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Posing Multibody Dynamics with Friction and Contact as a Differential Complementarity Problem

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Abstract

We use a complementarity approach to pose the Coulomb friction model, combine it with a non-penetration condition, and append to a differential algebraic problem to characterize the dynamics of multibody systems with friction and contact. The resulting problem is relaxed to a Cone Complementarity Problem, whose solution is shown to represent the first order optimality condition of a quadratic program with conic constraints.

Keywords: friction, contact, unilateral constraints, bilateral constraints, complementarity conditions, differential complementarity problem

Contents

1 Notation. Problem setup

The time-evolution of a collection of n_b rigid bodies interacting through friction and contact is described herein using Cartesian coordinates associated with each body j, where $1 \leq j \leq n_b$. The array of generalized coordinates $\mathbf{q} = [\mathbf{r}_1^T, \boldsymbol{\epsilon}_1^T, \ldots, \mathbf{r}_{n_b}^T, \boldsymbol{\epsilon}_{n_b}^T]^T \in \mathbb{R}^{7n_b}$, and its time derivative $\dot{\mathbf{q}} = [\dot{\mathbf{r}}_1^T, \dot{\boldsymbol{\epsilon}}_1^T, \ldots, \dot{\mathbf{r}}_{n_b}^T, \dot{\boldsymbol{\epsilon}}_{n_b}^T]^T \in \mathbb{R}^{7n_b}$, are used to represent the state of the system, where for body j, $\mathbf{r}_j$ and $\epsilon_{\mathbf{j}}$ are the absolute position of the center of mass and the body orientation Euler parameters, respectively. The derivative of the Euler parameters $\dot{\boldsymbol{\epsilon}}$ can be replaced with a different set of unknowns; i.e., the angular velocity in local coordinates $\bar{\omega}$, formulating the generalized velocity $\mathbf{v} = [\dot{\mathbf{r}}_1^T, \bar{\boldsymbol{\omega}}_1^T, \ldots, \dot{\mathbf{r}}_{n_b}^T, \bar{\boldsymbol{\omega}}_{n_b}^T]^T \in \mathbb{R}^{6n_b}$, which can be mapped to $\dot{\mathbf{q}}$ via [4],

$$\dot{\mathbf{q}} = \mathbf{L}(\mathbf{q})\mathbf{v}. \tag{1}$$

Assume two bodies of index A and B, $0 \leq A < B$ are in contact. Ground is assigned index 0. As in Fig. 1, let i identify the contact event between these two bodies. A collision detection process produces the point of contact P, a signed distance function Φ_i, and a set of three orthonormal vectors: $\mathbf{n}_i$, $\mathbf{u}_i$, and $\mathbf{w}_i$. By convention, the normal vector $\mathbf{n}_i$ is oriented from the body of lower index; i.e., A, to the body of high index; i.e., B.

Any two bodies that are closer than a prescribed $\delta_K \geq 0$ are considered to produce an active contact event. The gap function Φ_i is negative if the two bodies share more than one point; it is zero, if they share one point; it is greater than zero, if they share no point. The geometry of the bodies is assumed to be convex in a neighborhood of the contact area.

In each configuration $\mathbf{q}(t)$, the collection of N_K contacts is denoted by $\mathcal{A}(\mathbf{q}(t), \delta_K)$; it potentially changes at each time step $t^{(l)}$. The force acting on body B at point P is then $\mathbf{F}_{i,B} = \hat{\gamma}_{i,n}\mathbf{n}_i + \hat{\gamma}_{i,u}\mathbf{u}_i + \hat{\gamma}_{i,w}\mathbf{w}_i = \mathbf{A}_i\hat{\boldsymbol{\gamma}}_i$. The location of point P on body B is $\mathbf{r}_B^P = \mathbf{r}_B + \mathbf{A}_B\bar{\mathbf{s}}_{i,B}$ and its virtual displacement is $\delta\mathbf{r}_B^P = \delta\mathbf{r}_B - \mathbf{A}_B\tilde{\bar{\mathbf{s}}}_{i,B}\delta\bar{\pi}_B$, where $\mathbf{r}_B$ is the location of center of mass of body B and $\mathbf{A}_B$ is the orientation matrix of body B. Moreover, the tilde operator produces the skew symmetric matrix associated with the vector it is used in conjunction with, and the vector $\delta\bar{\pi}_B$ is the virtual rotation associated with body B. The virtual work associated with the frictional contact force $\mathbf{F}_{i,B}$ is

$$\delta\mathcal{W}_{i,B} = [\delta\mathbf{r}_B^P]^T\ \mathbf{F}_{i,B} = \delta\mathbf{r}_B^T\mathbf{A}_i\hat{\boldsymbol{\gamma}}_i + \delta\bar{\pi}_B^T\tilde{\bar{\mathbf{s}}}_{i,B}\mathbf{A}_B^T\mathbf{A}_i\hat{\boldsymbol{\gamma}}_i$$

