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Technical Report TR-2016-14

An Overview of an SPH Technique to Maintain Second-Order Convergence for 2D and 3D Fluid Dynamics

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Version: March 1, 2017

Contents

1 Introduction

Smoothed particle hydrodynamics (SPH) is a Lagrangian particle based method for solving partial differential equations (PDEs) describing momentum, mass, and energy conservation laws [2]. In this report, we show how to use second-order operators, which were first proposed in [4], to discretize the governing equations of fluid in two- (2D) and three-dimensional (3D) space. Detailed steps of the discretization process are shown. Finally, we state the SPH governing equations in a form that displays second-order convergence properties.

2 Governing equations

According to the theory of continuum mechanics [1], the momentum equations for incompressible flows can be written as

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \ , \tag{1}$$

where $\mathbf{v}$ is the velocity vector, ρ is the density, ν is the viscosity coefficient, and p is the pressure. The pressure p can be determined from the state equation [3] as

$$p = \rho c^2 \ , \tag{2}$$

where c is the velocity of sound.

3 Relevant SPH minutia

According to the SPH method, a continuous function f defined at point i can be approximated as

$$f_i = \sum_j f_j W_{ij} V_j \ , \tag{3}$$

where V_i is the particle volume,

$$V_i = (\sum_j W_{ij})^{-1} \ , \tag{4}$$

The kernel function, $W_{ij} = W(\mathbf{r}_{ij})$, is expressed as

$$W_{ij} = \alpha_d \times \begin{cases} (3-R)^5 - 6(2-R)^5 + 15(1-R)^5 & 0 \leqq R < 1 \\ (3-R)^5 - 6(2-R)^5 & 1 \leqq R < 2 \\ (3-R)^5 & 2 \leqq R < 3 \\ 0 & R \geqq 3 \end{cases} \tag{5}$$

and the partial differentiation of the kernel function with respect to $\mathbf{x}_i$ is

$$\nabla_i W_{ij} = \alpha_d \frac{1}{h} \frac{\mathbf{r}_{ij}}{r_{ij}} \begin{cases} -5(3-R)^4 + 30(2-R)^4 - 75(1-R)^4 & 0 \leqq R < 1 \\ -5(3-R)^4 + 30(2-R)^4 & 1 \leqq R < 2 \\ -5(3-R)^4 & 2 \leqq R < 3 \\ 0 & R \geqq 3 \end{cases} \tag{6}$$

where $R = \frac{r_{ij}}{h}$, h is the kernel length, $r_{ij} = |\mathbf{r}_{ij}|$, $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, $\alpha_d = 120/h$, $7/478\pi h^2$ and $3/359\pi h^3$ respectively in one-, two- and 3D.

The formulation reviewed herein alleviates a drawback of the standard SPH solution methodology in which a non-uniform distribution of SPH particles reduces the quality of the numerical solution. In this discussion, the concept of "quality of the numerical solution" is tied to, or is a proxy for, the common expectation that as the number of SPH markers increases, the quality of the solution improves. Specifically, in 1D, the expectation is that if one doubles the number of SPH particles, the error will go down by a factor of four. The same improvement in the quality of the solution would call for a particle count increase by a factor of four and eight for 2D and 3D problems, respectively. What the discussed solution methodology ensures is that this quadratic SPH convergence order is maintained. Elaborating on the concept of convergence and accuracy, if the numerical solution matches the first m terms of the Taylor expansion of the solution, then the numerical approximation is said to be $(m+1)$th-order accurate and has mth-order consistency. In practice, the consistency of the SPH formulation can be deteriorated due to factors such as (i) truncation of the support domain of the kernel near the boundaries, or (ii) irregular distribution of the particles. More specifically, if the Taylor series expansion of the kernel approximation is written with two terms, it will emerge that the SPH approximation is 2nd-order accurate (or first-order consistent) under the kernel normalization condition $\int_S W(|x-x'|,h)dx' = 1$, where S is the support domain of the kernel function. However, when the kernel has a compact support, κh, the discrete form of the normalization condition is expressed using the n SPH particles that are inside the support domain as $\sum_{j=1}^{n} W_{ij}\ \Delta x_j$, which may or may not be 1. The error associated with the inconsistency of this discretization can be scaled by $O(1/\sqrt{n})$ for unfavorable particle distributions, or by $O(1/n)$ for a quasi-ordered distribution of the n particles. This scaling shows that (a) the error is reduced by increasing the number of particle in the support domain regardless of particle distribution, and, equally important, (b) the particle distribution in a quasi-ordered setting leads to better error scaling. Therefore, in practical application where Lagrangian particles are suboptimally distributed, *renormalization techniques* are used to improve upon such defects in order to retain the consistency of the normalization condition and ultimately the second-order accuracy of the formulation.

