Technical Report TR05

An Introduction to the Floating Frame of Reference Formulation

for Small Deformation in Flexible Multibody Dynamics

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Abstract

This technical report describes the basis of the nonlinear finite element floating frame of reference formulation.

Keywords: Floating Frame of Reference Formulation, Nonlinear Finite Elements, Flexible Multibody Systems

Contents

1 Introduction

The idea of a floating frame of reference (FFR) may have its origin in the analysis of astrodynamic systems. Such systems experience strong coupling of inertia forces, due to high accelerations in its reference (rigid body) motion, and flexibility due to the light nature of the structures involved (see Ref. [2]). The idea of a body-wise co-rotated frame or floating frame can be summarized as the splitting of the overall motion of bodies that experience small deformation into a frame that captures the dynamics of mean rigid body motion and a superposed flexible motion that defines the deformation state. This method requires one flexible body to have one single FFR to describe its dynamics, in contrast with the co-rotational formulation. In addition, the fact that the body's (structure's) elasticity can be defined via a constant stiffness matrix opens up the possibility of using general model order reduction techniques to reduce the order of a linear finite element model. In what follows, the basis of the FFR formulation is presented, whereas many derivations are left to the reader as an exercise or given as a reference.

2 Floating Frame of Reference

2.1 Kinematics of the FFR

From now on, we will assume that Euler parameters will be our choice of rotation coordinates. Even though other choices can be made, it is, in any case, important to guarantee that singularities will be avoided. The kinematic state of the frame of reference of a body i may be written as

$$\boldsymbol{q}_r^i = \begin{bmatrix} \boldsymbol{R}^{i\mathrm{T}} & \boldsymbol{\theta}^{i\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{1}$$

where $\boldsymbol{q}_r^i$ is a set of Cartesian coordinates that locates the origin of the FFR of body i, O^i, with respect to the global frame, $\boldsymbol{\theta}^i$ is a set of Euler parameters that describe the orientation of such a frame in the global coordinate system. A point P in body i can be described, using FFR kinematics, as

$$\boldsymbol{r}_P^i = \boldsymbol{R}^i + \boldsymbol{A}^i\left(\boldsymbol{\theta}^i\right)\bar{\boldsymbol{u}}^P, \tag{2}$$

where $\boldsymbol{A}^i$ denotes the orientation matrix of the FFR and $\bar{\boldsymbol{u}}^P$ is the vector that defines the position of point P with respect to the FFR (superscript i is dropped for simplicity). This vector can be split into its reference position and its elastic displacement as follows

$$\bar{\boldsymbol{u}}^P = \bar{\boldsymbol{u}}_o^P + \bar{\boldsymbol{u}}_f^P. \tag{3}$$

$\bar{\boldsymbol{u}}_o^P$ represents the undeformed position of P, whereas its elastic displacement, measured in the FFR, is denoted by the vector $\bar{\boldsymbol{u}}_f^P$ (see Fig. 1). The elastic displacement vector $\bar{\boldsymbol{u}}_f^P$ may be decomposed into the product of a space-dependent matrix and a vector of time-dependent flexible coordinates, as follows

$$\bar{\boldsymbol{u}}_f^P = \boldsymbol{S}^i\left(\bar{\boldsymbol{u}}_o^P\right)\boldsymbol{q}_f^i, \tag{4}$$

where $\boldsymbol{S}^i(\bar{\boldsymbol{u}}_o^P)$ is a shape function matrix referred to the reference (undeformed) configuration and $\boldsymbol{q}_f^i$ is the vector of flexible coordinates of body i. By spelling out the kinematic equations in (3) and (4), the expression of the deformed position of the point P in body i may be written as

$$\boldsymbol{r}_P^i = \boldsymbol{R}^i + \boldsymbol{A}^i \left(\bar{\boldsymbol{u}}_o^P + \boldsymbol{S}^i \boldsymbol{q}_f^i \right) \tag{5}$$

Velocity expressions of a deformed point P within the FFR formulation are given below:

$$\dot{\bar{\boldsymbol{u}}}_f^P = \boldsymbol{S}^i \left(\bar{\boldsymbol{u}}_o^P \right) \dot{\boldsymbol{q}}_f^i, \tag{6}$$

$$\dot{\boldsymbol{r}}_P^i = \dot{\boldsymbol{R}}^i + \dot{\boldsymbol{A}}^i \left(\bar{\boldsymbol{u}}_o^P + \boldsymbol{S}^i \boldsymbol{q}_f^i \right) + \boldsymbol{A}^i \boldsymbol{S}^i \dot{\boldsymbol{q}}_f^i. \tag{7}$$

Note that Eqs. (6)-(7) take into account that $\bar{\boldsymbol{u}}_o^P$ and $\boldsymbol{S}^i$ do not depend on time.

