP||



- Remarks
  - Geometric vectors are entities that are independent of any reference frame
  - Basis for the 3D Euclidian space: any collection of three independent vectors

Geometric Entities: Their Relevance

- Kinematics (and later Dynamics) of systems of rigid bodies:
  - Requires the ability to describe as function of time the position, velocity, and acceleration of each rigid body in the system

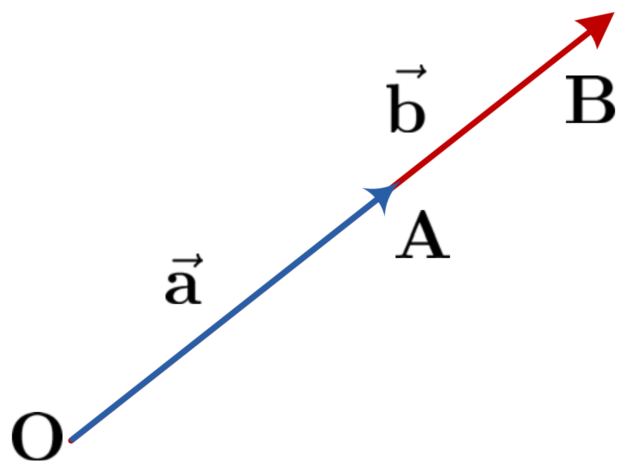
  - In the Euclidian 3D space, geometric vectors and second order tensors are extensively used to this end
    - Geometric vectors - used to locate points on a body or the center of mass of a rigid body
    - Second order tensors - used to describe the orientation of a body

Geometric Vectors: Operations

- What geometric vectors operations are defined out there?
  - Scaling by a scalar –
  - Measuring the angle θ between two geometric vectors
  - Addition of geometric vectors (the parallelogram rule)
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
- A review these definitions follows over the next couple of slides

G. Vector Operation : Scaling by –

- By definition, scaling one geometric vector $\vec{\mathbf{a}}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{\mathbf{b}} \equiv \alpha\vec{\mathbf{a}}$ that has the following three attributes:
  - $\vec{\mathbf{b}}$ has the same direction as the vector $\vec{\mathbf{a}}$
  - $\vec{\mathbf{b}}$ has the sense of $\vec{\mathbf{a}}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{\mathbf{b}}$ is $b = |\alpha| a$
- Note that if $\alpha = 0$, then $\vec{\mathbf{b}}$ is the null vector.



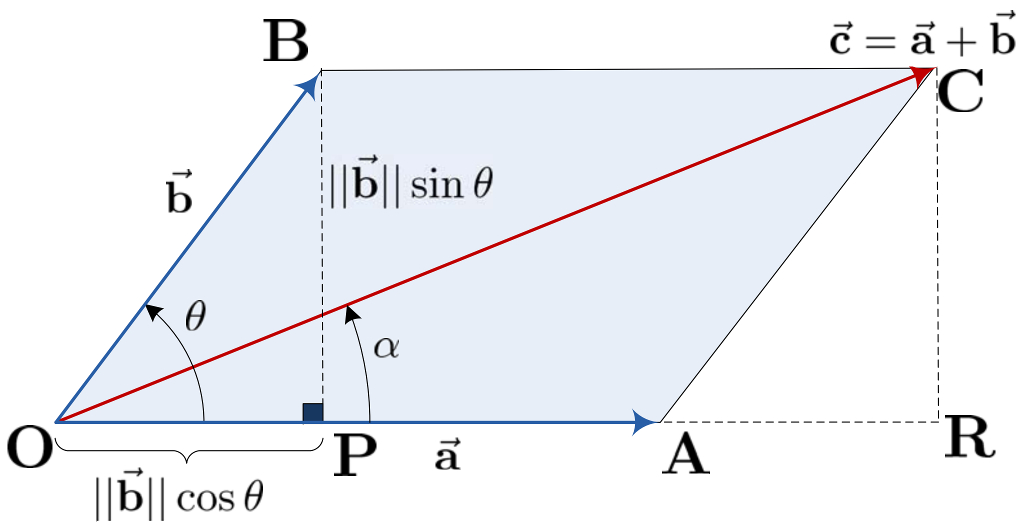
G. Vector Operation: Angle Between Two G. Vectors