In this context, for second order SPH we use the gradient and Laplacian operators [4]

$$\nabla f_i = \sum_j (f_j - f_i)\mathbf{G}_i \nabla_i W_{ij} V_j \ , \tag{7}$$

$$\nabla^2 f_i = 2\sum_j \left[\mathbf{L}_i : (\mathbf{e}_{ij} \otimes \nabla_i W_{ij})\right] \left(\frac{f_i - f_j}{r_{ij}} - \mathbf{e}_{ij} \cdot \nabla f_i\right) V_j \ , \tag{8}$$

where $\mathbf{e}_{ij} = \mathbf{r}_{ij}/r_{ij}$, and $\mathbf{G}_i$ and $\mathbf{L}_i$ are both symmetric $n \times n$ matrices defined to achieve the required order consistency. In terms of notation, r_{ij}^k, e_{ij}^k, x_i^k, $\nabla_{i,k} W_{ij}$ are each the k^{th} component of vectors $\mathbf{r}_{ij}$, $\mathbf{e}_{ij}$, $\mathbf{x}_i$ and $\nabla_i W_{ij}$, respectively. The (m,n) component of the inverse of the gradient correction matrix $\mathbf{G}_i$ is

$$(\mathbf{G}_i^{-1})^{mn} = -\sum_j r_{ij}^m \nabla_{i,n} W_{ij} V_j \ . \tag{9}$$

To define the Laplacian correction matrix $\mathbf{L}_i$, we adopt the Einstein summation convention and seek a solution of the linear system [4]

$$-\delta^{mn} = \sum_j (A_i^{kmn} e_{ij}^k + r_{ij}^m e_{ij}^n)(L_i^{ps} e_{ij}^p \nabla_{i,s} W_{ij} V_j) \ , \tag{10}$$

where δ^{mn} is the Kronecker symbol, $\nabla_{i,s}$ represents the partial derivative with respect to x_i^s, and the third order tensor $\mathbf{A}_i$ is defined as

$$A_i^{kmn} = \sum_j r_{ij}^m r_{ij}^n G_i^{kq} \nabla_{i,q} W_{ij} V_j \ . \tag{11}$$

4 Evaluating the correction matrices

In this section, we detail the steps for obtaining the matrices $\mathbf{G}_i$ and $\mathbf{L}_i$ in 2D an 3D. According to Eq. (9), the inverse matrix of $\mathbf{G}_i$ in 2D can be expressed as

$$\mathbf{G}_i^{-1} = -\begin{pmatrix} \sum\limits_j r_{ij}^1 \nabla_{i,1} W_{ij} V_j & \sum\limits_j r_{ij}^1 \nabla_{i,2} W_{ij} V_j \\ \sum\limits_j r_{ij}^2 \nabla_{i,1} W_{ij} V_j & \sum\limits_j r_{ij}^2 \nabla_{i,2} W_{ij} V_j \end{pmatrix} \ , \tag{12}$$

and the inverse matrix of $\mathbf{G}_i$ in 3D can be expressed as following

$$\mathbf{G}_i^{-1} = -\begin{pmatrix} \sum\limits_j r_{ij}^1 \nabla_{i,1} W_{ij} V_j & \sum\limits_j r_{ij}^1 \nabla_{i,2} W_{ij} V_j & \sum\limits_j r_{ij}^1 \nabla_{i,3} W_{ij} V_j \\ \sum\limits_j r_{ij}^2 \nabla_{i,1} W_{ij} V_j & \sum\limits_j r_{ij}^2 \nabla_{i,2} W_{ij} V_j & \sum\limits_j r_{ij}^2 \nabla_{i,3} W_{ij} V_j \\ \sum\limits_j r_{ij}^3 \nabla_{i,1} W_{ij} V_j & \sum\limits_j r_{ij}^3 \nabla_{i,2} W_{ij} V_j & \sum\limits_j r_{ij}^3 \nabla_{i,3} W_{ij} V_j \end{pmatrix} \ . \tag{13}$$

