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An Overview of the Theoretical Foundation of the Smoothed-Particle Hydrodynamics Method

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Abstract

This technical report provides a summary of the key elements that anchor the smoothed particle hydrodynamics (SPH) solution for the fluid dynamics problem.

Keywords: Smoothed-Particle Hydrodynamics, Fluid Mechanics, Fluid-Solid-Interaction

Contents

1 Introduction

Smoothed particle hydrodynamics (SPH) is a mesh-free method to simulate fluid flow using a Lagrangian approach to the governing equations.

2 The Numerical Method

The SPH method is a particle method in which a grid is not needed in order to compute the spatial derivatives. Unlike finite element, discontinuous Galerkin (DG), or finite volume, based approaches, the derivatives are calculated by analytically differentiating an interpolation formula. The governing equations at hand reduce to a system of ordinary differential equations (ODE).

3 Function Approximation

3.1 Kernel Function Approximation

The kernel approximation of a function $f(\mathbf{x})$ is defined as

$$f(\mathbf{x}) = \int_D f(\mathbf{x}')\delta(\|\mathbf{x} - \mathbf{x}'\|)d\mathbf{x}' \tag{1}$$

where $\mathbf{x}$ represents the position vector and $\delta(\|\mathbf{x} - \mathbf{x}'\|)$ is the Dirac delta function. This relation holds given $f(\mathbf{x})$ is a continuous function in $D \subset \mathbb{R}^3$. If we replace the Dirac delta function by a smoothing function $W(\|\mathbf{x} - \mathbf{x}'\|, h)$ then we get

$$f(x) = \int_D f(\mathbf{x}')W(\|\mathbf{x} - \mathbf{x}'\|, h)d\mathbf{x}' \tag{2}$$

where h is defined as the smoothing length. We can rewrite this as

$$\langle f(x)\rangle = \int_D f(\mathbf{x}')W(\|\mathbf{x} - \mathbf{x}'\|, h)d\mathbf{x}' \tag{3}$$

where $\langle\cdot\rangle$ is the kernel approximation operator. Choosing $W(d, h)$ requires it to have the following properties

1. $W(d, h)$ is an even function
2. $W(d, h)$ satisfies the normalization condition

$$\int_D W(\|\mathbf{x} - \mathbf{x}'\|)d\mathbf{x}' = 1. \tag{4}$$

3. $W(d, h)$ satisfies

$$\lim_{h\to 0} W(\|\mathbf{x} - \mathbf{x}'\|, h) = \delta(\|\mathbf{x} - \mathbf{x}'\|). \tag{5}$$

4. $W(d, h)$ has compact support

$$W(v, h) = 0, \|\mathbf{x} - \mathbf{x}'\| > \epsilon h. \tag{6}$$

5. $W(d, h) \geq 0$
6. $W(d, h)$ is monotonically decreasing given the distance away from the particle is increasing.
7. $W(d, h)$ is sufficently smooth.

3.2 Derivative Approximation

3.2.1 Integral Representation

The derivative of a function $f(\mathbf{x})$ can be approximated by the following

$$\langle \nabla \cdot f(\mathbf{x}) \rangle = \int_D [\nabla \cdot f(\mathbf{x})] \, W(\|\mathbf{x} - \mathbf{x}'\|, h) d\mathbf{x}' \tag{7}$$

The divergence operator in the integral is operating on the $\mathbf{x}'$ rather than $\mathbf{x}$. This is because

$$\nabla \cdot f(\mathbf{x}) W(\|\mathbf{x} - \mathbf{x}'\|, h) = \nabla \cdot [f(\mathbf{x}') W(\|\mathbf{x} - \mathbf{x}'\|, h)] - f(\mathbf{x}') \cdot \nabla W(\|\mathbf{x} - \mathbf{x}'\|, h) \tag{8}$$

Using equation (7) and equation (8) the following equation is obtained

$$\langle \nabla \cdot f(\mathbf{x}) \rangle = \int_D \nabla \cdot [f(\mathbf{x}') W(\|\mathbf{x} - \mathbf{x}'\|, h)] d\mathbf{x}' - \int_D f(\mathbf{x}') \cdot \nabla W(\|\mathbf{x} - \mathbf{x}'\|, h) d\mathbf{x}' \tag{9}$$