- Regarding the angle between two vectors, note that

$$\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \neq \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}}) \qquad \theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = 2\pi - \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}})$$

  - Important: Angles are positive counterclockwise

G. Vector Operation: Addition of Two G. Vectors



- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to see visualize, pretty messy to summarize in an analytical fashion:

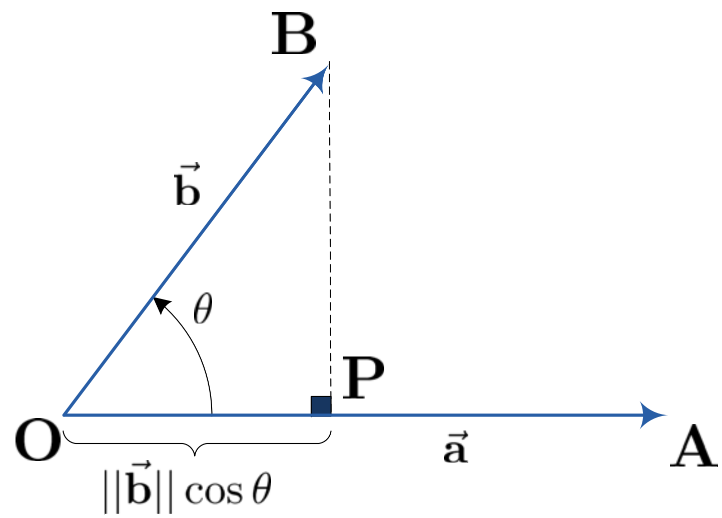
$$c = \sqrt{||\mathbf{OR}||^2 + ||\mathbf{RC}||^2} = \sqrt{(a + b\cos\theta)^2 + b^2\sin^2\theta}$$

$$\alpha = \tan^{-1}\frac{b\sin\theta}{a + b\cos\theta}$$

G. Vector Operation: Inner Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \cos(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$



- Note that operation is commutative

- Don’t call this the “dot product” of the two vectors
  - This name reserved for algebraic vectors

G. Vector Operation: Outer Product of Two G. Vectors

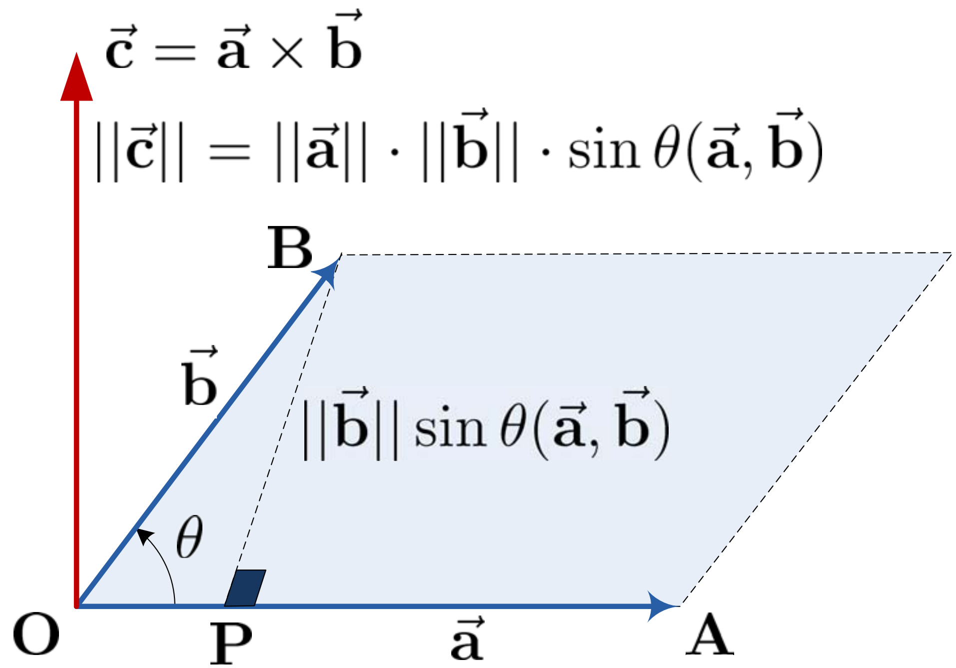
- Direction: perpendicular to the plane determined by the two geometric vectors