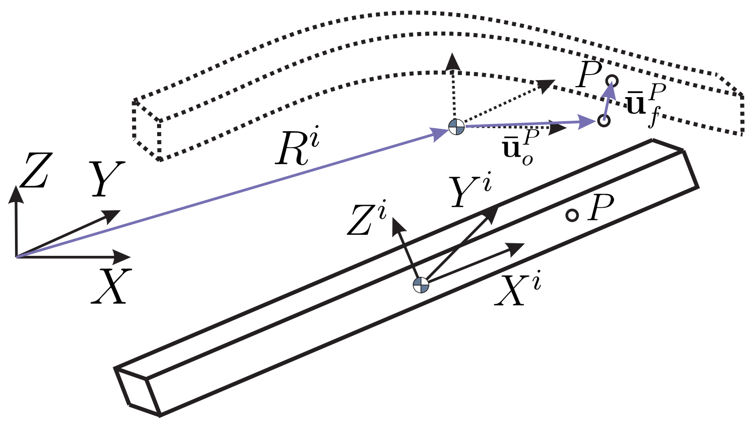

Figure 1: Basic kinematics of the FFR formulation. The floating frame axes are denoted by $\langle X^i Y^i Z^i \rangle$, the initial, unstressed configuration is depicted with solid lines, whereas the current configuration –translated, rotated, and deformed– is shown in dashed lines.

2.2 Equations of Motion

Here, we present the derivation equations of motion of the system in abbreviated manner, omitting some derivations. The interested reader can consult the book by Shabana [8] to find out more on some detailed derivations of the FFR formulation equations.

Before moving on with derivations, we arrange, for convenience, the velocity vector in the following form:

$$\dot{\boldsymbol{r}}_P^i = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^i & \boldsymbol{A}^i \boldsymbol{S}^i \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{R}}^i \\ \dot{\boldsymbol{\theta}}^i \\ \dot{\boldsymbol{q}}_f^i \end{bmatrix} = \boldsymbol{L}^i \dot{\boldsymbol{q}}^i, \tag{8}$$

where matrix $\boldsymbol{B}^i$ is a linear operator that acts on the rotation coordinates, as follows $\boldsymbol{A}^i \bar{\boldsymbol{u}}^P = \boldsymbol{B}^i \dot{\boldsymbol{\theta}}^i$, and $\boldsymbol{B}^i = -\boldsymbol{A}^i \tilde{\bar{\boldsymbol{u}}}^P \bar{\boldsymbol{G}}^i$. The tilde $\tilde{\star}$ denotes the skew symmetric matrix operation on a vector $\star$, and $\boldsymbol{G}^i$ is a linear operator that transforms time derivatives of rotation parameters to the angular velocity vector of the reference motion of body i. The form of $\boldsymbol{G}^i$ depends upon the selection of rotational parameters.

The equations of motion of the FFR formulations are to be derived using Lagrange's equations. For a flexible body i described using a floating frame, these equations take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^i}{\partial \dot{\boldsymbol{q}}^i}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \boldsymbol{q}^i}\right)^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{q}^i}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{Q}_e^i + \boldsymbol{Q}_a^i, \tag{9}$$

where T is the kinetic energy, $\boldsymbol{q}^i$ and $\dot{\boldsymbol{q}}^i$ are the generalized coordinates and velocities of body i, respectively, $\boldsymbol{C}_{\boldsymbol{q}^i}$ is the Jacobian of the constraint equations, $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers, and $\boldsymbol{Q}_e^i$ and $\boldsymbol{Q}_a^i$ are the vectors of elastic and applied forces, respectively. Note that, in general, the vectors of Lagrange multipliers and applied forces may link the flexible body i with other (flexible) bodies and/or other interacting systems.