Using the divergence theorem on the first term on the right-hand side of (9) gives

$$\langle \nabla \cdot f(\mathbf{x}) \rangle = \int_S f(\mathbf{x}') W(\|\mathbf{x} - \mathbf{x}'\|, h) \cdot \mathbf{n} dS - \int_D f(\mathbf{x}') \cdot \nabla W(\|\mathbf{x} - \mathbf{x}'\|, h) d\mathbf{x}' \tag{10}$$

Due to the requirement that the smoothing function $W(\mathbf{x})$ has compact support the surface integral above becomes 0, which yields the following expression

$$\langle \nabla \cdot f(\mathbf{x}) \rangle = -\int_D f(\mathbf{x}') \cdot \nabla W(\|\mathbf{x} - \mathbf{x}'\|, h) d\mathbf{x}' \tag{11}$$

3.2.2 Particle Approximation

In SPH, the system is represented by a finite number of particles, where each particle has individual mass. Equation 3 and equation 11 are continuous representations of $f(\mathbf{x})$ and $\nabla f(\mathbf{x})$. The discrete analog consists of summations over the particles, which is known as the particle approximation. The relation between the finite volume of particle ΔV_j and and it's mass m_j,

$$m_j = \Delta V_j \rho_j. \tag{12}$$

ρ_j is the density of the particle. With this in hand we can reformulate the continuous representation of $f(\mathbf{x})$ as

$$\begin{aligned}
\int_D f(\mathbf{x}')W(\|\mathbf{x}-\mathbf{x}'\|,h)d\mathbf{x}' &\approx \sum_{j=1}^{N} f(\mathbf{x_j})W(\|\mathbf{x}-\mathbf{x}_j\|,h)\Delta V_j && (13)\\
&= \sum_{j=1}^{N} f(\mathbf{x_j})W(\|\mathbf{x}-\mathbf{x}'\|,h)\frac{1}{\rho_j}(\rho_j \Delta V_j) && (14)\\
&= \sum_{j=1}^{N} f(\mathbf{x_j})W(\|\mathbf{x}-\mathbf{x}'\|,h)\frac{1}{\rho_j}m_j. && (15)
\end{aligned}$$

We can rewrite (15) as

$$\int_D f(\mathbf{x}')W(\|\mathbf{x}-\mathbf{x}'\|,h)d\mathbf{x}' \approx \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x}_j)W(\|\mathbf{x}-\mathbf{x}'\|,h). \tag{16}$$

The particle approximation for a function at particle i can be written as

$$\langle f(\mathbf{x}_i)\rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x}_j)W_{ij}, \tag{17}$$

where $W_{ij} = W(\|\mathbf{x}_i - \mathbf{x}_j\|,h)$. With this in hand the derivative of equation (17) is

$$\langle \nabla \cdot f(\mathbf{x_i})\rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(\mathbf{x}_j)\cdot \nabla W_{ij}. \tag{18}$$

In this case $\nabla_i W_{ij} = \frac{\|\mathbf{x}_i - \mathbf{x}_j\|}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}} = \frac{x_{ij}}{r_{ij}} \frac{\partial W_{ij}}{\partial r_{ij}}$ If we substitute the function $f(\mathbf{x})$ with a denstiy function $\rho(\mathbf{x},t)$ in equation (17)

$$\rho_i = \sum_{i=1}^{N} m_{ij} W_{ij}. \tag{19}$$

In literature this is known as the summation density approach.

3.3 Construction of Smooth Functions

A smooth function can be constructed given it satisfies conditions described in section 3.1. A classic function approximation is the bell-shaped function

$$W(\|\mathbf{x}-\mathbf{x}'\|,\mathbf{h}) = \alpha_d \begin{cases} (1+3R)(1-R)^3 & R \leq 1 \\ 0 & R > 1 \end{cases},$$

where $\alpha_d = (5/4h, 5/\pi h^2, 106/16\pi h^3)$. d refers to the dimension. We define R as the relative distance between two particles at points $\mathbf{x}$ and $\mathbf{x}'$. Another function approximation is the Gaussian kernel

$$W(R,h) = \alpha_d e^{-R^2} \tag{20}$$

$$\alpha_d = \begin{cases} 1/\sqrt{\pi}h & d=1 \\ 1/\pi h^2, & d=2 \\ 1/\pi^{3/2}h^3, & d=3 \end{cases}, \tag{21}$$