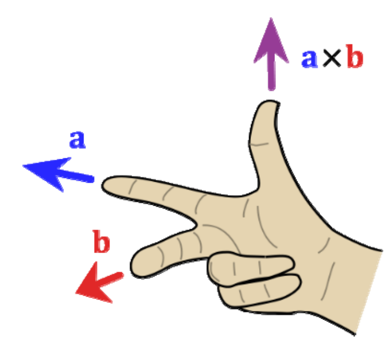
- Sense: provided by the “right-hand rule”

- Magnitude:

$$||\vec{\mathbf{c}}|| = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \sin\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$

- Operation is not commutative, think right-hand rule





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Combining Basic GV Operations

- P1 – The sum of geometric vectors is associative

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$$

- P2 – Multiplication with a scalar is distributive with respect to the sum:

$$k \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = k \cdot \vec{\mathbf{a}} + k \cdot \vec{\mathbf{b}}$$

- P3 – The inner and outer products are distributive with respect to sum:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}}$$

- P4:

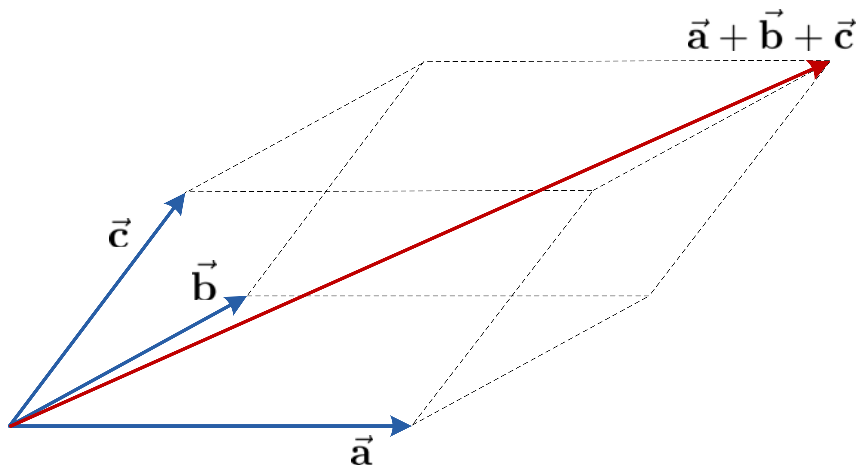
$$\vec{\mathbf{b}}(\alpha + \beta) = \alpha \cdot \vec{\mathbf{b}} + \beta \cdot \vec{\mathbf{b}}$$

- Look innocent, but rather hard to prove true

P1:

- Prove that the sum of geometric vectors is associative:

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$$



[AO]

Exercise, P2:

- Prove that multiplication by a scalar is distributive with respect to the sum:

$$k \cdot (\vec{\mathbf{a}} + \vec{\mathbf{b}}) = k \cdot \vec{\mathbf{a}} + k \cdot \vec{\mathbf{b}}$$

Use slides 16 and 18

Geometric Vectors: Making Things Simpler

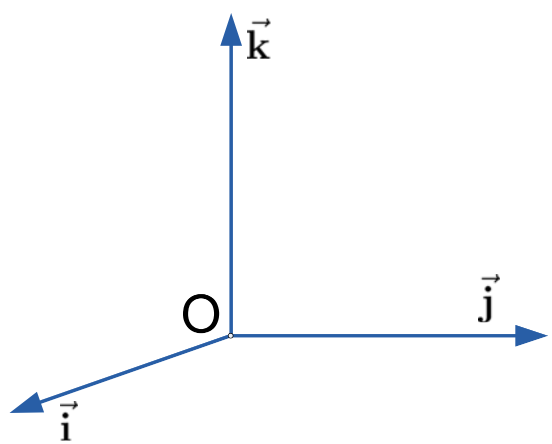
- Geometric vectors:
  - Easy to visualize but cumbersome to work with
  - The major drawback: hard to manipulate
    - Was very hard to carry out simple operations (see proving P2, for instance)
    - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entities is cumbersome
- Addressing cumbersome approach:
  - Introduce a reference frame (RF), express all vectors in that RF
- Useful observation:
  - Three geometric vectors are *independent* if the third one does not belong to the plan defined by the first two ones