Note that both for 2D and 3D, $\mathbf{G}_i$ is indeed symmetric. The gradient correction matrix $\mathbf{G}_i$ is obtained by inverting the matrix above. Finally, the symmetric correction matrix $\mathbf{G}_i$ of particle i is denoted in 2D as

$$\mathbf{G}_i = \begin{pmatrix} G_i^{11} & G_i^{12} \\ G_i^{21} & G_i^{22} \end{pmatrix} \ , \tag{14}$$

where $G_i^{21} = G_i^{12}$ due to $\mathbf{G}_i$ being symmetric, and in 3D as

$$\mathbf{G}_i = \begin{pmatrix} G_i^{11} & G_i^{12} & G_i^{13} \\ G_i^{21} & G_i^{22} & G_i^{23} \\ G_i^{31} & G_i^{32} & G_i^{33} \end{pmatrix} \ , \tag{15}$$

where $G_i^{21} = G_i^{12}$, $G_i^{31} = G_i^{13}$ and $G_i^{23} = G_i^{32}$.

Before calculating the Laplacian correction matrix $\mathbf{L}_i$, we need access to the entries of the third order tensor in Eq. (11). In 2D, upon accounting on the symmetry attribute in the A_i^{1jk} and A_i^{2jk} entries, the components of the tensor $\mathbf{A}_i$ are:

$$
\begin{aligned}
A_i^{111} &= \sum_j r_{ij}^1 r_{ij}^1 (G_i^{11} \nabla_{i,1} W_{ij} + G_i^{12} \nabla_{i,2} W_{ij}) V_j \\
A_i^{112} &= \sum_j r_{ij}^1 r_{ij}^2 (G_i^{11} \nabla_{i,1} W_{ij} + G_i^{12} \nabla_{i,2} W_{ij}) V_j \\
A_i^{122} &= \sum_j r_{ij}^2 r_{ij}^2 (G_i^{11} \nabla_{i,1} W_{ij} + G_i^{12} \nabla_{i,2} W_{ij}) V_j \\
A_i^{211} &= \sum_j r_{ij}^1 r_{ij}^1 (G_i^{21} \nabla_{i,1} W_{ij} + G_i^{22} \nabla_{i,2} W_{ij}) V_j \\
A_i^{212} &= \sum_j r_{ij}^1 r_{ij}^2 (G_i^{21} \nabla_{i,1} W_{ij} + G_i^{22} \nabla_{i,2} W_{ij}) V_j \\
A_i^{222} &= \sum_j r_{ij}^2 r_{ij}^2 (G_i^{21} \nabla_{i,1} W_{ij} + G_i^{22} \nabla_{i,2} W_{ij}) V_j \ .
\end{aligned}
\tag{16}
$$

In 3D, the relevant entires are (recall the symmetry attribute of $\mathbf{A}_i$)

$$\begin{aligned}
A_i^{111} &= \sum_j r_{ij}^1 r_{ij}^1 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{112} &= \sum_j r_{ij}^1 r_{ij}^2 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{113} &= \sum_j r_{ij}^1 r_{ij}^3 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{122} &= \sum_j r_{ij}^2 r_{ij}^2 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{123} &= \sum_j r_{ij}^2 r_{ij}^3 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{133} &= \sum_j r_{ij}^3 r_{ij}^3 (G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij} + G_i^{13}\nabla_{i,3}W_{ij})V_j \\
A_i^{211} &= \sum_j r_{ij}^1 r_{ij}^1 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{212} &= \sum_j r_{ij}^1 r_{ij}^2 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{213} &= \sum_j r_{ij}^1 r_{ij}^3 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{222} &= \sum_j r_{ij}^2 r_{ij}^2 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{223} &= \sum_j r_{ij}^2 r_{ij}^3 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{233} &= \sum_j r_{ij}^3 r_{ij}^3 (G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij} + G_i^{23}\nabla_{i,3}W_{ij})V_j \\
A_i^{311} &= \sum_j r_{ij}^1 r_{ij}^1 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \\
A_i^{312} &= \sum_j r_{ij}^1 r_{ij}^2 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \\
A_i^{313} &= \sum_j r_{ij}^1 r_{ij}^3 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \\
A_i^{322} &= \sum_j r_{ij}^2 r_{ij}^2 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \\
A_i^{323} &= \sum_j r_{ij}^2 r_{ij}^3 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \\
A_i^{333} &= \sum_j r_{ij}^3 r_{ij}^3 (G_i^{31}\nabla_{i,1}W_{ij} + G_i^{32}\nabla_{i,2}W_{ij} + G_i^{33}\nabla_{i,3}W_{ij})V_j \ .
\end{aligned} \tag{17}$$