(22)

A disadvantage of this kernel is that it has no compact support. Therefore, one must have to take into account all the particles in the summation interpolant. As a result, this kernel is computationally expensive as the influence of far away particles is negligible. Another widely used kernel function is the cubic spline

$$W(R,h) = \alpha_d S_4 \tag{23}$$

$$S_4 = \begin{cases} 1-\frac{3}{2}R^2+\frac{3}{4}R^3 & 0 \leq R \leq 1 \\ \frac{1}{4}(2-R)^3 & 1 < q \leq 2 \\ 0, & R > 2 \end{cases} \tag{24}$$

$$\alpha_d = \begin{cases} 2/3h & d=1 \\ 10/(7\pi)h^2, & d=2 \\ 1/\pi h^3, & d=3 \end{cases}, \tag{25}$$

(26)

An advantage of the cubic spline when compared to the Gaussian kernel is the computational costs in a simulation. In a simulation with N particles the computational expensive in calculating the summation of the interpolants is $O(N^2)$. When using cubic splines, the computational expense of summing the interpolants is $O(N_c N)$ where N_c refers to the particles that are within the support of W.

4 Incompressible Fluids

The incompressible Navier-Stokes equations in Lagrangian formulation is defined as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \tag{27}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{28}$$

Equation (27) is the momentum equation and equation (28) is the continuity equation, or conservation of mass.

4.1 Spatial Discretization

4.1.1 Projection Method

A canonical method for numerically solving the incompressible Navier-Stokes equations in both Eulerarian and Lagrangian communities is the "Projection Method." The SPH formulation of the Projection Method was introduced by Cummins in 1999 [1].

In the SPH approach to hydrodynamics, the fluid is represented by a set of particles that follow the fluid motion and advect the fluid quantities, such as mass and momentum. As the SPH method is set in a Lagrangian framework, the incompressible Navier-Stokes equations are reduced to a system of ordinary differential equations (ODEs) for each particle. The smoothness of the numerical solution are ensured by the requirements set on the choice of the function approximation, W. As such we define the function approximation as

$$f(\mathbf{x}) = \sum_{i}^{N} m_i \frac{f_i}{\rho_i} W(\|\mathbf{x} - \mathbf{x}_i\|, h) \tag{29}$$

$$\nabla f(\mathbf{x}) = \sum_{i=1}^{N} m_i \frac{f_i}{\rho_i} \nabla W(\|\mathbf{x} - \mathbf{x}_i\|), h). \tag{30}$$

h refers to the smooth length. With this, we can define the non-dimensional momentum equation at particle a as

$$\frac{d\mathbf{u}_i}{dt} = -\sum_{j=1}^{N} m_i \left[\left(\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2} \right) \nabla_a W_{ij} + \chi_{ij} \right] + \frac{\mathbf{g}}{Fr^2}. \tag{31}$$

$\mathbf{g}$ refers to a gravity term and Fr is the Froude number. In addition, χ_{ij} refers to the viscous stresses. The pressure gradient term is formulated to conserve total linear and total angular momentum. One possible way to treat viscosity, as formulated in [1], is

$$\chi_{ij} = -\frac{1}{Re}\frac{1}{\xi}\left(\frac{4}{\rho_j + \rho_i}\right)\frac{\mathbf{u}_{ij} \cdot \mathbf{x}_{ij}}{|\mathbf{x}_{ij}|^2 + \eta^2}\nabla_i W_{ij}. \tag{32}$$

where ξ is defined as the calibration factor and η is a small parameter to ensure that the denominator remains non-zero. The continuity equation is discretized in the following way

$$\frac{d\rho_i}{dt} = \sum_{j=1} m_i \left(\mathbf{u_i} - \mathbf{u_j}\right) \cdot \nabla_i W_{ij}. \tag{33}$$

We note that

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_i \tag{34}$$
$$\mathbf{u}_{ij} = \mathbf{u}_i - \mathbf{u}_j \tag{35}$$
$$W_{ij} = W(\|\mathbf{x}_{ij}\|, h) = \frac{1}{h^d} f\left(\frac{\|\mathbf{x}_{ij}\|}{h}\right). \tag{36}$$