Using Reference Frames (RFs)

- 3D Euclidian vector space: three independent vectors are enough to represent all the other vectors

- It's convenient to choose these three vectors to be mutually orthonormal
  - Length 1.0
  - Angle between them: `/2
  - Denoted by: $\vec{\mathbf{i}}, \quad \vec{\mathbf{j}}, \quad \vec{\mathbf{k}}$
  - Defined such that the following relations hold (right hand RF) :

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}$$
$$\vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}}$$
$$\vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}}$$



Representing a G. Vector in a RF; Cartesian Coordinates

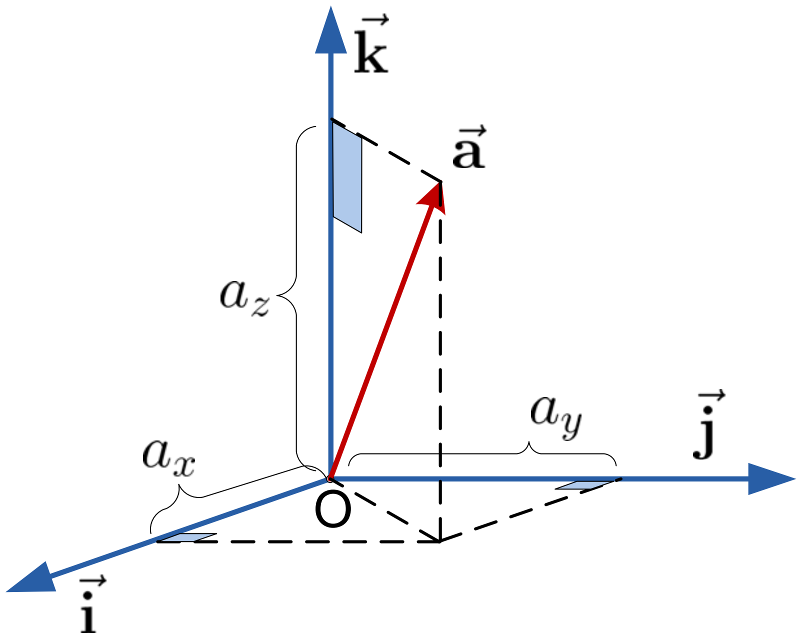
- Together, $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ define a right-hand **reference frame**
- The geometric vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ form a basis for the Euclidian space $\Rightarrow$ Any other vector can be expressed as a linear combination of the basis vectors:

$$\vec{\mathbf{a}} = a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}$$

- The scalars a_x, a_y, and a_z are called the Cartesian coordinates of $\vec{\mathbf{a}}$ in the reference frame defined by $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$
- Since $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ are mutually orthogonal we can compute easily the Cartesian coordinates:

$$a_x = \vec{\mathbf{i}} \cdot \vec{\mathbf{a}} \qquad a_y = \vec{\mathbf{j}} \cdot \vec{\mathbf{a}} \qquad a_z = \vec{\mathbf{k}} \cdot \vec{\mathbf{a}}$$

Representing a G. Vector in a RF



$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}}$$

- Inner product of two GVs, recall: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a\, b \cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$

- Since the angle between basis vectors is π/2:

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} = 1 \qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{i}} = 0$$

  - Therefore, the Cartesian coordinates are computed as

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}} \qquad a_z = \vec{\mathbf{a}} \cdot \vec{\mathbf{k}}$$

Geometric Vectors and RFs: Revisiting the Basic Operations

- Assume that $\alpha \in \mathbb{R}$, and we work with two arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

$$\vec{\mathbf{a}} = a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}} \qquad \& \qquad \vec{\mathbf{b}} = b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}} + b_z\vec{\mathbf{k}}$$

- Sum of two geometric vectors can be shown to be computed as (HOMEWORK):

$$\vec{\mathbf{a}}+\vec{\mathbf{b}} = (a_x\vec{\mathbf{i}}+a_y\vec{\mathbf{j}}+a_z\vec{\mathbf{k}})+(b_x\vec{\mathbf{i}}+b_y\vec{\mathbf{j}}+b_z\vec{\mathbf{k}}) = (a_x+b_x)\vec{\mathbf{i}}+(a_y+b_y)\vec{\mathbf{j}}+(a_z+b_z)\vec{\mathbf{k}}$$

