ME451
Kinematics and Dynamics of Machine Systems

Introduction

September 4, 2014

- Quote of the day: "History will be kind to me for I intend to write it" -Winston Churchill

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Before we get started…

- Last time
  - Discussed Syllabus
  - In ME451 we are interested in figuring out how bodies move
  - Key observation: We pursue a general approach that works for all mechanisms
    - This approach might not be the simplest but computers are super fast these days

- Today
  - Review of vector operations and linear algebra

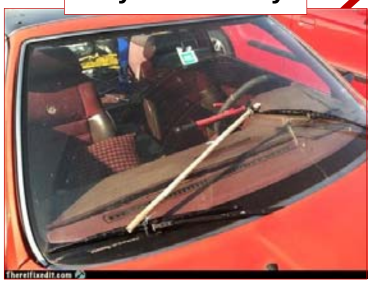
- HW: 2.2.5, 2.2.8, 2.2.10
  - Due in one week, at 9:30 am

Modeling & Simulation (1)

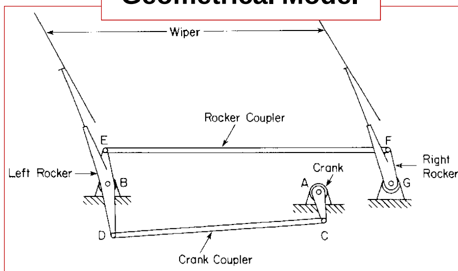
- M&S applies to many (most?) disciplines: engineering, physics, chemistry, biology, economics, etc.
- The goal is to figure out how "something" happens without having to actually (build it and) test it in real-life.
- Modeling is the abstraction of reality while simulation is the execution of the model.
- Computer M&S:
  - Start with a physical phenomenon
  - Use laws, principles, scientific theories to extract a mathematical model (a set of equations that describe the salient features of the particular problem)
  - Figure out a numerical solution algorithm for solving your math problem
  - Implement computer code
  - Simulate, that is run the code (also called "carrying out the analysis")
  - Post-process to turn raw data into useful information for analysis, visualization, and animation
  - Interpret results
- Side note: as close as you'll get in Mechanical Engineering to video gaming

Modeling & Simulation (2)

Physical Reality



Geometrical Model



Mathematical Model

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}\lambda = \mathbf{Q}^{A}$$
$$\mathbf{\Phi} = \mathbf{0}$$

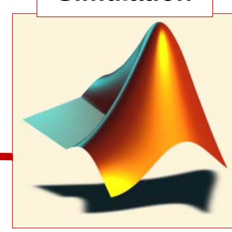
Solution Algorithm

$$\mathbf{q}_1 = \mathbf{q}_0 + h\dot{\mathbf{q}}_0 + \frac{h^2}{2}\left[(1-2\beta)\ddot{\mathbf{q}}_0 + 2\beta\ddot{\mathbf{q}}_1\right]$$
$$\dot{\mathbf{q}}_1 = \dot{\mathbf{q}}_0 + h\left[(1-\gamma)\ddot{\mathbf{q}}_0 + \gamma\ddot{\mathbf{q}}_1\right]$$

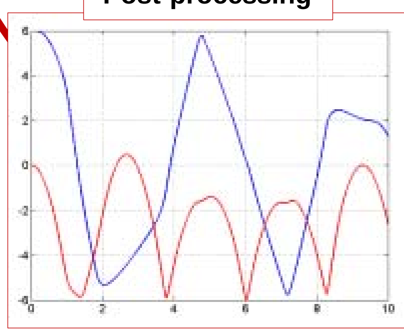
Computer Implementation

```
% Update position and velocity (using Newmark)
q = q_prev + h*qd_prev + 0.5*h^2*((1-2*beta)*qdd_prev + 2*beta*qdd);
qd = qd_prev + h*((1-gam)*qdd_prev + gam*qdd);
```

Simulation



Post-processing



"Essentially, all models are wrong, but some are useful."
George Box & Norman Draper

More on the Computational Perspective…

- Everything that we do in ME451 is governed by Newton's Second Law.