Similarly, the virtual work for body A, is

$$\delta\mathcal{W}_{i,A} = [\delta\mathbf{r}_A^P]^T\ (-\mathbf{F}_{i,B}) = -\delta\mathbf{r}_A^T\mathbf{A}_i\hat{\boldsymbol{\gamma}}_i - \delta\bar{\pi}_A^T\tilde{\bar{\mathbf{s}}}_{i,A}\mathbf{A}_A^T\mathbf{A}_i\hat{\boldsymbol{\gamma}}_i$$

Then the virtual work that the presence of the frictional contact force $\mathbf{F}_{i,B}$ imparts is

$$\delta\mathcal{W}_i = \mathcal{W}_{i,A} + \mathcal{W}_{i,B} = \delta\mathbf{r}^T\mathbf{D}_i\hat{\boldsymbol{\gamma}}_i\ ,$$

where

$$\delta\mathbf{r} = \begin{bmatrix} \delta\mathbf{r}_1 \\ \delta\bar{\pi}_1 \\ \vdots \\ \delta\mathbf{r}_{N_b} \\ \delta\bar{\pi}_{N_b} \end{bmatrix} \in \mathbb{R}^{6N_b} \qquad \mathbf{D}_i = \begin{bmatrix} \mathbf{0}_{3\times 3} \\ \vdots \\ \mathbf{0}_{3\times 3} \\ -\mathbf{A}_i \\ -\tilde{\bar{\mathbf{s}}}_{i,A}\mathbf{A}_A^T\mathbf{A}_i \\ \mathbf{0}_{3\times 3} \\ \vdots \\ \mathbf{0}_{3\times 3} \\ \mathbf{A}_i \\ \tilde{\bar{\mathbf{s}}}_{i,B}\mathbf{A}_B^T\mathbf{A}_i \\ \mathbf{0}_{3\times 3} \\ \vdots \\ \mathbf{0}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{6N_b\times 3}\,, \tag{2}$$

and therefore the generalized force associated with the frictional contact force is $\mathbf{D}_i\boldsymbol{\gamma}_i$.

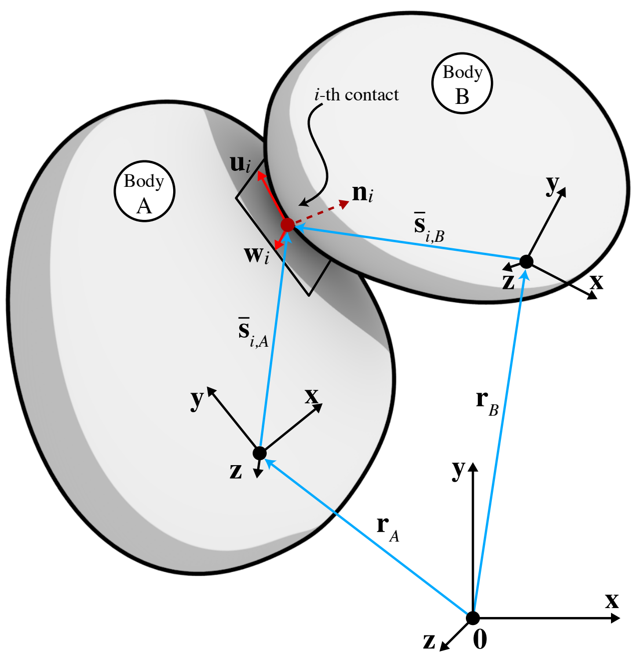

Figure 1: Bodies A and B in contact; a local reference frame $\{\mathbf{n}_i,\ \mathbf{u}_i,\ \mathbf{w}_i\}$ is generated at the contact point based on contact detection information. The contact point is located in the ceontroidal and principal reference frames via the $\mathbf{s}_{i,A}$ and $\mathbf{s}_{i,B}$ constant vectors.