Mass Matrix

The total kinetic energy of body i may be expressed as a volume integral:

$$T^i = \frac{1}{2}\int_{V^i} \rho^i \dot{\boldsymbol{r}}_P^{i\mathrm{T}} \dot{\boldsymbol{r}}_P^i \mathrm{d}V^i \tag{10}$$

which, in terms of the generalized coordinates, may be written as

$$T^i = \frac{1}{2}\boldsymbol{q}^{i\mathrm{T}}\boldsymbol{M}^i\boldsymbol{q}^i. \tag{11}$$

The mass matrix $\boldsymbol{M}^i$ is nonlinear in the rotation coordinates and takes the following form

$$\boldsymbol{M}^i = \int_{V^i} \rho^i \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^i & \boldsymbol{A}^i\boldsymbol{S}^i \\ & \boldsymbol{B}^{i\mathrm{T}}\boldsymbol{B}^i & \boldsymbol{B}^{i\mathrm{T}}\boldsymbol{A}^i\boldsymbol{S}^i \\ sym. & & \boldsymbol{S}^{i\mathrm{T}}\boldsymbol{S}^i \end{bmatrix} \mathrm{d}V^i = \begin{bmatrix} \boldsymbol{m}_{RR} & \boldsymbol{m}_{R\theta} & \boldsymbol{m}_{Rf} \\ & \boldsymbol{m}_{\theta\theta} & \boldsymbol{m}_{\theta f} \\ sym. & & \boldsymbol{m}_{ff} \end{bmatrix}^i, \tag{12}$$

where submatrices have been named for convenience. Cross terms in the mass matrix, e.g. $\boldsymbol{m}_{Rf}$ and $\boldsymbol{m}_{\theta f}$, indicate that coupling between reference and flexible coordinates is captured.

The Quadratic Velocity Vector

The FFR formulation introduces complex inertia terms which represent Coriolis and centrifugal forces. Deriving the inertia terms of Lagrange equations in Eq. (9), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^i}{\partial \dot{\boldsymbol{q}}^i}\right)^{\mathrm{T}} - \left(\frac{\partial T^i}{\partial \boldsymbol{q}^i}\right)^{\mathrm{T}} = \boldsymbol{M}^i\ddot{\boldsymbol{q}}^i + \underbrace{\dot{\boldsymbol{M}}^i\dot{\boldsymbol{q}}^i - \left[\frac{\partial}{\partial \boldsymbol{q}^i}\left(\dot{\boldsymbol{q}}^{i\mathrm{T}}\boldsymbol{M}^i\dot{\boldsymbol{q}}^i\right)\right]^{\mathrm{T}}}_{\boldsymbol{Q}_v^i\text{(Quadratic velocity vector)}} \tag{13}$$

The final expression for the quadratic velocity vector $\boldsymbol{Q}_v^i = \left[(\boldsymbol{Q}_v)_R^{i\mathrm{T}} \quad (\boldsymbol{Q}_v)_\theta^{i\mathrm{T}} \quad (\boldsymbol{Q}_v)_f^{i\mathrm{T}}\right]^\mathrm{T}$, in terms of generalized coordinates, reads

$$\left(\boldsymbol{Q}_v^i\right)_R = -\boldsymbol{A}^i \rho^i \int_{V^i} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\boldsymbol{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \boldsymbol{S}^i \dot{\boldsymbol{q}}_f^i\right] \mathrm{d}V^i \tag{14}$$

$$\left(\boldsymbol{Q}_v^i\right)_\theta = \bar{\boldsymbol{G}}^{i\mathrm{T}} \rho^i \int_{V^i} \left[\tilde{\bar{\boldsymbol{u}}}^{i\mathrm{T}} \left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\boldsymbol{u}}^i + 2\tilde{\bar{\boldsymbol{u}}}^{i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}_i \boldsymbol{S}^i \dot{\boldsymbol{q}}_f^i\right] \mathrm{d}V^i \tag{15}$$

$$\left(\boldsymbol{Q}_v^i\right)_f = -\rho^i \int_{V^i} \boldsymbol{S}^{i\mathrm{T}} \left[\left(\tilde{\bar{\boldsymbol{\omega}}}_i\right)^2 \bar{\boldsymbol{u}}^i + 2\tilde{\bar{\boldsymbol{\omega}}}_i \boldsymbol{S}^i \dot{\boldsymbol{q}}_f^i\right] \mathrm{d}V^i \tag{16}$$

$$\tag{17}$$

and may be obtained by deriving equation (13) or via the Principle of Virtual Work. The Euler parameter identity $\dot{\bar{\boldsymbol{G}}}^i \dot{\boldsymbol{\theta}}^i = \boldsymbol{0}$ is used above to simplify the expressions. Checking the correctness of (14) is left to the reader as an exercise. As an advanced topic, the ready may learn on the accurate calculation of FFR velocity-dependent inertia terms in [9].