- We pose the problem so that it is suited for solution using a computer:

  1. Identify in a simple and general way the data that is needed to formulate the equations of motion.

  2. Automatically solve the set of nonlinear equations of motion using appropriate numerical solution algorithms: e.g. Newton-Raphson, Newmark Numerical Integration Method, etc.

  3. Provide post-processing support for analysis of results: e.g. plot time curves for quantities of interest, animate the mechanism, etc.

Overview of the Class
[Chapter numbers according to Haug's book]

- Chapter 1 – general considerations regarding the scope and goal of Kinematics and Dynamics (with a computational slant)
- Chapter 2 – review of basic Linear Algebra and Calculus
  - Linear Algebra: Focus on geometric vectors and matrix-vector operations
  - Calculus: Focus on taking partial derivatives (a lot of this), handling time derivatives, chain rule (a lot of this too)
- Chapter 3 – introduces the concept of kinematic constraint as the mathematical building block used to represent joints in mechanical systems
  - This is the hardest part of the material covered
  - Basically poses the Kinematics problem
- Chapter 4 – quick discussion of the numerical algorithms used to solve Kinematics problem formulated in Chapter 3
- Chapter 5 – applications (Kinematics)
  - Only tangentially touching it; ADAMS assignments
- Chapter 6 – formulation of the Dynamics problem: derivation of the equations of motion (EOM)
- Chapter 7 – numerical methods for solving the Dynamics problem formulated in Chapter 6

Haug's book is available online at the class website

ADAMS

- Automatic Dynamic Analysis of Mechanical Systems

- It says Dynamics in name, but it does a whole lot more
  - Kinematics, Statics, Quasi-Statics, etc.

- Philosophy behind software package
  - Offer a pre-processor (ADAMS/View) for people to be able to generate models
  - Offer a solution engine (ADAMS/Solver) for people to be able to find the time evolution of their models
  - Offer a post-processor (ADAMS/PPT) for people to be able to animate and plot results

- It now has a variety of so-called vertical products, which all draw on the ADAMS/Solver, but address applications from a specific field:
  - ADAMS/Car, ADAMS/Rail, ADAMS/Controls, ADAMS/Linear, ADAMS/Hydraulics, ADAMS/Flex, ADAMS/Engine, etc.

ADAMS tutorial: October 9, Room TBA (given by Justin Madsen)

2.1, 2.3

PLANAR VECTORS

Geometric Entities
in Kinematics and Dynamics

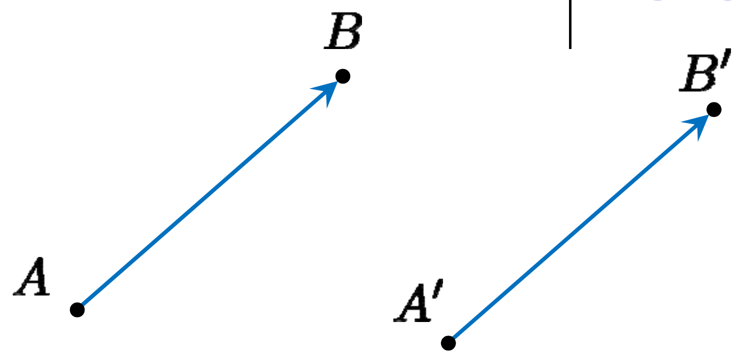
- Kinematics & Dynamics of systems of rigid bodies require the ability to describe the position, velocity, and acceleration of each body in the system

- Geometric vectors and orthogonal matrices provide the most natural description:

  - Geometric vectors - used to locate points on a body and the center of mass of a rigid body, to describe forces and torques, etc.
  - Orthogonal matrices - used to describe the orientation of a body

- ME451, **Planar** Kinematics and Dynamics
  - Calls exclusively for use of 2D vectors and 2x2 orthogonal matrices