The equations of motion assume the form [4]

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{F}(\mathbf{q},\mathbf{v},t) + \mathbf{G}\hat{\boldsymbol{\lambda}}^B + \mathbf{D}\hat{\boldsymbol{\gamma}}^K\,, \tag{3}$$

where $\mathbf{M} = diag\{m_1\mathbf{I}_{3\times3}, \bar{\mathbf{J}}_1, \ldots, m_{N_b}\mathbf{I}_{3\times3}, \bar{\mathbf{J}}_{N_b}\}$ is the constant mass matrix, $\mathbf{F}(\mathbf{q}, \mathbf{v}, t)$ is the generalized applied and Corriolis forces/torques, $\mathbf{G}\hat{\boldsymbol{\lambda}}^B$ is the constraint reaction force associated with bilateral constraints, and $\mathbf{D}\hat{\boldsymbol{\gamma}}^K$ is the frictional contact force associated with the presence of N_K contact events. In terms of notation, a 3D vector quantity with an overbar, such as the angular velocity $\bar{\boldsymbol{\omega}}_A$ of body A, indicates a representation of a geometric vector in the local (body-attached) centroidal and principal reference frame.

$$\mathbf{v} = \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \bar{\boldsymbol{\omega}}_1 \\ \cdots \\ \dot{\mathbf{r}}_{N_b} \\ \bar{\boldsymbol{\omega}}_{N_b} \end{bmatrix} \in \mathbb{R}^{6N_b} \ , \quad \mathbf{D} \equiv [\mathbf{D}_1 \ldots \mathbf{D}_{N_K}] \in \mathbb{R}^{6N_b \times 3N_K} \ , \quad \mathbf{G} \equiv [\mathbf{G}_1 \ldots \mathbf{G}_{N_B}] \in \mathbb{R}^{6N_b \times N_B} \tag{4}$$

and

$$\hat{\boldsymbol{\gamma}}^K \equiv \begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_{N_K} \end{bmatrix} \in \mathbb{R}^{3N_K} \qquad \hat{\boldsymbol{\lambda}}^B \equiv \begin{bmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_{N_B} \end{bmatrix} \in \mathbb{R}^{N_B} \tag{5}$$

The bilateral constraint reaction forces are associated with the presence of a collection of bilateral constraints. These can be holonomic or non-holonomic; without loss of generality, they are assume here to be holonomic and assume the form

$$g_j(\mathbf{q}, t) = 0 \tag{6a}$$

where $1 \leq j \leq N_B$. The time derivative of the constraints in Eq. (6a) assume the form

$$\dot{g}_j(\mathbf{q}, \mathbf{v}, t) \equiv \mathbf{G}_j^T \mathbf{v} + \frac{\partial g_j}{\partial t} = 0 \tag{6b}$$

Finally, note that

$$\begin{aligned} \mathbf{D}_i^T \mathbf{v} = & - \mathbf{A}_i^T \dot{\mathbf{r}}_A + \mathbf{A}_i^T \dot{\mathbf{r}}_B + \mathbf{A}_i^T \mathbf{A}_A \tilde{\bar{\mathbf{s}}}_{i,A} \bar{\boldsymbol{\omega}}_A - \mathbf{A}_i^T \mathbf{A}_B \tilde{\bar{\mathbf{s}}}_{i,B} \bar{\boldsymbol{\omega}}_B \\ = & \mathbf{A}_i^T (\dot{\mathbf{r}}_B + \mathbf{A}_B \tilde{\bar{\boldsymbol{\omega}}}_B \bar{\mathbf{s}}_{i,A} - \dot{\mathbf{r}}_A - \mathbf{A}_A \tilde{\bar{\boldsymbol{\omega}}}_A \bar{\mathbf{s}}_{i,A}) \\ = & \mathbf{A}_i^T (\dot{\mathbf{r}}_{i,B} - \dot{\mathbf{r}}_{i,A}) \equiv \begin{bmatrix} v_{i,n} \\ v_{i,u} \\ v_{i,w} \end{bmatrix} \ , \end{aligned} \tag{7}$$

represents the relative velocity at the contact point between the two bodies expressed in the local reference frame $\{\mathbf{n}_i, \, \mathbf{u}_i, \, \mathbf{w}_i\}$.

2 Frictional Contact Model

The frictional contact model considered is the Coulomb dry friction model. It states that for a contact event i, the friction force can assume any value between 0 and $\mu_i \hat{\gamma}_{i,n}$, where $\hat{\gamma}_{i,n}$

is the local normal contact force. Impacts are considered to be inelastic; i.e., the restitution coefficient is zero. The Coulomb dry friction model states that the following three conditions hold simultaneously:

$$
\begin{aligned}
0 \le \Phi_i \quad &\perp \quad \hat{\gamma}_{i,n} \ge 0 && \text{(8a)}\\
0 \le \sqrt{v_{i,u}^2 + v_{i,w}^2} \quad &\perp \quad (\mu_i \hat{\gamma}_{i,n} - \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2}) \ge 0 && \text{(8b)}\\
\exists\, \alpha_i \ge 0 \quad &: \quad \begin{cases} v_{i,u} = -\alpha_i \hat{\gamma}_{i,u} \\ v_{i,w} = -\alpha_i \hat{\gamma}_{i,w} \end{cases} && \text{(8c)}
\end{aligned}
$$