Applied Forces

To include forces and moments in the system it is necessary to obtain their generalized counterparts by using FFR kinematics. As an example, here we describe how to compute the generalized force associated with a point force. The virtual work of a force described in the global frame, $\boldsymbol{F}_a^i$, applied at P may be expressed as

$$\delta W_a^i = \boldsymbol{F}_a^i \delta \boldsymbol{r}_P^i = \boldsymbol{Q}_a^i \delta \boldsymbol{q}^i. \tag{18}$$

As a first step, we write the variation of the position vector of a point P in an FFR body i as

$$\delta \boldsymbol{r}_P^i = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{A}^i \tilde{\bar{\boldsymbol{u}}}^i \bar{\boldsymbol{G}}^i & \boldsymbol{A}^i \boldsymbol{S}^i \end{bmatrix} \begin{bmatrix} \delta \boldsymbol{R}^i \\ \delta \boldsymbol{\theta}^i \\ \delta \boldsymbol{q}_f^i \end{bmatrix}. \tag{19}$$

Equation (19) can be plugged into (18) to yield the following generalized force vector

$$\left(\boldsymbol{Q}_a^i\right)_R = \boldsymbol{F}_a^P, \; \left(\boldsymbol{Q}_a^i\right)_\theta = -\bar{\boldsymbol{G}}^{i\mathrm{T}} \tilde{\bar{\boldsymbol{u}}}^{i\mathrm{T}} \boldsymbol{A}^{i\mathrm{T}} \boldsymbol{F}_a^P, \; \left(\boldsymbol{Q}_a^i\right)_f = \boldsymbol{S}^{i\mathrm{T}} \boldsymbol{A}^{i\mathrm{T}} \boldsymbol{F}_a^P, \tag{20}$$

where space-dependent vector and matrix, $\bar{\boldsymbol{u}}^i$ and $\boldsymbol{S}^i$, respectively, must be particularized at the position of point P. Note that the FFR formulation does not impose limitations as to the location or nature of the load.

Equations of Motion

In general form, the FFR equations of motion of a body can be written as follows

$$\boldsymbol{M}^i\ddot{\boldsymbol{q}}^i + \boldsymbol{D}^i\dot{\boldsymbol{q}}^i + \boldsymbol{K}^i\boldsymbol{q}^i + \boldsymbol{C}_{\boldsymbol{q}^i}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{Q}_a + \boldsymbol{Q}_v, \tag{21}$$

which, in turn, may be expressed in terms of submatrices as

$$\begin{bmatrix} \boldsymbol{m}_{RR} & \boldsymbol{m}_{R\theta} & \boldsymbol{m}_{Rf} \\ & \boldsymbol{m}_{\theta\theta} & \boldsymbol{m}_{\theta f} \\ sym. & & \boldsymbol{m}_{ff} \end{bmatrix}^i \begin{bmatrix} \ddot{\boldsymbol{R}} \\ \ddot{\boldsymbol{\theta}} \\ \ddot{\boldsymbol{q}}_f \end{bmatrix}^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{D}_{ff} \end{bmatrix}^i \begin{bmatrix} \dot{\boldsymbol{R}} \\ \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{q}}_f \end{bmatrix}^i + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{K}_{ff} \end{bmatrix}^i \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{\theta} \\ \boldsymbol{q}_f \end{bmatrix}^i +$$
$$\begin{bmatrix} \boldsymbol{C}_{\boldsymbol{R}^i}^{\mathrm{T}} \\ \boldsymbol{C}_{\boldsymbol{\theta}^i}^{\mathrm{T}} \\ \boldsymbol{C}_{\boldsymbol{q}_f^i}^{\mathrm{T}} \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} (\boldsymbol{Q}_a)_{\boldsymbol{R}^i} \\ (\boldsymbol{Q}_a)_{\boldsymbol{\theta}^i} \\ (\boldsymbol{Q}_a)_{\boldsymbol{q}_f^i} \end{bmatrix} + \begin{bmatrix} (\boldsymbol{Q}_v)_{\boldsymbol{R}^i} \\ (\boldsymbol{Q}_v)_{\boldsymbol{\theta}^i} \\ (\boldsymbol{Q}_v)_{\boldsymbol{q}_f^i} \end{bmatrix}.$$

The new terms $\boldsymbol{K}_{ff}$ and $\boldsymbol{D}_{ff}$ refer to the stiffness and damping matrices of the structure. These two matrices may be obtained by modeling the structure as a linear system using, for instance, the finite element method.

