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A Comparison of Several Low order Integration Methods vis-a-vis Dissipation and Stability Properties

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1 Background

In certain DEM simulations, such as a settling test of spheres in a bucket, there is an undesirable numerical artifact wherein even if enough time is given to allow the system to settle: the spheres continue to vibrate with small amplitudes. This phenomenon is considered as spurious oscillation arising from the choice of integrator. In order for the system to dissipate this “noise” energy faster and to combat unwanted oscillations, we take a look at several numerical integration methods that might be used to handle this scenario. Specifically, without projecting this discussion in the context of DEM, we analyze the behavior of four schemes: explicit Euler, half-implicit Euler, implicit Euler, and explicit Chung. The first three schemes are discussed in [1]. The last is presented in [2].

2 Problem Setup

As a test case for comparing these numerical methods, a single degree-of-freedom oscillator (mass-spring-damper) problem is solved. Considering free response only, the governing equation of the oscillator is

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0, \tag{1}$$

where ω is the natural frequency and ζ is the damping ratio of the oscillator. The second order ODE problem in Eq.(1) is solved subject to initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. To solve the above system with given initial conditions, let $v = \dot{x}$, and thus

$$\dot{x} = v \tag{2}$$

$$\dot{v} = -\omega^2 x - 2\zeta\omega v \tag{3}$$

Discretizing over time t with a step size of $\Delta t = t_{n+1} - t_n$ and defining $x_n = x(t_n)$, $v_n = v(t_n)$, we have

$$\frac{x_{n+1} - x_n}{\Delta t} = g_1(x, v, t) \tag{4}$$

$$\frac{v_{n+1} - v_n}{\Delta t} = g_2(x, v, t) \tag{5}$$

where functions g_1 and g_2 depend on the choice of numerical method.

3 Numerical Schemes

3.1 Explicit Euler

For Explicit Euler, the gradients on the right-hand-side of equations (4, 5) are evaluated using the information from the current time step. Therefore,

$$\frac{x_{n+1} - x_n}{\Delta t} = v_n \tag{6}$$

$$\frac{v_{n+1} - v_n}{\Delta t} = -\omega^2 x_n - 2\zeta\omega v_n \tag{7}$$

The position and velocity at the next time step x_{n+1} and v_{n+1} are evaluated as

$$x_{n+1} = x_n + \Delta t v_n \tag{8}$$

$$v_{n+1} = -\omega^2 \Delta t x_n + (1 - 2\zeta\omega\Delta t)v_n. \tag{9}$$

In matrix form, we have

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n \quad n = 0, 1, \ldots, N \ , \tag{10}$$

where $\mathbf{X}_n = [x_n, \Delta t v_n]^T$, and $\mathbf{A}$ is called the amplification matrix, written as

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -\Omega^2 & 1 - 2\zeta\Omega \end{bmatrix} \tag{11}$$

with $\Omega = \omega\Delta t$. Since $\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n = \mathbf{A}^{n+1}\mathbf{X}_0$, the stability region of the algorithm is governed by the eigenvalues λ of the amplification matrix $\mathbf{A}$, which are computed as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 2(\zeta\Omega - 1)\lambda + \Omega^2 - 2\zeta\Omega + 1 = 0. \tag{12}$$

To achieve a stable algorithm, the roots of the characteristic equation (12), λ_1 and λ_2, should be inside or on the unit circle of the complex plane. This is equivalent to having the spectral radius of the amplification matrix, $\rho(\mathbf{A})$, to be less than or equal to 1,

$$\rho(\mathbf{A}) = \max(\|\lambda_1\|, \|\lambda_2\|) \tag{13}$$

$$\rho \leq 1 \tag{14}$$

3.2 Implicit Euler

Using Implicit Euler, g_1 and g_2 in (4) and (5) are approximated using the unknown information from the next step,

$$\frac{x_{n+1} - x_n}{\Delta t} = v_{n+1} \tag{15}$$

$$\frac{v_{n+1} - v_n}{\Delta t} = -\omega^2 x_{n+1} - 2\zeta\omega v_{n+1}. \tag{16}$$

With $\mathbf{X}_n = [x_n, \Delta t v_n]^T$, we have

$$\begin{bmatrix} 1 & -1 \\ \Omega^2 & 1 + 2\zeta\Omega \end{bmatrix} \mathbf{X}_{n+1} = \mathbf{X}_n, \tag{17}$$

and the amplification matrix for Implicit Euler defined as,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ \Omega^2 & 1 + 2\zeta\Omega \end{bmatrix}^{-1}. \tag{18}$$