Then rewrite Eq. (10) as

$$\mathbf{B}_i \begin{pmatrix} L_i^{11} \\ L_i^{12} \\ L_i^{22} \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \tag{18}$$

in 2D, and

$$\mathbf{B}_i \begin{pmatrix} L_i^{11} \\ L_i^{12} \\ L_i^{13} \\ L_i^{22} \\ L_i^{23} \\ L_i^{33} \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} , \tag{19}$$

in 3D. Matrix $\mathbf{B}_i$ assumes the form

$$\mathbf{B}_i = \begin{pmatrix} B_i^{11} & B_i^{12} & B_i^{13} \\ B_i^{21} & B_i^{22} & B_i^{23} \\ B_i^{31} & B_i^{32} & B_i^{33} \end{pmatrix} , \tag{20}$$

in 2D, and

$$\mathbf{B}_i = \begin{pmatrix} B_i^{11} & B_i^{12} & B_i^{13} & B_i^{14} & B_i^{15} & B_i^{16} \\ B_i^{21} & B_i^{22} & B_i^{23} & B_i^{24} & B_i^{25} & B_i^{26} \\ B_i^{31} & B_i^{32} & B_i^{33} & B_i^{34} & B_i^{35} & B_i^{36} \\ B_i^{41} & B_i^{42} & B_i^{43} & B_i^{44} & B_i^{45} & B_i^{46} \\ B_i^{51} & B_i^{52} & B_i^{53} & B_i^{54} & B_i^{55} & B_i^{56} \\ B_i^{61} & B_i^{62} & B_i^{63} & B_i^{64} & B_i^{65} & B_i^{66} \end{pmatrix} , \tag{21}$$

in 3D. The components of the 3×3 matrix $\mathbf{B}_i$ in 2D are expressed as

$$\begin{aligned}
B_i^{11} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + r_{ij}^1 e_{ij}^1)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{12} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + r_{ij}^1 e_{ij}^1)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{13} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + r_{ij}^1 e_{ij}^1)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{21} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + r_{ij}^1 e_{ij}^2)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{22} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + r_{ij}^1 e_{ij}^2)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{23} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + r_{ij}^1 e_{ij}^2)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{31} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + r_{ij}^2 e_{ij}^2)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{32} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + r_{ij}^2 e_{ij}^2)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{33} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + r_{ij}^2 e_{ij}^2)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j
\end{aligned} \tag{22}$$

The components of the 6×6 matrix $\mathbf{B}_i$ in 3D are expressed as

$$\begin{aligned}
B_i^{11} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{12} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{13} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{14} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{15} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{16} &= \sum_j (A_i^{111} e_{ij}^1 + A_i^{211} e_{ij}^2 + A_i^{311} e_{ij}^3 + r_{ij}^1 e_{ij}^1)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned} \tag{23}$$

$$
\begin{aligned}
B_i^{21} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{22} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{23} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{24} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{25} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{26} &= \sum_j (A_i^{112} e_{ij}^1 + A_i^{212} e_{ij}^2 + A_i^{312} e_{ij}^3 + r_{ij}^1 e_{ij}^2)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
B_i^{31} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{32} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{33} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{34} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{35} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{36} &= \sum_j (A_i^{113} e_{ij}^1 + A_i^{213} e_{ij}^2 + A_i^{313} e_{ij}^3 + r_{ij}^1 e_{ij}^3)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
B_i^{41} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{42} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{43} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{44} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{45} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{46} &= \sum_j (A_i^{122} e_{ij}^1 + A_i^{222} e_{ij}^2 + A_i^{322} e_{ij}^3 + r_{ij}^2 e_{ij}^2)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
B_i^{51} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{52} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{53} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{54} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{55} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{56} &= \sum_j (A_i^{123} e_{ij}^1 + A_i^{223} e_{ij}^2 + A_i^{323} e_{ij}^3 + r_{ij}^2 e_{ij}^3)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
B_i^{61} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^1 \nabla_{i,1} W_{ij}) V_j \\
B_i^{62} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^1 \nabla_{i,2} W_{ij} + e_{ij}^2 \nabla_{i,1} W_{ij}) V_j \\
B_i^{63} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^1 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,1} W_{ij}) V_j \\
B_i^{64} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^2 \nabla_{i,2} W_{ij}) V_j \\
B_i^{65} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^2 \nabla_{i,3} W_{ij} + e_{ij}^3 \nabla_{i,2} W_{ij}) V_j \\
B_i^{66} &= \sum_j (A_i^{133} e_{ij}^1 + A_i^{233} e_{ij}^2 + A_i^{333} e_{ij}^3 + r_{ij}^3 e_{ij}^3)(e_{ij}^3 \nabla_{i,3} W_{ij}) V_j
\end{aligned}
\tag{28}
$$