3 The Discretized Equations of Motion

The discretization scheme adopted is a half implicit symplectic Euler method [3], like the one used in [7]. It is used to discretize the kinematic differential equations in Eq. (1), the Newton-Euler equations of motion in Eq. (3), and the Coulomb friction model stated in Eq. (8). This yields the following nonlinear complementarity problem:

$$
\begin{aligned}
\overbrace{\mathbf{q}^{(l+1)}}^{\text{generalized positions}} &= \mathbf{q}^{(l)} + \overbrace{\Delta t}^{\text{step size}} \underbrace{\mathbf{L}(\mathbf{q}^{(l)})}_{\text{velocity transformation matrix}} \mathbf{v}^{(l+1)} && \text{(9a)}\\
\mathbf{M}(\overbrace{\mathbf{v}^{(l+1)}}^{\text{gen. speeds}} - \mathbf{v}^{(l)}) &= \underbrace{\mathbf{f}^{(l)}}_{\text{applied impulse}} + \overbrace{\mathbf{G}^{(l)} \boldsymbol{\lambda}^{B,(l+1)}}^{\text{reaction impulse}} + \underbrace{\mathbf{D}^{(l)} \boldsymbol{\gamma}^{K,(l+1)}}_{\text{frictional contact impulse}} && \text{(9b)}\\
0 &= \underbrace{\frac{1}{\Delta t} \mathbf{g}^{(l)}}_{\text{stabilization term}} + \mathbf{G}^{(l),T} \mathbf{v}^{(l+1)} + \mathbf{g}_t^{(l)} && \text{(9c)}
\end{aligned}
$$

$$
\left.
\begin{aligned}
0 \le \gamma_{i,n}^{(l+1)} \quad &\perp \quad (\overbrace{\frac{1}{\Delta t}\Phi_i^{(l)}}^{\text{stabilization term}} + v_{i,n}^{(l+1)} \overbrace{- \mu_i \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}}}^{\text{relaxation term}}) \ge 0\\
0 \le \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}} \quad &\perp \quad (\mu_i \gamma_{i,n}^{(l+1)} - \sqrt{\gamma_{i,u}^{2,(l+1)} + \gamma_{i,w}^{2,(l+1)}}) \ge 0\\
\exists\, \alpha_i \ge 0 \quad &: \begin{cases} v_{i,u}^{(l+1)} = -\alpha_i \gamma_{i,u}^{(l+1)} \\ v_{i,w}^{(l+1)} = -\alpha_i \gamma_{i,w}^{(l+1)} \end{cases}
\end{aligned}
\right\} \forall\, i \in \mathcal{A}^{(l)} \text{(9d)}
$$

where $\mathbf{f}^{(l)} \equiv \Delta t\, \mathbf{F}(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)})$; $\boldsymbol{\gamma}^{K,(l+1)} \equiv \Delta t\, \hat{\boldsymbol{\gamma}}^{K,(l+1)}$; and, $\boldsymbol{\lambda}^{B,(l+1)} \equiv \Delta t\, \hat{\boldsymbol{\lambda}}^{B,(l+1)}$. Moreover, $\mathbf{G}^{(l)} \equiv \mathbf{G}(\mathbf{q}^{(l)}, t^{(l)})$, and $\mathbf{D}^{(l)} \equiv \mathbf{D}(\mathbf{q}^{(l)}, t^{(l)})$. In Eq. (9c), $\mathbf{g}^{(l)} \equiv \mathbf{g}(\mathbf{q}^{(l)}, t^{(l)})$ and $\mathbf{g}_t^{(l)} \equiv \frac{\partial \mathbf{g}(\mathbf{q}^{(l)}, t^{(l)})}{\partial t}$. Finally, in Eq. (9d), $\Phi_i^{(l)} \equiv \Phi_i(\mathbf{q}^{(l)})$ and $\mathcal{A}^{(l)}$ is the set of active contact events produced by the collision detection step carried out at $t^{(l)}$ in the configuration $\mathbf{q}^{(l)}$.

There are two notable aspects tied to the discretization of the differential variational inclusion problem above.

1. The bilateral kinematic constraint equations, see Eq. (6a), are not used. Instead, we use the velocity-level set of kinematic constraints in Eq. (6b). However, the latter are modified in two respects. First, to account for violation in satisfying the kinematic constraints at position level, a "stabilization term" is considered in the discretized form of the equation. Applying this type of stabilization is inspired by Baugmarte's method [1]. A similar idea has been recently used in [5]. Second, since the method is half-implicit, we chose to evaluate the partial time derivate $\mathbf{g}_t$ in the configuration $(\mathbf{q}^{(l)}, t^{(l)})$

2. When discretizing the expression of the signed gap function in Eq. (9d), there are two approximations involved in the process. First, the complementarity conditions is imposed using an approximation of the signed gap function at $t^{(l+1)}$:

   $$\forall\, i \in \mathcal{A}^{(l)} \;:\quad \Phi_i^{(l+1)} \approx \Phi_i^{(l)} + \Delta t\, v_{i,n}^{(l+1)} \;.$$

   Second, to render this nonlinear complementarity problem tractable, a relaxation of the approximation above is introduced via the term $-\mu_i\sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}}$. As shown below, this change allows us to pose the problem in Eq. (9) as a Cone Complementarity Problem (CCP).