3 Finite Element FFR Formulation

As shown in the previous section, the FFR formulation leads to a local linear problem that enables the use of classical finite element technology: Flexible coordinates can be considered as finite element degrees of freedom. Within the context of FFR, finite elements containing infinitesimal angles still accurately describe rigid body motion. This fact, together with the use of *intermediate element coordinate systems*, allows to obtain exact modeling of rigid body inertia for flexible bodies. The introduction of intermediate element frames, as described by Shabana in [8], eliminates the issue of ensuring that actual small strains are seen as small strains when described in the body's floating frame of reference. Complex geometric shapes can be modeled by using one single, co-rotated floating frame per body due to the introduction of intermediate frames. The co-rotational formulation, which uses element-wise coordinates to describe the state of finite elements, also appeared due to need for modeling complex geometries [4].

The floating frame of reference must be attach to the flexible body by prescribing a set of conditions that will ensure that no rigid body's degrees of freedom remain in the structure. That is, the FFR's degrees of freedom must describe reference motion, whereas the finite elements coordinates must only describe displacements that lead to deformation. This is accomplished by imposing *reference conditions*, which link the finite element model to the FFR.

The success of the FFR in practical multibody applications owes much to the fact the it enables the use of linear *model order reduction* techniques, that is, numerical methods aiming at reducing the number of coordinates of a linear system while preserving static/dynamic information of the original, large-scale model. This idea will be elaborated on in a later subsection.

3.1 Local Deformation and Reference Frames

We will introduce a more complete notation in this subsection to refer to finite elements. The position of a finite element j in a body i is expressed as follows:

$$\boldsymbol{e}^{ij} = \boldsymbol{e}_0^{ij} + \boldsymbol{e}_f^{ij}, \tag{22}$$

where $\boldsymbol{e}_0^{ij}$ and $\boldsymbol{e}_f^{ij}$ denote the rigid body and flexible coordinates of a finite element j in a body i. Several coordinate systems needed for a general problem are introduced: 1) $\langle \boldsymbol{X}^{\boldsymbol{i}} \boldsymbol{Y}^i \boldsymbol{Z}^i \rangle$ is a **body** coordinate system that represents the overall motion of the body but need not be rigidly attached to the flexible body. 2) An **element-wise** reference system $\langle \boldsymbol{X}^{ij} \boldsymbol{Y}^{ij} \boldsymbol{Z}^{ij} \rangle$ is rigidly attached to the finite element j. 3) Finally, at the location of the body coordinate system, an **intermediate element coordinate system** with axes initially parallel to $\langle \boldsymbol{X}^{ij} \boldsymbol{Y}^{ij} \boldsymbol{Z}^{ij} \rangle$ is placed, which is denoted by $\langle \boldsymbol{X}_i^{ij} \boldsymbol{Y}_i^{ij} \boldsymbol{Z}_i^{ij} \rangle$. This intermediate frame is necessary to guarantee exact modeling of rigid body kinematics [8]. A way to interpret the need for this system is the following: Since the FFR formulation assumes one single reference frame per body, there is a need to ensure that we account for the reference orientation at the unstressed configuration. Note that the intermediate element coordinate system does not introduce additional coordinates in the formulation since it is defined by the geometry of the flexible body at the initial configuration.