Next we define an equation of state

$$P = \frac{c^2 \rho_0}{\Gamma}\left(\left(\frac{\rho}{\rho_0}\right)^{\Gamma} - 1\right). \tag{37}$$

c refers to the sound speed, $\Gamma = 7$, and ρ_0 is the initial reference density. In order to approximation incompressibility a large value of c is required. This results in Mach number of $M \approx 0.1$. Due to the effects of incompressibility being $O(M^2)$, use of this particular Mach number should theoretically result in maximum density variations of order 1

The next step in the Projection Method is the temporal integration. A popular method is the use of the forward Euler method. We define the intermediate particle position in which particle positions $\mathbf{x}_i^n$ are advected with velocity $\mathbf{u}_j^n$ to positions $\mathbf{x}_i^*$

$$\mathbf{x}_i^* = \mathbf{x}_i^n + \Delta t \left(\mathbf{u}_j^n\right). \tag{38}$$

At these positions an intermediate velocity step $\mathbf{u}_i^*$ is employed to temporally march the momentum equation

$$\mathbf{u}_i^* = \mathbf{u}_i^n - \Delta t \left(\sum_{j=1}^{N} m_i \chi_{ij}^n(\mathbf{x}^*) + \frac{\mathbf{g}}{Fr^2}\right). \tag{39}$$

We note that the momentum equation is temporally marched without the pressure term. This has to do with the decoupling of the pressure Poisson equation, which is solved independently. The pressure Poisson equation is solved to obtain the pressure needed to enforce conservation of mass, or incompressibility

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P\right)_i = \frac{\nabla \cdot \mathbf{u}_i^*}{\Delta t}. \tag{40}$$

The next step is adding the pressure gradient to obtain a divergent-free velocity field

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^* - \Delta t \sum_{j=1}^{N} m_i \left(\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_j^2}\right) \nabla_i W_{ij}. \tag{41}$$

Having solved for the velocity field, the particle positions are now marched forward in time

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta t \left(\frac{\mathbf{u}_i^{n+1} + \mathbf{u}_i^n}{2} \right). \tag{42}$$

We note that this approximation is centered in time. Given the method of temporal integration, the scheme is $O(\Delta t)$. Since the intermediate velocity field $\mathbf{u}_i^*$ does not depend on the pressure gradient from the previous time-step, this type of projection method is a "full pressure projection."

In a projection method, the pressure needed to enforce incompressibility is found by projecting an estimate of the velocity onto a divergence-free space. Mathematically, this projection approach is based on the Helmholtz decomposition, which states that "Given a vector field in $\mathcal{R}^3$ that is sufficiently smooth and monotonically decaying, it can be decomposed into a sum of a curl-free vector field and a divergence-free vector field."

The projection operator $\mathbf{P}$ for a non-constant density flow is defined as

$$\mathbf{P} = \mathbf{I} - \sigma \mathbf{G}(D\sigma\mathbf{G})^{-1}D, \tag{43}$$

where we define D is a divergence operator,$\mathbf{G}$ is a gradient operator, and $\sigma = \frac{1}{\rho}$. The projection $\mathbf{P}$ will project the intermediate velocity field $\mathbf{u}^*$ onto the space of solenoidal vector fields provided $D = -(\sigma\mathbf{G})^T$. we note that the the intermediate velocity $\mathbf{u}^*$ is computed from momentum equation. In most projection methods, the Helmholtz decomposition is assured by solving for the curl-free component and subtracting it from the intermediate velocity term $\mathbf{u}^*$. This subsequently leads to solving the Pressure Poisson Equation (PPE)

$$D\sigma\mathbf{G}P = \frac{D\mathbf{u}^*}{dt}. \tag{44}$$

for pressure P and subtracting $\Delta t \mathbf{G}P$ from $\mathbf{u}^*$ to give us the incompressible velocity field $\mathbf{u}^{n+1}$ at the next step

$$\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t(\sigma\mathbf{G}P). \tag{45}$$

The Pressure Poisson Equation is formulated as follows

$$D\sigma\mathbf{G}P_i = \sum_j \frac{m_j}{\rho_j} \left(\frac{4}{\rho_i + \rho_j} \right) \frac{P_{ij}\mathbf{r_{ij}} \cdot \nabla_a W_{ij}}{|\mathbf{r^2}_{ij}| + \eta^2}, \tag{46}$$

where $P_{ij} = P_i - P_j$. [1]