4 The Cone Complementarity Problem

4.1 Posing the problem

The "unilateral constraints" case. The friction cone $\mathcal{K}_i$ associated with contact event i is defined as

$$\mathcal{K}_i \equiv \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \quad 0 \le x \;\text{ and }\; \mu_i x - \sqrt{y^2+z^2} \ge 0 \;.\} \tag{10a}$$

Similarly, the polar cone $\mathcal{K}_i^\circ$ associated with the friction cone $\mathcal{K}_i$ is defined as

$$\mathcal{K}_i^\circ \equiv \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : ax + by + cz \le 0 \;\; \forall \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathcal{K}_i \} \;. \tag{10b}$$

Based on Eq. (9), we have that

$$\boldsymbol{\gamma}_i^{(l+1)} \equiv \begin{bmatrix} \gamma_{i,n}^{(l+1)} \\ \gamma_{i,u}^{(l+1)} \\ \gamma_{i,w}^{(l+1)} \end{bmatrix} \in \mathcal{K}_i \;.$$

Define next

$$\mathbf{d}_i \equiv \begin{bmatrix} \frac{1}{\Delta t}\Phi_i^{(l)} + v_{i,n}^{(l+1)} \\ v_{i,u}^{(l+1)} \\ v_{i,w}^{(l+1)} \end{bmatrix} .$$

Then, using Eq. (9),

$$\begin{aligned}
\mathbf{d}_i^T \cdot \boldsymbol{\gamma}_i^{(l+1)} &= \boldsymbol{\gamma}_i^{(l+1)} (\frac{1}{\Delta t}\Phi_i^{(l)} + v_{i,n}^{(l+1)}) + \gamma_{i,u}^{(l+1)}\, v_{i,u}^{(l+1)} + \gamma_{i,w}^{(l+1)}\, v_{i,w}^{(l+1)} \\
&= \mu_i \gamma_{i,n}^{(l+1)} \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}} + v_{i,u}^{(l+1)} \gamma_{i,u}^{(l+1)} + v_{i,w}^{(l+1)} \gamma_{i,w}^{(l+1)} \\
&= \sqrt{\gamma_{i,u}^{2,(l+1)} + \gamma_{i,w}^{2,(l+1)}}\, \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}} + v_{i,u}^{(l+1)} \gamma_{i,u}^{(l+1)} + v_{i,w}^{(l+1)} \gamma_{i,w}^{(l+1)} \\
&= \alpha_i \sqrt{\gamma_{i,u}^{2,(l+1)} + \gamma_{i,w}^{2,(l+1)}} - \alpha_i \sqrt{\gamma_{i,u}^{2,(l+1)} + \gamma_{i,w}^{2,(l+1)}} \\
&= 0 ,
\end{aligned}$$

and therefore $\boldsymbol{\gamma}_i^{(l+1)} \perp \mathbf{d}_i$.

Next, we show that $-\mathbf{d}_i \in \mathcal{K}_i^\circ$; i.e., that $\mathbf{d}_i^T \cdot \mathbf{p} \geq 0$. To this end, take an arbitrary $\mathbf{p} = [x\ y\ z]^T \in \mathcal{K}_i$. If $x = 0$, then $y = z = 0$ and $\mathbf{d}_i^T \cdot \mathbf{p} \geq 0$. If $x > 0$, then we can scale $\mathbf{p}$ by a constant $\beta > 0$ such that $x = \gamma_{i,n}^{(l+1)}$. Note that this scaling does not change the sign of the dot product $\mathbf{d}_i^T \cdot \mathbf{p}$. We assume $\gamma_{i,n}^{(l+1)} > 0$ since the case $\gamma_{i,n}^{(l+1)} = 0$ is trivial. Then, using that $\mu_i \gamma_{i,n}^{(l+1)} \geq \sqrt{b^2 + c^2}$ and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\mathbf{d}_i^T \cdot \mathbf{p} &= \gamma_{i,n}^{(l+1)}\, (\frac{1}{\Delta t}\Phi_i^{(l)} + v_{i,n}^{(l+1)}) + v_{i,u}^{(l+1)}\, b + v_{i,w}^{(l+1)}\, c \\
&= \mu_i \gamma_{i,n}^{(l+1)} \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}} + v_{i,u}^{(l+1)}\, b + v_{i,w}^{(l+1)}\, c \\
&\geq \sqrt{b^2 + c^2}\, \sqrt{v_{i,u}^{2,(l+1)} + v_{i,w}^{2,(l+1)}} + v_{i,u}^{(l+1)}\, b + v_{i,w}^{(l+1)}\, c \\
&\geq 0 .
\end{aligned}$$