3.3 Half-Implicit Euler

For Half-Implicit Euler integration, at the new time step, velocity is updated explicitly while the position is updated implicitly. Therefore, the gradients can be approximated as,

$$\frac{x_{n+1} - x_n}{\Delta t} \approx g_1(x_{n+1}, v_{n+1}, t) = v_{n+1} \tag{19}$$

$$\frac{v_{n+1} - v_n}{\Delta t} \approx g_2(x_n, v_n, t) = -\omega^2 x_n - 2\zeta\omega v_n \tag{20}$$

Using $\mathbf{X}_n = [x_n, \Delta t v_n]^T$, we have,

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{X}_{n+1} = \begin{bmatrix} 1 & 0 \\ -\Omega^2 & 1 - 2\zeta\Omega \end{bmatrix} \mathbf{X}_n. \tag{21}$$

The amplification matrix of the Half-Implicit algorithm is,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -\Omega^2 & 1 - 2\zeta\omega \end{bmatrix} = \begin{bmatrix} 2 - \Omega^2 & 1 - 2\zeta\Omega \\ -\Omega^2 & 1 - 2\zeta\Omega \end{bmatrix} \tag{22}$$

3.4 Second-order Chung's method

A second-order explicit time integration method is introduced in [2] specifically for non-linear structural dynamics, and the update scheme is given by

$$a_{n+1} + 2\zeta\omega v_n + \omega^2 x_n = 0 \tag{23}$$

$$v_{n+1} = v_n + \Delta t(\hat{\gamma} a_n + \gamma a_{n+1}) \tag{24}$$

$$x_{n+1} = x_n + \Delta t v_n + \Delta t^2(\hat{\beta} a_n + \beta a_{n+1}) \tag{25}$$

With $\mathbf{X}_n = [x_n, \Delta t v_n, \Delta t^2 a_n]^T$ and $\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n$, we have

$$\mathbf{A} = \begin{bmatrix} 1 - \beta\Omega^2 & 1 - 2\beta\zeta\Omega & \hat{\beta} \\ -\gamma\Omega^2 & 1 - 2\gamma\zeta\Omega & \hat{\gamma} \\ -\Omega^2 & -2\zeta\Omega & 0 \end{bmatrix} \tag{26}$$

In order for the algorithm to achieve stability and second-order accuracy, the parameters are determined as

$$1 \leq \beta \leq 28/27, \quad \hat{\beta} = 1/2 - \beta, \quad \gamma = 3/2, \quad \hat{\gamma} = -1/2 \tag{27}$$

The detailed derivations can be found in Chung's work.

4 Results

4.1 Oscillator simulation

An oscillator with natural frequency $\omega = \pi \text{ rad s}^{-1}$ and damping ratio $\zeta = 0.2$ is simulated for 8s with time step size $\Delta t = 0.01$s. The initial value of the problem has $x_0 = 1$ and $v_0 = 1$. The displacement x_n of different schemes in Sec. 3 over time, including the analytical solution, are plotted in Fig. 1. For Chung's method, $\beta = 28/27$ is used here.

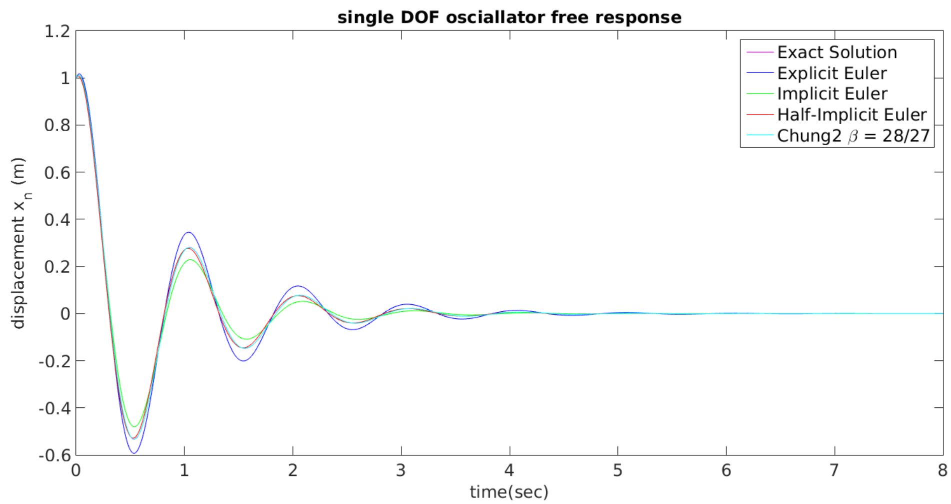

Figure 1: Displacement of a free response oscillator using various numerical schemes

Fig. 2 gives the absolute error $|x_n - x(t_n)|$ of the displacement over time using each method compared with the analytical solution. As expected, Chung's second-order method is more accurate than the others, which are all first-order.