An important topic of discussion of the projection method is the soleability of the Pressure Poisson equation (PPE). Numann boundary conditions are enforced on the boundary of the PPE. This leads to a singular matrix which admits a non-unique solution. In order to have a unique solution, a constraint condition needs to be placed in which the source term of the PPE is related to the boundary conditions of the PPE. In order to do this consider integrating the PPE

$$\int_V \nabla \cdot \left(\frac{1}{\rho}\nabla P\right) dV \quad = \quad \int_V \frac{\nabla \cdot \mathbf{u}^*}{\Delta t} dV. \tag{47}$$

Using the divergence theorem on the right-hand side of the above expression gives

$$\int_V \nabla \cdot \left(\frac{1}{\rho}\nabla P\right) = \int_S \mathbf{n} \cdot \mathbf{u}^* dS = 0. \tag{48}$$

This expression holds provided the intermediate velocity $\mathbf{u}^*$ satisfies the correct boundary conditions. This implies that for each time update the sum of the discrete source term $\sum_{i=1}^{N} \nabla \cdot \mathbf{u}^*$ must be 0. We can rewrite this PPE formulation in matrix form

$$A\mathbf{P} = \mathbf{b}, \tag{49}$$

where

$$A \quad = \quad D\sigma G \tag{50}$$

$$\mathbf{b} \quad = \quad \frac{D\mathbf{u}^*}{\Delta t}. \tag{51}$$

We note that matrix A is symmetric [1]. Furthermore, we require $\mathbf{b}$ is in the column space of matrix A, i.e. $\mathbf{b} \in R(A)$. As a result of A being constructed from using homogeneous Neumann boundary conditions for pressure, then there exists a constant vector, $\mathbf{c}$ in the left null space of A

$$A\mathbf{c} = 0 = A^T c = c^T A. \tag{52}$$

If we multiply the PPE matrix system by $\mathbf{c}^T$ gives

$$\mathbf{c}^T\mathbf{b} = 0. \tag{53}$$

This implies that $\sum_{i=1}^{N} \nabla \cdot \mathbf{u}^* = 0$. [1]

4.1.2 Implicit Incompressible SPH

Standard SPH (SSPH) calculates the density of particle i at time n as the following

$$\rho_i^n = \sum_{j=1} m_j W_{ij}^n, \tag{54}$$

where m_j is the mass of a particle j and $W_{ij}^n = W(\mathbf{x}_i^n - \mathbf{x}_j^n)$ is the kernel function with finite support. Pressure p_i^n is computed with an equation of state(EOS)

$$p_i^n = \frac{\kappa\rho_0}{\gamma}\left[\left(\frac{\rho_i^n}{\rho_0}\right)^\gamma - 1\right], \tag{55}$$

where ρ_0 refers to the rest density of the fluid. κ and γ control the stiffness. The momentum-preserving pressure forces are calculated as

$$\mathbf{F}_i^{p,n} = -m_i \sum_{j=1} m_j \left(\frac{p_i^n}{\rho_i^{2,n}} + \frac{\rho_j^n}{\rho_j^{2,n}}\right)\nabla W_{ij}^n. \tag{56}$$

This term serves as the pressure Poisson equation (PPE) in terms of the pressure computation. The pressure forces in SSPH penalize compression, however they do not guarantee an incompressible state at time $n+1$. To resolve this problem, the IISPH method computes the pressure through iteratively solving a linear system. In order to formulate this linear system of unknown pressure forces, a projection method concept is employed. The linear system for pressure can be efficiently solved with a matrix-free implementation.

Implicit Incompressible SPH (IISPH) is formulated on a semi-implicit discretization of the density prediction using the time rate of change of the density. As such, consider the continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v}. \tag{57}$$

Integrating the density term and divergence of the velocity field in time using Forward Euler gives

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \sum_{j=1} m_j \mathbf{v}_{ij}^{n+1}\nabla W_{ij}^n. \tag{58}$$

This particular process of discretization yields unknown relative velocities $\mathbf{v}_{ij}^{n+1} = \mathbf{v}_i^{n+1} - \mathbf{v}_j^{n+1}$. The unknown relative velocities depend on unknown linear pressure forces at time t. The above expression can be integrated using a semi-implicit Euler method for position and velocity update, i.e.