We thus conclude that we can equivalently express the conditions in Eq. (9) as the following CCP:

$$\mathcal{K}_i \ni \boldsymbol{\gamma}_i^{(l+1)} \quad \perp \quad -\mathbf{d}_i \in \mathcal{K}_i^\circ \tag{11}$$

The "bilateral constraints" case. Let $0 \leq j \leq N_B$. For each bilateral kinematic constraint equation j, define

$$b_j \equiv \frac{1}{\Delta t}\, g_j^{(l)} + \mathbf{G}_j^{(l),T} \mathbf{v}^{(l+1)} + \frac{\partial g_j^{(l)}}{\partial t} .$$

In the light of Eq. (9c), one has

$$0 = b_j \perp \lambda_j^{(l+1)} \in \mathbb{R}$$

Following in the steps of the argument made for the unilateral constraints, one can define the cone $\mathcal{B}_j \equiv \mathbb{R}$ and the polar cone $\mathcal{B}_j^\circ \equiv \{y : \quad x \cdot y \leq 0 \;\; \forall\, x \in \mathcal{B}_j\}$. Note that this polar cone set has only one element: $\mathcal{B}_j^\circ = \{0\}$. Therefore, we can reformulate the condition in Eq. (9c) as

$$\mathcal{B}_j \ni \lambda_j^{(l+1)} \quad \perp \quad -b_j \in \mathcal{B}_j^\circ \;, \tag{12}$$

which is the "bilateral constraint" CCP analog of the condition in Eq. (11).

4.2 Reformulating the CCP

The next goal is to eliminate any dependency on the unknown velocity $\mathbf{v}^{(l+1)}$ in the Cone Complementarity Problems of Eqs. (11) and (12). To this end, using the force balance condition stated in Eq. (9b), one has that

$$\mathbf{v}^{(l+1)} = \mathbf{v}^{(l)} + \mathbf{M}^{-1}\,\mathbf{f}^{(l)} + \mathbf{M}^{-1}\,\mathbf{G}^{(l)}\boldsymbol{\lambda}^{B,(l+1)} + \mathbf{M}^{-1}\,\mathbf{D}^{(l)}\boldsymbol{\gamma}^{K,(l+1)} \;. \tag{13}$$

Let

$$\begin{aligned}
\mathbf{d}_{i,0} &\equiv \begin{bmatrix} \frac{1}{\Delta t}\Phi_i^{(l)} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3 \\
\mathbf{d}_{i,1} &\equiv \mathbf{d}_{i,0} + \mathbf{D}_i^{(l),T}(\mathbf{v}^{(l)} + \mathbf{M}^{-1}\,\mathbf{f}^{(l)}) \in \mathbb{R}^3 \\
b_{j,0} &\equiv \frac{1}{\Delta t}\,g_j^{(l)} + \frac{\partial g_j^{(l)}}{\partial t} + \mathbf{G}_j^{(l),T}(\mathbf{v}^{(l)} + \mathbf{M}^{-1}\,\mathbf{f}^{(l)}) \in \mathbb{R} \;.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{d}_i &= \mathbf{d}_{i,0} + \mathbf{D}_i^{(l),T}\mathbf{v}^{(l+1)} \\
&= \mathbf{d}_{i,1} + \mathbf{D}_i^{(l),T}\mathbf{M}^{-1}\,\mathbf{G}^{(l)}\boldsymbol{\lambda}^{B,(l+1)} + \mathbf{D}_i^{(l),T}\mathbf{M}^{-1}\,\mathbf{D}^{(l)}\boldsymbol{\gamma}^{K,(l+1)} \;, \text{and} \\
b_j &= b_{j,0} + \mathbf{G}_j^{(l)}\mathbf{M}^{-1}\,\mathbf{G}^{(l)}\boldsymbol{\lambda}^{B,(l+1)} + \mathbf{G}_j^{(l)}\mathbf{M}^{-1}\,\mathbf{D}^{(l)}\boldsymbol{\gamma}^{K,(l+1)} \;.
\end{aligned}$$