4.2 Settling

To better understand how the oscillator settles, the kinetic energy is computed over the last 4s, as plotted in Fig. 3. The Explicit Euler method takes longer to settle, while Implicit Euler introduces more numerical damping and settles much faster.

4.3 Stability Properties

For a stable algorithm, $\rho(\mathbf{A}) \leq 1$, where $\mathbf{A}$ is the amplification matrix and ρ is the spectral radius. Fig. 4 shows the stability region of each algorithm. Both implicit and half-implicit Euler methods are unconditionally stable. Explicit Euler has the smallest stability region. The stability region of Chung's method decreases as the damping ratio ζ increases, as shown in Fig. 5.

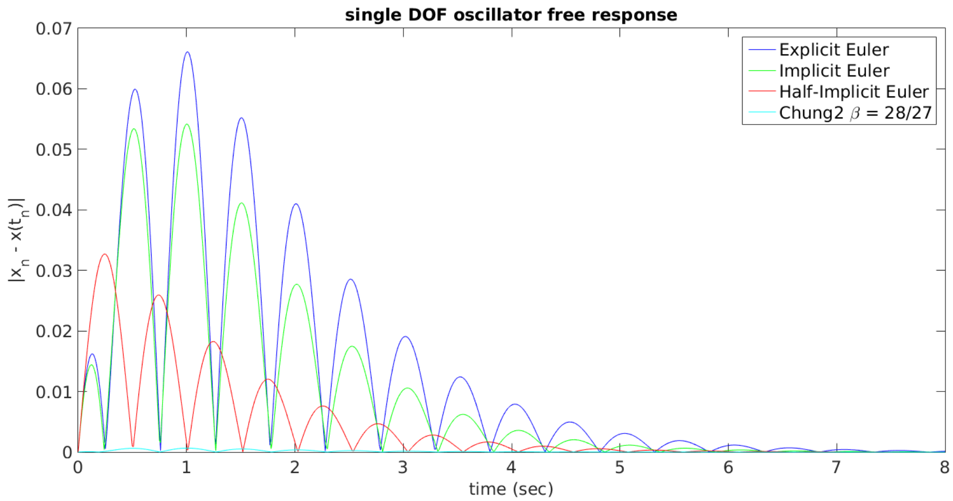

Figure 2: Displacement error of a free response oscillator using various numerical schemes