In 2D, L_i^{11}, L_i^{12} and L_i^{22} can be obtained via Eq. (18). In 3D, L_i^{11}, L_i^{12}, L_i^{13}, L_i^{22}, L_i^{23} and L_i^{33} can be obtained via Eq. (19). Finally, we can get the symmetric correction matrix $\mathbf{L}$ of particle i in 2D as

$$
\mathbf{L}_i = \begin{pmatrix} L_i^{11} & L_i^{12} \\ L_i^{12} & L_i^{22} \end{pmatrix} , \tag{29}
$$

and in 3D as

$$
\mathbf{L}_i = \begin{pmatrix} L_i^{11} & L_i^{12} & L_i^{13} \\ L_i^{12} & L_i^{22} & L_i^{23} \\ L_i^{13} & L_i^{23} & L_i^{33} \end{pmatrix} . \tag{30}
$$

5 SPH discretization of the governing equations

According to Eq. (3), the evolution of fluid density can be expressed as

$$
\rho_i = \sum_j m_j W_{ij} . \tag{31}
$$

In 2D, according to Eq. (1), the gradient of pressure p of particle i can be expressed as

$$
\nabla p_i = \frac{\partial p_i}{\partial x} \mathbf{i} + \frac{\partial p_i}{\partial y} \mathbf{j} , \tag{32}
$$

where

$$\frac{\partial p_i}{\partial x} = \sum_j (p_j - p_i)(G_i^{11}\nabla_{i,1}W_{ij} + G_i^{12}\nabla_{i,2}W_{ij})V_j \ , \tag{33}$$

$$\frac{\partial p_i}{\partial y} = \sum_j (p_j - p_i)(G_i^{21}\nabla_{i,1}W_{ij} + G_i^{22}\nabla_{i,2}W_{ij})V_j \ . \tag{34}$$

The Laplacian of velocity $\mathbf{v}$ of particle i can be expressed as

$$\nabla^2 \mathbf{v}_i = \nabla^2 v_i^1 \mathbf{i} + \nabla^2 v_i^2 \mathbf{j} \ . \tag{35}$$

In order to calculate $\nabla^2 v_i^1$ and $\nabla^2 v_i^2$, first calculate ∇v_i^1 and ∇v_i^2 as

$$\nabla v_i^1 = \frac{\partial v_i^1}{\partial x}\mathbf{i} + \frac{\partial v_i^1}{\partial y}\mathbf{j} \tag{36}$$

$$\nabla v_i^2 = \frac{\partial v_i^2}{\partial x}\mathbf{i} + \frac{\partial v_i^2}{\partial y}\mathbf{j} \ . \tag{37}$$

By replacing p_i and p_j with v_i^1 and v_j^1 in Eq. (33), and replacing p_i and p_j with v_i^1 and v_j^1 in Eq. (34), $\frac{\partial v_i^1}{\partial x}$ and $\frac{\partial v_i^1}{\partial y}$ can be easily obtained. Repeat the same procedures and get $\frac{\partial v_i^2}{\partial x}$ and $\frac{\partial v_i^2}{\partial y}$. Then calculate $\nabla^2 v_i^1$ and $\nabla^2 v_i^2$ according to Eq. (8)

$$\begin{aligned} \nabla^2 v_i^1 = 2\sum_j (&L_i^{11} e_{ij}^1 \nabla_{i,1}W_{ij} + L_i^{12} e_{ij}^1 \nabla_{i,2}W_{ij} + \\ &L_i^{21} e_{ij}^2 \nabla_{i,1}W_{ij} + L_i^{22} e_{ij}^2 \nabla_{i,2}W_{ij}) \\ &(\frac{v_i^1 - v_j^1}{r_{ij}} - e_{ij}^1 \nabla_1 v_i^1 - e_{ij}^2 \nabla_2 v_i^1) \end{aligned} \tag{38}$$