Geometric Vectors

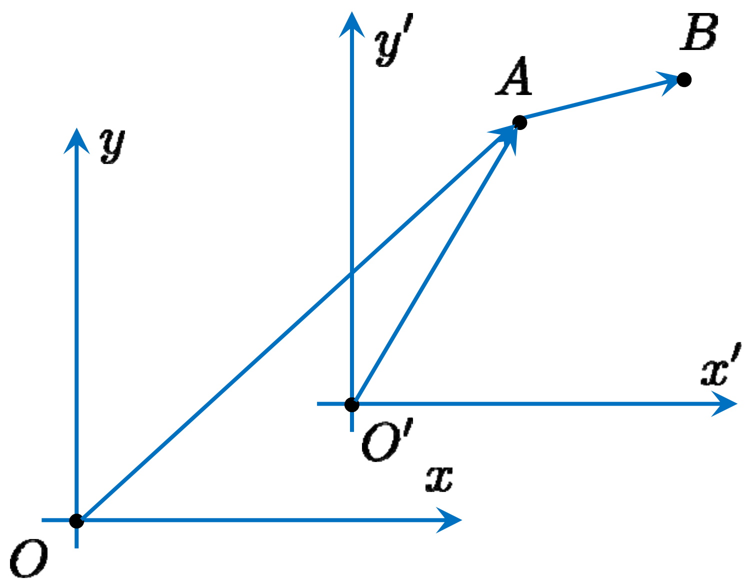


- What is a "Geometric Vector"?
  - A geometrical entity that has:
    - A support line
    - A direction
    - A magnitude (length)
  - A directed line segment (arrow) and therefore has a start point and an end point such that:
    - Its support line is the line defined by A and B
    - Its direction is "from the start point A to the end point B"
    - Its magnitude is the distance between the points A and B
- IMPORTANT:
  - If the start point is fixed in space, called **bound vector**
  - If the start point is of no importance, called **free vector** (therefore $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ represent the exact same free vector)

G. Vectors:

Free vectors and Bound vectors

- Bound vectors: $\overrightarrow{OA}$ and $\overrightarrow{O'A}$
- As a free vector, $\overrightarrow{AB}$ is independent of a translation of the reference frame
- We associate the **position of a point** A to the *bound vector* whose start point is the origin O of some reference frame and whose end point is the point A. Therefore, the position of point A is **different** in the two reference frames.



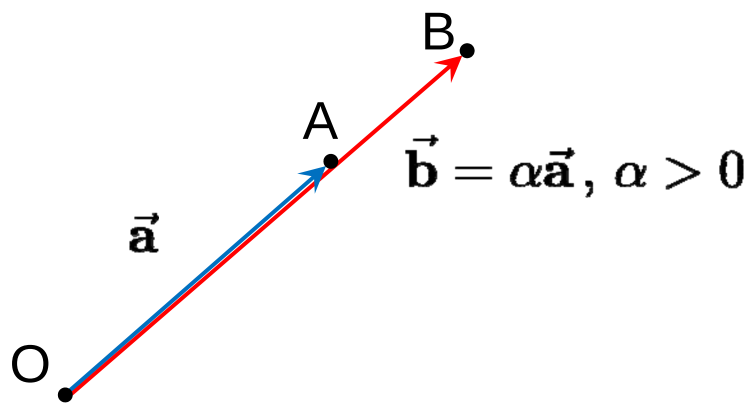
Unless specified otherwise:
vector will mean a free vector
position vector will mean a bound vector