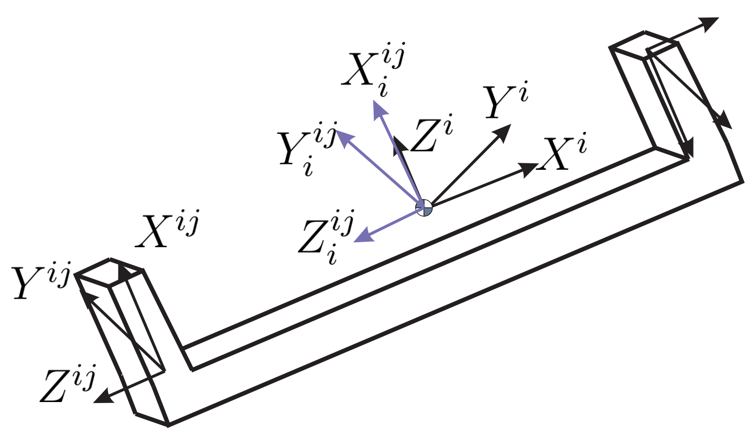

Figure 2: The intermediate frame of reference $\langle \boldsymbol{X}_i^{ij} \boldsymbol{Y}_i^{ij} \boldsymbol{Z}_i^{ij} \rangle$ is shown in the figure as a rotated coordinated system which is parallel to the initial finite element frame, $\langle \boldsymbol{X}^{ij} \boldsymbol{Y}^{ij} \boldsymbol{Z}^{ij} \rangle$. The reference frame $\langle \boldsymbol{X}_i^{ij} \boldsymbol{Y}_i^{ij} \boldsymbol{Z}_i^{ij} \rangle$ allows the transformation of the finite element coordinates in the element system, $\boldsymbol{e}_i^{ij}$, into the body reference frame, $\boldsymbol{q}_n^{ij}$.

3.2 Description of Local Deformation

Connectivity conditions in the flexible body are established, as in any finite element model, by identifying the nodes shared by elements. From now on, the connectivity problem will be disregarded and the derivations will be presented element-wise, without loss of generality. To describe the kinematics of the deformation of a finite element j in a body i, we need first to define the orientation matrix describing the orientation of the intermediate element coordinate system with respect to the body coordinate system or floating frame, $\boldsymbol{C}^{ij}$. The assumed displacement field $\boldsymbol{w}_i^{ij}$ of an element can be expressed in the body frame as

$$\bar{\boldsymbol{u}}^{ij} = \boldsymbol{C}^{ij}\boldsymbol{w}_i^{ij} = \boldsymbol{C}^{ij}\boldsymbol{S}^{ij}\boldsymbol{e}_i^{ij}, \tag{23}$$

where $\boldsymbol{S}^{ij}$ is the shape function matrix of an element j in a body i and $\boldsymbol{e}_i^{ij}$ is the vector of element j's coordinates. Note that $\boldsymbol{w}_i^{ij}$ is defined with respect to the intermediate element coordinate system (see Fig. 2). The vector of nodal coordinates $\boldsymbol{e}_i^{ij}$ may be defined in the body coordinate system as

$$\boldsymbol{e}_i^{ij} = \boldsymbol{C}^{ij\mathrm{T}}\boldsymbol{q}_n^{ij}, \tag{24}$$

in which $\boldsymbol{q}_n^{ij}$ is the vector of element j's nodal coordinates expressed in the body coordinate system. Making use of Eqs. (23)-(24), one gets

$$\bar{\boldsymbol{u}}^{ij} = \boldsymbol{C}^{ij}\boldsymbol{w}_i^{ij} = \boldsymbol{C}^{ij}\boldsymbol{S}^{ij}\boldsymbol{C}^{ij\mathrm{T}}\boldsymbol{q}_n^{ij}. \tag{25}$$

Equation (25) is key since it expresses the position coordinates of an arbitrary point on the finite element with respect to the origin of the body coordinate system [7].

3.3 Reference Conditions

Cartesian and rotational coordinates describe the location and attitude of the floating frame of reference. Flexible coordinates, on the other hand, describe small deformation of the body. However, so far, we have skipped describing how to eliminate the rigid body modes associated with the finite element coordinates. In other words, the model describing the deformation of the flexible body must be attached to the floating frame by using a set of reference conditions that must fully eliminate relative rigid body motion. Reference conditions can imposed in different ways; some of them are summarized below:

- **Free body**. If the flexible body is not connected to other bodies, a sensible solution to the problem of assigning reference conditions to the flexible body is to keep the origin of the coordinate system at the instantaneous mass center and, simultaneously, ensure that its axes are principal, that is, they define the three products of inertia remain zero as the body deforms [1]. The reference conditions for mean axes may be imposed by selecting a proper set of algebraic constraints [6].