$$\nabla^2 v_i^2 = 2\sum_j (L_i^{11} e_{ij}^1 \nabla_{i,1} W_{ij} + L_i^{12} e_{ij}^1 \nabla_{i,2} W_{ij} + L_i^{21} e_{ij}^2 \nabla_{i,1} W_{ij} + L_i^{22} e_{ij}^2 \nabla_{i,2} W_{ij}) (\frac{v_i^2 - v_j^2}{r_{ij}} - e_{ij}^1 \nabla_1 v_i^2 - e_{ij}^2 \nabla_2 v_i^2) \ . \tag{39}$$

Finally, substitute Eqs. (32) and (35) into Eq. (1) when handling the momentum equation in 2D.

In 3D, according to Eq. (1), the gradient of pressure p of particle i can be expressed as

$$\nabla p_i = \frac{\partial p_i}{\partial x}\mathbf{i} + \frac{\partial p_i}{\partial y}\mathbf{j} + \frac{\partial p_i}{\partial z}\mathbf{k} \ , \tag{40}$$

where

$$\frac{\partial p_i}{\partial x} = \sum_j (p_j - p_i)(G_i^{11} \nabla_{i,1} W_{ij} + G_i^{12} \nabla_{i,2} W_{ij} + G_i^{13} \nabla_{i,3} W_{ij}) V_j \tag{41}$$

$$\frac{\partial p_i}{\partial y} = \sum_j (p_j - p_i)(G_i^{21} \nabla_{i,1} W_{ij} + G_i^{22} \nabla_{i,2} W_{ij} + G_i^{23} \nabla_{i,3} W_{ij}) V_j \tag{42}$$

$$\frac{\partial p_i}{\partial z} = \sum_j (p_j - p_i)(G_i^{31} \nabla_{i,1} W_{ij} + G_i^{32} \nabla_{i,2} W_{ij} + G_i^{33} \nabla_{i,3} W_{ij}) V_j \ . \tag{43}$$

The Laplacian of velocity $\mathbf{v}$ of particle i can be expressed as

$$\nabla^2 \mathbf{v}_i = \nabla^2 v_i^1 \mathbf{i} + \nabla^2 v_i^2 \mathbf{j} + \nabla^2 v_i^3 \mathbf{k} \ . \tag{44}$$

In order to calculate $\nabla^2 v_i^1$, $\nabla^2 v_i^2$ and $\nabla^2 v_i^3$, first we need to evaluate ∇v_i^1, ∇v_i^2 and ∇v_i^3:

$$\nabla v_i^1 = \frac{\partial v_i^1}{\partial x}\mathbf{i} + \frac{\partial v_i^1}{\partial y}\mathbf{j} + \frac{\partial v_i^1}{\partial z}\mathbf{k} \tag{45}$$

$$\nabla v_i^2 = \frac{\partial v_i^2}{\partial x}\mathbf{i} + \frac{\partial v_i^2}{\partial y}\mathbf{j} + \frac{\partial v_i^2}{\partial z}\mathbf{k} \tag{46}$$

$$\nabla v_i^3 = \frac{\partial v_i^3}{\partial x}\mathbf{i} + \frac{\partial v_i^3}{\partial y}\mathbf{j} + \frac{\partial v_i^3}{\partial z}\mathbf{k}\ . \tag{47}$$

By replacing p_i and p_j with v_i^1 and v_j^1 in Eq. (41), and replacing p_i and p_j with v_i^1 and v_j^1 in Eq. (42), and replacing p_i and p_j with v_i^1 and v_j^1 in Eq. (43), one can obtain $\frac{\partial v_i^1}{\partial x}$, $\frac{\partial v_i^1}{\partial y}$ and $\frac{\partial v_i^1}{\partial z}$. Repeat the same procedures and get $\frac{\partial v_i^2}{\partial x}$, $\frac{\partial v_i^2}{\partial y}$ and $\frac{\partial v_i^2}{\partial z}$, and also $\frac{\partial v_i^3}{\partial x}$, $\frac{\partial v_i^3}{\partial y}$ and $\frac{\partial v_i^3}{\partial z}$. Then calculate $\nabla^2 v_i^1$, $\nabla^2 v_i^2$ and $\nabla^2 v_i^3$ according to Eq. (8)