G. Vectors: Operations

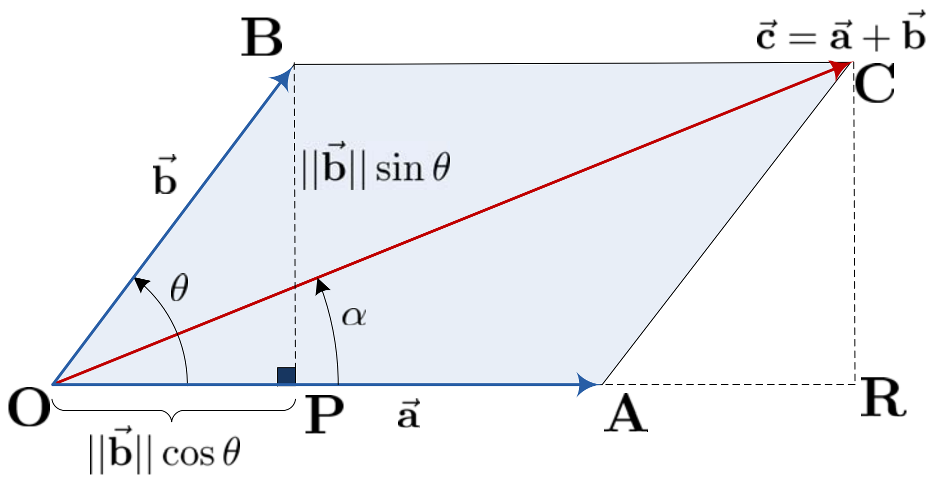
- What operations are <u>defined</u> for 2D geometric vectors?
  - Scaling by a scalar α
  - Addition of 2D geometric vectors (the parallelogram rule)
  - Multiplication of 2D geometric vectors (result in scalars!)
    - The inner product (a.k.a. dot product)
    - The outer product
  - The angle θ between two geometric vectors
- A review these definitions follows next

G. Vector Operation :
Scaling by α

- By definition, scaling a geometric vector $\vec{\mathbf{a}}$ by a scalar $\alpha \neq 0$ results in a new vector $\vec{\mathbf{b}} \equiv \alpha\vec{\mathbf{a}}$ with the following attributes:
  - $\vec{\mathbf{b}}$ has the same support line as $\vec{\mathbf{a}}$
  - $\vec{\mathbf{b}}$ has the same direction as $\vec{\mathbf{a}}$ if $\alpha > 0$ and opposite direction if $\alpha < 0$
  - The magnitude of $\vec{\mathbf{b}}$ is $b = |\alpha| a$
- Note that if $\alpha = 0$, then $\vec{\mathbf{b}}$ is the null vector $\vec{\mathbf{0}}$



G. Vector Operation:
Addition of Two G. Vectors



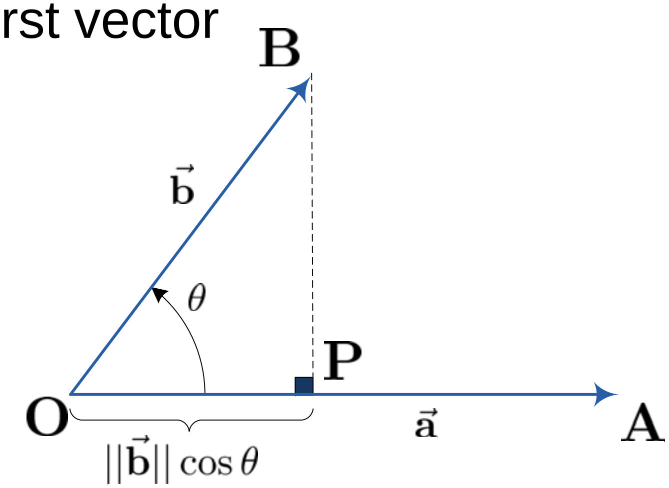
- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to visualize, somewhat messy to summarize in an analytical fashion:

$$c = \sqrt{\|OR\|^2 + \|RC\|^2} = \sqrt{(a + b\cos\theta)^2 + b^2 \sin^2\theta}$$

$$\alpha = \tan^{-1} \frac{b \sin\theta}{a + b\cos\theta}$$

G. Vector Operation:
Dot Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector



$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\| \|\vec{\mathbf{b}}\| \cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \equiv ab \cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$