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^n + \Delta t\frac{\mathbf{F}_i^{adv,n} + \mathbf{F}_i^{p,n}}{m_i}. \tag{59}$$

The term $\mathbf{F}^{adv,n}$ refers to known non-pressure forces, such as surface tension, gravity and viscosity. $\mathbf{F}^{p,n}$ refers to unknown pressure forces. Using the predictor-corrector approach of the projection method, we define an intermediate velocities

$$\mathbf{v}_i^{adv} = \mathbf{v}_i^n + \Delta t \frac{\mathbf{F}_i^n}{m_i}. \tag{60}$$

Subsequently this leads to an intermediate density

$$\rho_i^{adv} = \rho_i^n + \Delta t \sum_{j=1} m_j \mathbf{v}_{ij}^{adv} \nabla W_{ij}^n. \tag{61}$$

This acts as the prediction step of our predictor-corrector method. In order to resolve compression, we search for pressure forces

$$(\Delta t)^2 \sum_{j=1} m_j \left(\frac{\mathbf{F}_i^{p,n}}{m_i} - \frac{\mathbf{F}_j^{p,n}}{m_j} \right) \nabla W_{ij}^n = \rho_0 - \rho_i^{adv}. \tag{62}$$

The relationship between continuity and the above expression is related through $\rho_i^{n+1} = \rho_0$. The correction step is calculated as follows

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^{adv} + \Delta t \left(\frac{\mathbf{F}_i^{p,n}}{m_i} \right). \tag{63}$$

If we substitute (56) in (62) we get

$$\sum_{j=1} a_{ij}^n p_j^n = b_i^n. \tag{64}$$

This is a linear system with one equation and one unknown pressure value p_j^n for each particle. As it is a linear system we can express it as

$$\mathbf{A}^n \mathbf{p}^n = \mathbf{b}^n, \tag{65}$$

where $\mathbf{b}^n = \rho_0 - \rho^{adv}$.

The source term for on the RHS is a statement on the density invariance condition. This is advantageous to the alternative divergence term tends to result in problematic compression. This particular IISPH formulation improves the convergence of the solver and temporal stability. This is attributed to treatment of the discrete Laplacian operator and continuity.

In this IISPH formulation, the relation between pressure and pressure force is considered. Furthermore, unlike other ISPH methods, this particular formulation does not start with the continuous PPE, but with the discretized form of the continuity equation. This discretization yields $\mathbf{v}^{n+1}$, which is expressed with the pressure term used in the final velocity update (corrector step.) With this formulation in hand we apply our particular form of the pressure force that has been introduced for calculating pressure. This particular IISPH method predicts the density based on

$$\rho_i^{adv} = \rho_i^n + \Delta t \sum_{j=1} m_j \mathbf{v}_{ij}^{adv} \nabla W_{ij}^n. \tag{66}$$

The term ∇W_{ij}^n is used instead of ∇W_{ij}^{n+1}, which avoids the update of the neighborhood. The above continuity equation formulation allows gives more robustness of our temporal integrator, i.e. allows for larger time steps. Due to this formulation, this system contains a large number of non-zero entires compared to other projection schemes. Due to equation (56) having a nested sum, the coefficients a_{ij} are non-zero for the neighbors j of particle i. Nevertheless, this system can be solved efficiently in a matrix-free way using Conjugate Gradient, SOR, Jacobi, and relaxed Jacobi.

For this particular system, relaxed Jacobi is more advantageous. As such we iteratively solve (**) for p_i

$$p_i^{m+1} = (1-\omega)p_i^m + \omega\left(\frac{\rho_0 - \rho_i^{adv} - \sum_{j\neq i} a_{ij}p_j^m}{a_{ii}}\right). \tag{67}$$

m refers to the iteration index and ω is the relaxtion factor. In order to compute the above expression we need to find $\mathbf{a}_{ii}$ and $\sum_{j\neq i} a_{ij}p_j^k$. In order to find the coefficients, displacement due to pressure is rewritten as

$$(\Delta t)^2 \frac{\mathbf{F}_i^p}{m_i} = -(\Delta t)^2 \sum_{j=1} m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right) \nabla W_{ij} \tag{68}$$

$$= \underbrace{\left(-(\Delta t)^2 \sum_{j=1} \frac{m_j}{\rho_i^2} \nabla W_{ij}\right)}_{\mathrm{d}_{ii}} p_i + \underbrace{\left(-(\Delta t)^2 \sum_{j=1} \frac{m_j}{\rho_j^2} \nabla W_{ij}\right)}_{\mathrm{d}_{ij}} p_j. \tag{69}$$