$$\begin{aligned}\nabla^2 v_i^1 = 2\sum_j (&L_i^{11} e_{ij}^1 \nabla_{i,1} W_{ij} + L_i^{12} e_{ij}^1 \nabla_{i,2} W_{ij} + \\ &L_i^{13} e_{ij}^1 \nabla_{i,3} W_{ij} + L_i^{21} e_{ij}^2 \nabla_{i,1} W_{ij} + \\ &L_i^{22} e_{ij}^2 \nabla_{i,2} W_{ij} + L_i^{23} e_{ij}^2 \nabla_{i,3} W_{ij} + \\ &L_i^{31} e_{ij}^3 \nabla_{i,1} W_{ij} + L_i^{32} e_{ij}^3 \nabla_{i,2} W_{ij} + \\ &L_i^{33} e_{ij}^3 \nabla_{i,3} W_{ij}) \\ &(\frac{v_i^1 - v_j^1}{r_{ij}} - e_{ij}^1 \nabla_1 v_i^1 - e_{ij}^2 \nabla_2 v_i^1 - e_{ij}^3 \nabla_3 v_i^1)\end{aligned} \tag{48}$$

$$
\begin{aligned}
\nabla^2 v_i^2 = 2\sum_j (&L_i^{11} e_{ij}^1 \nabla_{i,1} W_{ij} + L_i^{12} e_{ij}^1 \nabla_{i,2} W_{ij} + \\
&L_i^{13} e_{ij}^1 \nabla_{i,3} W_{ij} + L_i^{21} e_{ij}^2 \nabla_{i,1} W_{ij} + \\
&L_i^{22} e_{ij}^2 \nabla_{i,2} W_{ij} + L_i^{23} e_{ij}^2 \nabla_{i,3} W_{ij} + \\
&L_i^{31} e_{ij}^3 \nabla_{i,1} W_{ij} + L_i^{32} e_{ij}^3 \nabla_{i,2} W_{ij} + \\
&L_i^{33} e_{ij}^3 \nabla_{i,3} W_{ij}) \\
&(\frac{v_i^2 - v_j^2}{r_{ij}} - e_{ij}^1 \nabla_1 v_i^2 - e_{ij}^2 \nabla_2 v_i^2 - e_{ij}^3 \nabla_3 v_i^2)
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
\nabla^2 v_i^3 = 2\sum_j (&L_i^{11} e_{ij}^1 \nabla_{i,1} W_{ij} + L_i^{12} e_{ij}^1 \nabla_{i,2} W_{ij} + \\
&L_i^{13} e_{ij}^1 \nabla_{i,3} W_{ij} + L_i^{21} e_{ij}^2 \nabla_{i,1} W_{ij} + \\
&L_i^{22} e_{ij}^2 \nabla_{i,2} W_{ij} + L_i^{23} e_{ij}^2 \nabla_{i,3} W_{ij} + \\
&L_i^{31} e_{ij}^3 \nabla_{i,1} W_{ij} + L_i^{32} e_{ij}^3 \nabla_{i,2} W_{ij} + \\
&L_i^{33} e_{ij}^3 \nabla_{i,3} W_{ij}) \\
&(\frac{v_i^3 - v_j^3}{r_{ij}} - e_{ij}^1 \nabla_1 v_i^3 - e_{ij}^2 \nabla_2 v_i^3 - e_{ij}^3 \nabla_3 v_i^3)
\end{aligned} \tag{50}
$$

Finally, substitute Eqs. (40) and (44) into the 3D momentum balance condition, see Eq. (1).

References

[1] Morton E Gurtin, Eliot Fried, and Lallit Anand. *The mechanics and thermodynamics of continua*. Cambridge University Press, 2010.

[2] J. J. Monaghan. Smoothed Particle Hydrodynamics. *Reports on Progress in Physics*, 68(1):1703–1759, 2005.

[3] J. P. Morris, P. J. Fox, and Y. Zhu. Modeling low Reynolds number incompressible flows using SPH. *Journal of Computational Physics*, 136(1):214–226, 1997.

[4] N. Trask, M. Maxey, K. Kimb, M. Perego, M. L. Parks, K. Yang, and J. Xu. A scalable consistent second-order SPH solver for unsteady low reynolds number flows. *Comput. Methods Appl. Mech. Engrg.*, 289:155–178, 2015.