- Note that operation is commutative

- Two immediate consequences
  - The dot product of a vector with itself gives the square of its magnitude:

    $$\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|^2 \equiv a^2$$

  - Two vector are orthogonal if and only if their dot product vanishes:

    $$\vec{\mathbf{a}} \text{ orthogonal to } \vec{\mathbf{b}} \quad \Leftrightarrow \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$$

G. Vector Operation:
Angle Between Two G. Vectors

- Note that

$$\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \neq \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}}) \qquad \theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = 2\pi - \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}})$$

- Important: Angles are positive <u>counterclockwise</u>
  - This is why when measuring the angle between two vectors it's important which one is the first (start) vector

- Calculating the angle between two vectors:

$$\cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{\|\vec{\mathbf{a}}\| \, \|\vec{\mathbf{b}}\|} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \cdot \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}$$

G. Vector Operations:
Properties

- P1 – The sum of geometric vectors is associative

$$\vec{\mathbf{a}} + \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) = \left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) + \vec{\mathbf{c}}$$

- P2 – Distributivity (multiplication with scalars)

$$\alpha\left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) = \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right)\alpha = \alpha\vec{\mathbf{a}} + \alpha\vec{\mathbf{b}}$$

$$(\alpha + \beta)\,\vec{\mathbf{a}} = \vec{\mathbf{a}}\,(\alpha + \beta) = \alpha\vec{\mathbf{a}} + \beta\vec{\mathbf{b}}$$

- P3 – Distributivity (dot product)

$$\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} + \vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$$

- NOTE: The dot product does not obey the cancellation law. In other words,

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} \text{ and } \vec{\mathbf{a}} \neq \vec{\mathbf{0}} \quad \not\Rightarrow \quad \vec{\mathbf{b}} = \vec{\mathbf{c}}$$

[handout]

Distributivity Property (geometric proof)

- Prove (geometrically) that the dot product is distributive with respect to the vector sum:

$$\left(\vec{\mathbf{a}}+\vec{\mathbf{b}}\right)\cdot\vec{\mathbf{c}}=\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}+\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}$$

Reference Frames.

[Making Things Simpler]

- Geometric vectors:
  - Intuitive and easy to visualize
  - Cumbersome to manipulate
    - Hard to carry out simple operations (recall proving the distributive property on previous slide)
    - Difficult to implement in computer code (operations on geometric vectors are difficult to be provided as recipes, unlike for operations on vectors and matrices)
- To address these drawbacks, we seek alternative ways for expressing geometric vectors: introduce a Reference Frame (RF) in which we express all our vectors

RF: Basis Vectors

- Basis (Unit Coordinate) Vectors: a set of two independent vectors used to express all other vectors in the 2D Euclidian space.

- Although not technically necessary, to simplify our life, we will always use a set of two *orthogonal* unit vectors, $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$, which define the x and y directions of the RF.

$$\|\vec{\mathbf{i}}\| = \vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = 1 \qquad \|\vec{\mathbf{j}}\| = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = 1 \qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = 0$$