- **Clamped conditions**. A simpler way of defining reference conditions is by fully constraining the body coordinate system to be located and oriented as one node at a finite element. For instance, the body frame may be fully constrained to the node linking a rotor blade to the rotor itself.

- **Simply supported**. For beams, it is possible to attach the body coordinate frame to the flexible body in such a way that its location is placed at one end of the beam and one of its axes always points to the other end. Note that this is the case of many robotic flexible manipulators.

The choice of reference conditions must be consistent with the problem at hand since, in a practical multibody system problem, the way flexible bodies are connected to other bodies may determine the way they deform. In other words, reference conditions must be selected such that they respect both the boundary conditions of the flexible body within the system and the separate model used to describe the linear deformation problem (e.g. the finite element model).

3.4 Kinematics of an FFR Finite Element

For convenience, we rewrite Equation (25) in the following manner

$$\bar{\boldsymbol{u}}^{ij} = \bar{\boldsymbol{N}}^{ij}\boldsymbol{q}_n^{ij}, \tag{26}$$

where $\bar{\boldsymbol{N}}^{ij}$ is a space-dependent matrix: $\bar{\boldsymbol{N}}^{ij} = \boldsymbol{C}^{ij}\boldsymbol{w}_i^{ij} = \boldsymbol{C}^{ij}\boldsymbol{S}^{ij}\boldsymbol{C}^{ij\mathrm{T}}$. Using the new kinematic description that accounts for the use of finite element techniques, the position vector $\boldsymbol{r}^{ij}$ of an arbitrary point on element j of a body i may be written as

$$\boldsymbol{r}^{ij} = \boldsymbol{R}^i + \boldsymbol{A}^i\bar{\boldsymbol{u}}^{ij} = \boldsymbol{R}^i + \boldsymbol{A}^i\boldsymbol{N}^{ij}\boldsymbol{q}_n^{ij} = \boldsymbol{R}^i + \boldsymbol{A}^i\boldsymbol{N}^{ij}\left(\boldsymbol{q}_o^i + \boldsymbol{B}_r^i\boldsymbol{q}_f^{ij}\right), \tag{27}$$

where $\boldsymbol{q}_o^i$ is the vector of nodal coordinates in the undeformed state, i.e. they locate material points with respect to the body frame in the unformed state, $\boldsymbol{q}_f^i$ is the vector of nodal deformations, and $\boldsymbol{B}_r^i$ is a matrix that may apply a set of reference conditions on the deformation coordinates, e.g., in order to attach a node to the location of the reference frame. Equation (27) may be differentiated with respect to time to obtain

$$\dot{\boldsymbol{r}}^{ij} = \dot{\boldsymbol{R}}^i + \boldsymbol{A}^i\big(\bar{\boldsymbol{\omega}}^i \times \bar{\boldsymbol{u}}^i\big)^i + \boldsymbol{A}^i\boldsymbol{N}^{ij}\boldsymbol{B}_r^i\dot{\boldsymbol{q}}_f^{ij}. \tag{28}$$

Note that in Eq. (28) only matrix $\boldsymbol{N}^{ij}$ has spatial dependency. The kinetic and elastic terms of the equations of motion of the finite element FFR formulation can therefore be obtained using Eqs. (27)-(28). A set of inertia shape integrals involving the finite element shape functions and their relative orientation with respect to the body frame of reference may be calculated at a preprocessing stage as described in Ref. [8]. Using the kinematic equations presented in this subsection, Eqs. (12), (14), (20), and (21) can be rederived for the finite element FFR formulation.

3.5 Model Order Reduction

The equations of motion of the finite element FFR formulation may contain an excessive number of coordinates owing to the detailed finite element models used to describe a flexible body's dynamics. The single frame of reference introduced to capture flexible body dynamics allows for a separation of rigid body and flexible coordinates. Since flexible coordinates in the FFR formulation describe linear deformation, any model order reduction technique may be applied. Traditionally, methods such as Component Mode Synthesis, Guyan condensation, and purely eigenmode analysis have been profusely employed to reduce the number of coordinates of the flexible body. In the past decade, there has been an interest for the use of *modern* model order reduction techniques in multibody system dynamics, see [3, 5]. Proper use of model order reduction allows to analyze complex system at a high level of accuracy and reduced computational cost.

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