The term, $\mathbf{d}_{ii}p_i$ refers to the displacement of i due to pressure. The term $d_{ij}p_j$ is the movement caused by pressure values p_j of neighboring particles j. Plugging in (69) into (62) gives

$$\rho_0 - \rho_i^{adv} = \sum_{j=1} m_j \left(\mathbf{d}_{ii}p_i + \sum_{j=1} \mathbf{d}_{ij}p_j - \mathbf{d}_{jj}p_j - \sum_{k=1} \mathbf{d}_{jk}p_k\right) \tag{70}$$

It is observed that $\sum_{k=1} \mathbf{d}_{jk}p_k$ includes pressure values p_i since i and j are neighbors. To separate p_i

$$\sum_k \mathbf{d}_{jk}p_k = \sum_{k\neq i} (\mathbf{d}_{jk}p_k + \mathbf{d}_{ji}p_i) \tag{71}$$

We can split up the right-hand side of (70) such that

$$\rho_0 - \rho_i^{adv} = p_i \sum_{j=1} m_j(\mathbf{d}_{ii} - \mathbf{d}_{ji})\nabla W_{ij} + \sum_{j=1} m_j \left(\sum_{j=1} \mathbf{d}_{ij}p_j - \mathbf{d}_{jj}p_j - \sum_{k\neq i} \mathbf{d}_{jk}p_k\right) \nabla W_{ij} \tag{72}$$

We can compute the coefficients of a_{ii} by

$$a_{ii} = \sum_{j=1} m_j(\mathbf{d}_{ii} - \mathbf{d}_{ij})\nabla W_{ij}. \tag{73}$$

With an expression for the diagonal elements, we can solve for pressure p_i^{k+1}

$$p_i^{m+1} = (1-\omega)p_i^m + \frac{\omega}{a_{ii}}\left(\rho_0 - \rho_i^{adv} - \sum_{j=1} m_j \left[\sum_{j=1} \mathbf{d}_{ij}p_j^m - \mathbf{d}_{jj}p_j^m - \sum_{k\neq i} \mathbf{d}_{jk}p_k^m\right] \nabla W_{ij}\right). \tag{74}$$

4.2 Weakly Compressible Method

The weakly compresible method is another method for solving the incompressible Navier-Stokes equations in a SPH formulation. Unlike the projection method [1], the weakly compressible method does not require the decomposition of pressure from the momentum equation. Instead it relates continuity (conservation of mass) and and pressure through an equation of state. In this case, the equation of state is for water

$$p = \frac{\rho_0 c_0^2}{\gamma}\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right), \tag{75}$$

where $\gamma = 7.0$. ρ_0 acts as a reference density and c_0 is the numerical speed of sound, which is chosen to be 10 times higher than the maximum fluid velocity. This chosen to reduce the density fluctuation to 1 [4]. Due to the term γ acting as a power coefficient, small density fluctuations lead to large pressure fluctuations. The noise induced in the pressure field does not generally contaminate the flow evolution. This approach typically keeps the particle distances roughly constant by imposing a repelling force to a pair of particles when they come too close to each other. The momentum equations are of the following form

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -\frac{1}{\rho}\nabla p^n + \nu\nabla^2\mathbf{u}^n + \mathbf{F}^n. \tag{76}$$

Rearranging the terms above gives us an expression for velocity

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\left(-\frac{1}{\rho}\nabla p^n + \nu\nabla^2\mathbf{u}^n + \mathbf{F}^n\right). \tag{77}$$

The position and density are updated at the next step by

$$\mathbf{r}^{n+1} = \mathbf{r}^n + \Delta t\mathbf{u}^{n+1} \tag{78}$$
$$\rho^{n+1} = \rho^n + \Delta t\left(\nabla \cdot \mathbf{u}^{n+1}\right) \tag{79}$$

Since we have the updated density we are finally able to find the pressure (75). We note that the the weakly compressible SPH method (WCSPH) is only first order in space and time.

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