Define

$$\mathbf{P} \equiv \begin{bmatrix}\mathbf{D}^{(l)} & \mathbf{G}^{(l)}\end{bmatrix} \in \mathbb{R}^{6n_b \times (3N_K + N_B)} \;, \;\; \boldsymbol{\nu}^{(l+1)} \equiv \begin{bmatrix}\boldsymbol{\gamma}^{K,(l+1)} \\ \boldsymbol{\lambda}^{B,(l+1)}\end{bmatrix} \in \mathbb{R}^{3N_K + N_B} \;, \;\; \mathbf{p} \equiv \begin{bmatrix} \mathbf{d}_{1,1} \\ \vdots \\ \mathbf{d}_{N_K,1} \\ b_{1,0} \\ \vdots \\ b_{N_B,0} \end{bmatrix} \in \mathbb{R}^{3N_K + N_B}.$$

Then, the terms entering the CCPs both for unilateral and bilateral constraints can be expressed without any recurse to the velocity $\mathbf{v}^{(l+1)}$:

$$\begin{aligned}
\mathbf{d}_i &= \mathbf{d}_{i,1} + \mathbf{D}_i^{(l),T}\mathbf{M}^{-1}\mathbf{P}\boldsymbol{\nu}^{(l+1)} \;, \qquad \text{and} \\
b_j &= b_{j,0} + \mathbf{G}_j^{(l)}\mathbf{M}^{-1}\mathbf{P}\boldsymbol{\nu}^{(l+1)} \;.
\end{aligned}$$

Therefore, we have a collection of CCPs that can be generically represented as

$$\mathcal{C}_k \ni \boldsymbol{\nu}_k^{(l+1)} \quad \perp \quad -(\mathbf{p} + \mathbf{N}\boldsymbol{\nu}^{(l+1)})_k \in \mathcal{C}_k^\circ \ , \tag{14}$$

where $\mathbf{N} \equiv \mathbf{P}^T\mathbf{M}^{-1}\mathbf{P}$, $\mathcal{C} \equiv \mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_{N_K} \oplus \mathcal{B}_1 \oplus \ldots \oplus \mathcal{B}_{N_B}$, and $\mathcal{C}^\circ \equiv \mathcal{K}_1^\circ \oplus \ldots \oplus \mathcal{K}_{N_K}^\circ \oplus \mathcal{B}_1^\circ \oplus \ldots \oplus \mathcal{B}_{N_B}^\circ$.

5 The Quadratic Problem Angle

We show next that the CCP stated in Eq. (14) represents the first order optimality conditions [2] for the convex quadratic optimization problem with conic constraints

$$\begin{aligned} \boldsymbol{\nu}^{(l+1)} = \min_{\boldsymbol{\nu}} \frac{1}{2}\boldsymbol{\nu}^T\mathbf{N}\boldsymbol{\nu} + \mathbf{p}^T\boldsymbol{\nu} \\ \text{subject to } \boldsymbol{\nu}_k \in \mathcal{C}_k \ . \end{aligned} \tag{15}$$

To prove this statement, formulate first the Karush-Kuhn-Tucker (KKT) optimality conditions for the problem above. To this end, the associated Lagrangian is defined as [2]

$$\mathcal{L}(\boldsymbol{\nu}, \boldsymbol{\psi}, \boldsymbol{\phi}) = \frac{1}{2}\boldsymbol{\nu}^T\mathbf{N}\boldsymbol{\nu} + \mathbf{p}^T\boldsymbol{\nu} + \sum_{i=1}^{N_K} \psi_i\left(\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} - \mu_i\gamma_{i,n}\right) + \sum_{j=1}^{N_B} \phi_j\lambda_i \ ,$$

where $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ are dummy Lagrange multipliers of appropriate dimensions. The first order optimality conditions assume the form

$$\text{KKT:} \quad \begin{cases} \qquad\qquad \nabla_{\boldsymbol{\nu}}\mathcal{L} \ = \mathbf{0}_{3N_K+N_B} \\ \text{For } 1 \le i \le N_K \ : \ 0 \le \psi_i \quad \perp \quad \mu_i\gamma_{i,n} - \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \ge 0 \\ \text{For } 1 \le j \le N_B \ : \ \phi_j \, \lambda_j = 0 \end{cases} . \tag{16}$$