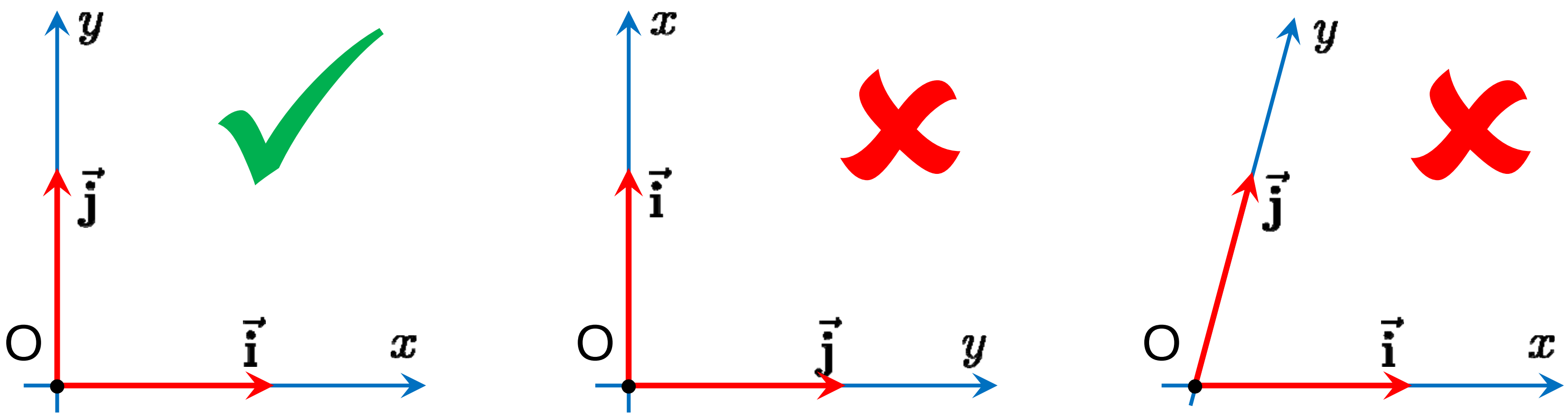
- Any given vector $\vec{\mathbf{a}}$ can then be resolved into its *components* a_x and a_y (a.k.a. its *Cartesian coordinates*), along the axes x and y, respectively:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}}$$

- A basis for $\mathbb{R}^2$ is a set of two independent vectors, such that any other vector can be written as a linear combination of the basis vectors.

RF: We will always…

… use right-handed orthogonal RFs



G. Vectors:
Operations in RF (1)

- Multiplication of a G. Vector by a scalar can be written in terms of the vector's Cartesian components as:

$$\alpha\vec{\mathbf{a}} = (\alpha a_x)\,\vec{\mathbf{i}} + (\alpha a_y)\,\vec{\mathbf{j}}$$

- The sum of two G. Vectors can be written in terms of their Cartesian components as:

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \left(a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}}\right) + \left(b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}}\right) = (a_x + b_x)\,\vec{\mathbf{i}} + (a_y + b_y)\,\vec{\mathbf{j}}$$

- The dot product of two G. Vectors can be written in terms of their Cartesian components as:

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left(a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}}\right) \cdot \left(b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}}\right) = a_x b_x + a_y b_y$$

where we have used the fact that the RF is *orthonormal*.

G. Vectors:
Operations in RF (2)

- The Cartesian components of a vector $\vec{\mathbf{a}}$ are the vector's projections on the RF axes:

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}}$$

- The length of a vector, in terms of its Cartesian components, can be obtained as:

$$a^2 = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = a_x a_x + a_y a_y \quad \Rightarrow \quad a = \sqrt{a_x^2 + a_y^2}$$

- Given a vector $\vec{\mathbf{a}}$, the vector $\vec{\mathbf{a}}^\perp$ orthogonal to $\vec{\mathbf{a}}$ is obtained as:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \quad \Rightarrow \quad \vec{\mathbf{a}}^\perp = -a_y \vec{\mathbf{i}} + a_x \vec{\mathbf{j}} \quad and \quad \vec{\mathbf{a}} \cdot \vec{\mathbf{a}}^\perp = 0$$

New Concept: Algebraic Vectors

- Given a RF, each vector can be represented by a pair of Cartesian components:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \quad \Leftrightarrow \quad \vec{\mathbf{a}} \longmapsto (a_x, a_y)$$

- It is therefore natural to associate to each geometric vector a two-dimensional algebraic vector:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} \quad \Leftrightarrow \quad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$

  Note that I dropped the arrow on **a** to indicate that we are talking about an algebraic vector.

- Convention: throughout this class, vectors and matrices are **bold**, while scalars are not (most often they are in *italics*).