- Multiplication by a real number (scaling) of a geometric vector – the Cartesian coordiantes of the resultsing vector are αa_x, αa_y, and αa_z (HOMEWORK):

$$\alpha\vec{\mathbf{a}} = \alpha \cdot (a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}) = (\alpha a_x)\vec{\mathbf{i}} + (\alpha a_y)\vec{\mathbf{j}} + (\alpha a_z)\vec{\mathbf{k}}$$

- Inner product of two geometric vectors can be shown to be computed as (HOMEWORK):

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = (a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}) \cdot (b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}} + b_z\vec{\mathbf{k}}) = a_x b_x + a_y b_y + a_z b_z$$

- Outer product of two geometric vectors can be shown to be computed as (HOMEWORK):

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = (a_x\vec{\mathbf{i}}+a_y\vec{\mathbf{j}}+a_z\vec{\mathbf{k}})\times(b_x\vec{\mathbf{i}}+b_y\vec{\mathbf{j}}+b_z\vec{\mathbf{k}}) = (a_yb_z-a_zb_y)\vec{\mathbf{i}}+(a_zb_x-a_xb_z)\vec{\mathbf{j}}+(a_xb_y-a_yb_x)\vec{\mathbf{k}}$$

Homework, Clarification

- Proving the relations on the previous slide means using
  - The fact that the vectors belong to the 3D Euclidian vector space
  - Properties P1 through P4 of several slides ago
  - The fact that you have a right-hand orthonormal reference frame

- In your proof, document each step of the way in proving the result
  - Example: now I'm using P2, now I'm using that $\vec{\mathbf{i}}$ and $\vec{\mathbf{k}}$ are orthonormal, etc.

New Concept: Algebraic Vectors

- Given a RF, each vector can be represented by a triplet

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}} \qquad \Leftrightarrow \qquad \vec{\mathbf{a}} \mapsto (a_x, a_y, a_z)$$

- It doesn't take too much imagination to associate to each <u>geometric</u> vector a tridimensional <u>algebraic</u> vector:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}} \qquad \Leftrightarrow \qquad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

- Note that I dropped the arrow on **a** to indicate that we are talking about an algebraic vector

Putting Things in Perspective…

- Step 1: We started with geometric vectors
- Step 2: We introduced a reference frame
- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a triplet (the Cartesian coordinates)
- Step 4: We generated an algebraic vector whose entries are provided by the triplet above
  - This vector is the algebraic representation of the geometric vector
- Note that the algebraic representations of the basis vectors are

$$\vec{\mathbf{i}} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{k}} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Revisiting the Basic Vector Operations
[An algebraic perspective]

- Based on conclusions drawn in slide “Geometric Vectors and RFs: Revisiting the Basic Operations” it’s easy to see that:

  - If you scale a geometric vector, the algebraic representation of the result is obtained by scaling of the original algebraic representation

$$\vec{\mathbf{a}} \mapsto \alpha\vec{\mathbf{a}} \qquad \Leftrightarrow \qquad \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \mapsto \begin{bmatrix} \alpha a_x \\ \alpha a_y \\ \alpha a_z \end{bmatrix}$$

  - If you add two geometric vectors and are curious about the algebraic representation of the result, you simply have to add the two algebraic representations of the original vectors

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \qquad \Leftrightarrow \qquad \mathbf{c} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix} = \mathbf{a} + \mathbf{b}$$

Revisiting the Basic Operations

[An algebraic perspective, Cntd.]

- Based on conclusions drawn in slide "Geometric Vectors and RFs: Revisiting the Basic Operations" it's easy to see that:

  - If you take an inner product of two geometric vectors you get the same results if you compute the dot product of their algebraic counterparts

$$c = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \qquad \Leftrightarrow \qquad c = \mathbf{a}^T \mathbf{b}$$

  - Dealing with the outer product of two geometric vectors is slightly less intuitive
    - Requires the concept of "cross product matrix" of a given algebraic vector **a**
      - A 3 X 3 matrix defined as :

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \mapsto \quad \tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Note the slight inconsistency:
I promised I'd have all the matrices in this class in bold upper case. This is the only exception.