The first condition above leads to two sets of equalities. First, for $1 \le i \le N_K$, the gradient with respect to $\boldsymbol{\gamma}_i$ yields

$$\mathbf{D}_i^{(l),T}\mathbf{M}^{-1}\mathbf{P}\boldsymbol{\nu} + \mathbf{d}_{i,1} + \psi_i \begin{bmatrix} -\mu_i \\ \frac{\gamma_{i,u}}{\sqrt{\gamma_{i,u}^2+\gamma_{i,w}^2}} \\ \frac{\gamma_{i,w}}{\sqrt{\gamma_{i,u}^2+\gamma_{i,w}^2}} \end{bmatrix} = \mathbf{0}_3 \ , \tag{17}$$

which leads to $\mathbf{d}_i^T = \psi_i[-\mu_i \quad \frac{\gamma_{i,u}}{\sqrt{\gamma_{i,u}^2+\gamma_{i,w}^2}} \quad \frac{\gamma_{i,w}}{\sqrt{\gamma_{i,u}^2+\gamma_{i,w}^2}}]$. Therefore, using the first set of complementarity conditions in Eq. (16),

$$\begin{aligned} \mathbf{d}_i^T\boldsymbol{\gamma}_i &= -\psi_i \left(-\mu_i\gamma_{i,n} + \frac{\gamma_{i,u}^2 + \gamma_{i,w}^2}{\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2}} \right) \\ &= \psi_i \left(\mu_i\gamma_{i,n} - \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \right) \\ &= 0 \ . \end{aligned}$$

Next, let a vector $\boldsymbol{\alpha} \equiv [a \;\; b \;\; c]^T \in \mathcal{K}_i$. We want to show that $-\boldsymbol{\alpha}^T \mathbf{d}_i \leq 0$, or equivalently, $\boldsymbol{\alpha}^T \mathbf{d}_i \geq 0$. First, note that if $a = 0$, then $b = c = 0$, and therefore $\boldsymbol{\alpha}^T \mathbf{d}_i = 0$. Otherwise, $a > 0$, and then

$$\mathbf{d}_i^T \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad a\mu_i + \frac{b\gamma_{i,u} + c\gamma_{i,w}}{\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2}} \geq 0 \quad \Leftrightarrow \quad \mu_i a \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} + b\gamma_{i,u} + c\gamma_{i,w} \geq 0 \, .$$

Since $\mu_i a \geq \sqrt{b^2 + c^2}$, using the Cauchy-Schwartz inequality,

$$\mu_i a \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} + b\gamma_{i,u} + c\gamma_{i,w} \geq \sqrt{b^2 + c^2} \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} + b\gamma_{i,u} + c\gamma_{i,w} \geq 0 \, ,$$

which proves that as far as $\boldsymbol{\gamma}_i$ is concerned, the following holds:

$$\mathcal{K}_i \ni \boldsymbol{\gamma}_i \quad \perp \quad -\mathbf{d}_i \in \mathcal{K}_i^\circ \, .$$

The condition above indicates that a $\boldsymbol{\gamma}_i$ that satisfies the KKT conditions in Eq. (16) is a solution of the CCP problem in Eq. (11).

A similar result can be obtained in the bilateral constraints case. Indeed, in this case $b_j = -\phi_j$. Using the last complementarity condition in Eq. (16), one has that whenever $\lambda_j \neq 0$, necessarily $b_j = 0$. In other words, we have that

$$\mathcal{B}_j \ni \lambda_j \quad \perp \quad -b_j \in \mathcal{B}_j^\circ \, ,$$

which indicates that a λ_j that satisfies the first KKT conditions in Eq. (16) is a solution of the CCP problem in Eq. (12).

Note that the dynamics step if essentially done once $\boldsymbol{\nu}_k^{(l+1)}$ is computed. Indeed, the new velocity is evaluated using Eq. (13), while the new position is obtained via Eq. (9a).

6 Conclusions and Future Work

A differential inclusion approach has been used to pose the equations of motion for a set of rigid bodies that mutually interact through friction and contact. The cornerstone of this approach is posing (*i*) a non-penetration condition, and (*ii*) the Coulomb dry friction model, using complementarity conditions. Upon discretization of the resulting differential inclusion one obtains a nonlinear complementarity problem that. The discretization is based on a half-implicit symplectic Euler scheme and has two salient attributes. First, the unilateral and bilateral kinematic constraints are imposed at the velocity level. Drift in the position-level constraints is prevented in Eq. (9) via "stabilization terms". Second, the non-penetration unilateral constraint, formulated at the velocity level, is further modified via a "relaxation term" to morph what would otherwise be a non-linear complementarity problem into a cone complementarity problem. The latter has a solution that is produced by solving of a convex quadratic optimization problem with conic constraints.

There are three aspects in which the method described can be improved. First, a better approach would enforce the unilateral and bilateral constraints at the position level. This would address numerous models in which the enforcement of the bilateral constraints was not tight enough. Second, the NCP-to-CCP relaxation is known to produce under certain scenarios numerical artifacts – see discussions in [6, 7]. Third, the rigid body assumption leads to scenarios in which, due to the presence of redundant constraints, the matrix $\mathbf{P}$ in Eq. (15) is symmetric positive semi-definite. As such, a solution of the convex optimization problem with conic constraints, while global, is not unique. There are early indications that these three limitations can be addressed. A discussion of this issue falls outside the scope of this document.

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