Putting Things in Perspective…

- Step 1: We started with geometric vectors
- Step 2: We introduced a reference frame
- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a pair of scalars (the Cartesian coordinates)
- Step 4: We generated an algebraic vector whose two entries are provided by the pair above
  - This vector is the algebraic representation of the geometric vector
- Note that the algebraic representations of the basis vectors are

$$\vec{\mathbf{i}} \longmapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} \longmapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

G. Vector Ops. $\longleftrightarrow$ A. Vector Ops.

- It is easy to see how you have to modify the algebraic representation of a geometric vector if you modify this geometric vector

  - Scaling a G. Vector $\rightarrow$ Scaling of corresponding A. Vector

$$\alpha\vec{v} \quad \rightarrow \quad \alpha\mathbf{v}$$

  - Adding two G. Vectors $\rightarrow$ Adding the corresponding two A. Vectors

$$\vec{v}+\vec{w} \quad \rightarrow \quad \mathbf{v}+\mathbf{w}$$

  - Dot product of two G. Vectors $\rightarrow$ Dot Product of the two A. Vectors

$$\vec{v}\cdot\vec{w} \quad \rightarrow \quad \mathbf{v}^T\mathbf{w}$$

  - Calculating the angle θ between two G. Vectors was based on their dot product $\rightarrow$ use the dot product of the corresponding A. Vectors

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{vw} \quad \rightarrow \quad \cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

2.2

MATRIX ALGEBRA

Recall Notation Conventions

- UPPERCASE **bold** quantities denote matrices
  Example: $\mathbf{A}$, $\mathbf{B}_i$, etc.
- lowercase **bold** quantities denote vectors
  Example: $\mathbf{v}$, $\mathbf{w}$, $\mathbf{r}^P$, etc.
- Quantities in *italics* indicate scalars
  Example: a, α, c_1, etc.

Matrix Review

- A matrix is a tableau of numbers ordered by rows and columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1^T \\ \mathbf{d}_2^T \\ \vdots \\ \mathbf{d}_m^T \end{bmatrix}$$

- A matrix with m rows and n columns is denoted by $\mathbf{A}_{mn}$ or alternatively we write $\mathbf{A} \in \mathbb{R}^{m \times n}$
- In terms of its elements, a matrix can be written as $\mathbf{A} = [a_{ij}]$, where $1 \leq i \leq m$ and $1 \leq j \leq n$
- The columns of the matrix $\mathbf{A}$ are the n arrays $\mathbf{a}_j$, where $1 \leq j \leq n$, each of dimension $m \times 1$; that is $\mathbf{a}_j \in \mathbb{R}^m$
- The rows of the matrix $\mathbf{A}$ are the m arrays $\mathbf{d}_i^T$, where $1 \leq i \leq m$, each of dimension $1 \times n$; that is $\mathbf{d}_i \in \mathbb{R}^n$

Matrix Addition

- Definition

$$\mathbf{A} = [a_{ij}], \quad \mathbf{A} \in \mathbb{R}^{m \times n}$$
$$\mathbf{B} = [b_{ij}], \quad \mathbf{B} \in \mathbb{R}^{m \times n}$$
$$\mathbf{C} = \mathbf{A} + \mathbf{B} = [c_{ij}], \quad \mathbf{C} \in \mathbb{R}^{m \times n}$$
$$c_{ij} = a_{ij} + b_{ij}$$

- Commutativity property: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Associativity property: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C}$

Matrix-Matrix Multiplication

- Definition

$$\mathbf{A} = [a_{ij}], \quad \mathbf{A} \in \mathbb{R}^{m \times n}$$
$$\mathbf{B} = [b_{ij}], \quad \mathbf{B} \in \mathbb{R}^{n \times p}$$
$$\mathbf{D} = \mathbf{AB} = [d_{ij}], \quad \mathbf{D} \in \mathbb{R}^{m \times p}$$
$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Note the condition: the number of columns of $\mathbf{A}$ must be equal to the number of rows of $\mathbf{B}$
- Matrix multiplication is **not** commutative
- Associativity property: $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC}) = \mathbf{AB} + \mathbf{C}$
- Distributivity property: $(\mathbf{A} + \mathbf{B})\,\mathbf{C} = \mathbf{AC} + \mathbf{BC}$