Revisiting the Basic Operations
[An algebraic perspective, Cntd.]

- Based on conclusions drawn in slide “Geometric Vectors: Revisiting the Basic Operations” it’s easy to see that:
  - If you take the outer product of two geometric vectors, then the algebraic vector representation of the result is obtained by left multiplying the second vector by the cross product matrix of the first vector:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\vec{\mathbf{i}} + (a_z b_x - a_x b_z)\vec{\mathbf{j}} + (a_x b_y - a_y b_x)\vec{\mathbf{k}} \quad \mapsto \quad \tilde{\mathbf{a}} \cdot \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

  - Note that the cross product matrix of a vector is a skew-symmetric matrix:

$$\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}$$

[New Topic]

Reference Frames: Nomenclature & Notation

- G-RF: Global Reference Frame (the “world” reference frame)
  - This RF is unique
  - This RF is fixed; that is, its location & orientation don’t change in time

- L-RF: Local Reference Frame
  - It typically represents a RF that is *rigidly* attached to a moving rigid body
  - Notation used
    - An algebraic vector represented in an L-RF has either a prime , as in $\mathbf{s}'$, or it has an overbar, like in $\bar{\mathbf{s}}$
    - The book *always* uses a prime, I will almost always use on overbar

Differentiation of Vectors

(pp.315, Haug book)

- Assumption: for the sake of this discussion on vector differentiation, the geometric vectors are assumed to be represented in a G-RF. Therefore:

$$\dot{\vec{\mathbf{i}}} = \dot{\vec{\mathbf{j}}} = \dot{\vec{\mathbf{k}}} = \mathbf{0}$$

- Due to the assumption above, one has:

$$\begin{aligned} \dot{\vec{\mathbf{a}}} &\equiv \tfrac{d}{dt}\vec{\mathbf{a}}(t) = \tfrac{d}{dt}[a_x(t)\vec{\mathbf{i}} + a_y(t)\vec{\mathbf{j}} + a_z(t)\vec{\mathbf{k}}] \\ &= \left[\tfrac{d}{dt}a_x(t)\right]\vec{\mathbf{i}} + \left[\tfrac{d}{dt}a_y(t)\right]\vec{\mathbf{j}} + \left[\tfrac{d}{dt}a_z(t)\right]\vec{\mathbf{k}} \\ &= \dot{a}_x(t)\vec{\mathbf{i}} + \dot{a}_y(t)\vec{\mathbf{j}} + \dot{a}_z(t)\vec{\mathbf{k}} \end{aligned}$$

- Therefore, the algebraic representation of the derivative of $\vec{\mathbf{a}}$ is

$$\dot{\mathbf{a}} \equiv \frac{d}{dt}\mathbf{a}(t) = \left[\frac{d}{dt}a_x(t), \frac{d}{dt}a_y(t), \frac{d}{dt}a_z(t)\right]^T = [\dot{a}_x(t), \dot{a}_y(t), \dot{a}_z(t)]^T$$

Differentiation of Vectors
(pp.315)

- Similarly, by taking one more time derivative, it is easy to see that the second time derivative of a geometric vector has the following algebraic vector representation

$$\ddot{\mathbf{a}} \equiv \frac{d}{dt}(\dot{\mathbf{a}}(t)) = \left[\frac{d^2}{dt^2}a_x(t), \frac{d^2}{dt^2}a_y(t), \frac{d^2}{dt^2}a_z(t)\right]^T = [\ddot{a}_x(t), \ddot{a}_y(t), \ddot{a}_z(t)]^T$$

- Likewise, consider the only operation introduced so far involving two geometric vectors that leads to a real number: the inner product

$$\frac{d}{dt}[\vec{\mathbf{a}}(t) \cdot \vec{\mathbf{b}}(t)] = \frac{d}{dt}[a_x(t)b_x(t) + a_y(t)b_y(t) + a_z(t)b_z(t)] = \frac{d}{dt}[\mathbf{a}^T(t) \cdot \mathbf{b}(t)]$$