$$k = \frac{1}{\sqrt{2}R} \cdot \sqrt{\frac{LHW}{2N}}$$

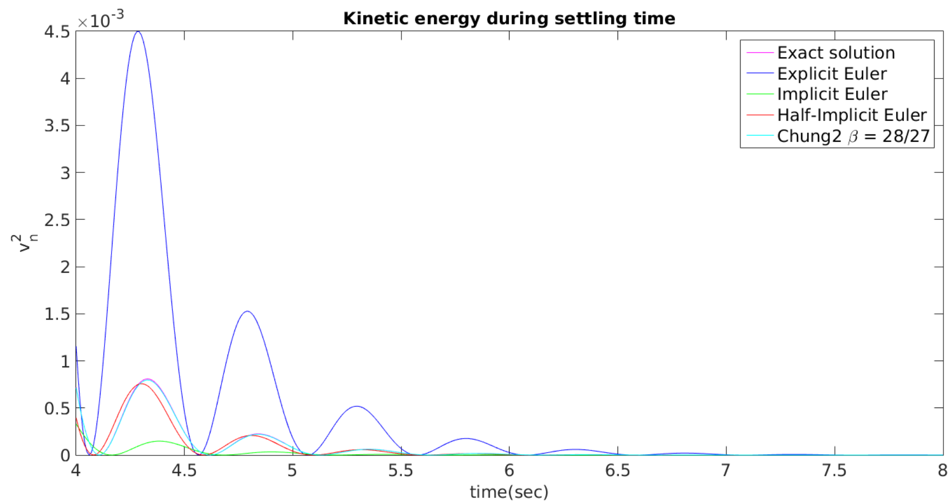

Figure 3: Kinetic energy of the oscillator during settling time

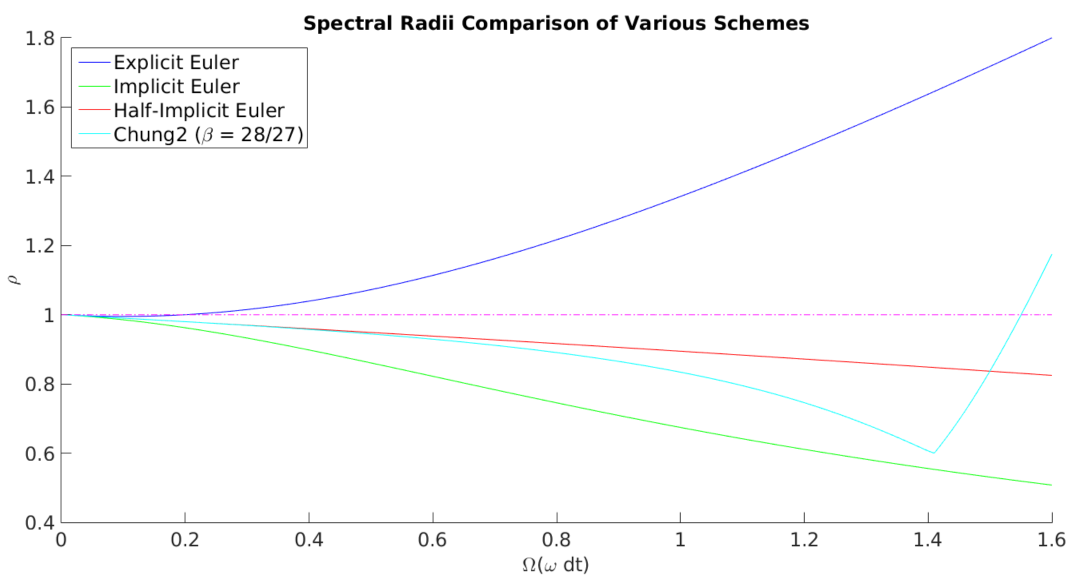

Figure 4: Spectral radii of various schemes

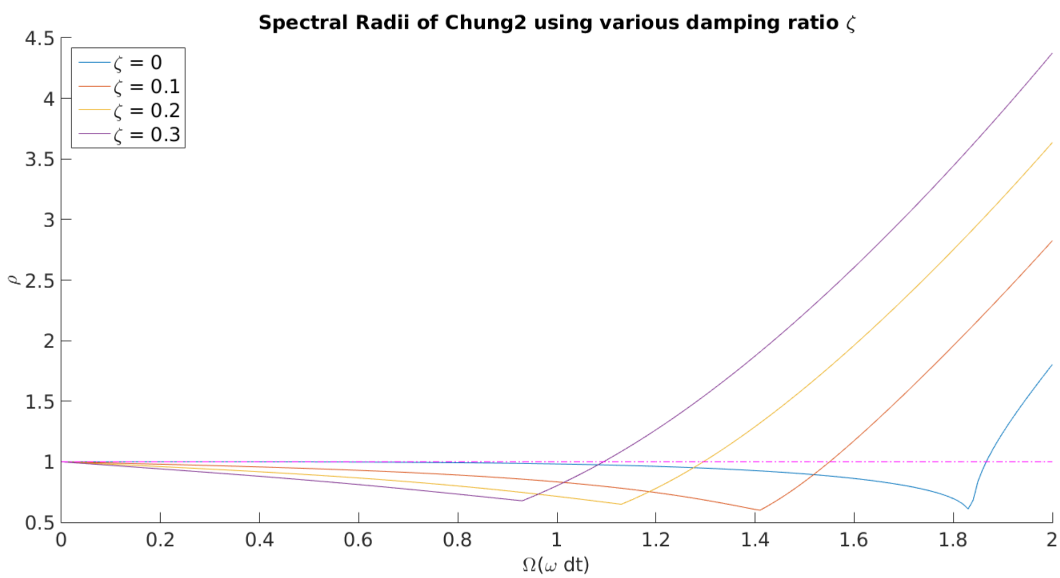

Figure 5: Effect of damping ratio ζ on stability region of Chung's method

References

[1] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems*. Springer, 1996.

[2] Jintai Chung and Jang Moo Lee. A new family of explicit time integration methods for linear and non-linear structural dynamics. *International Journal for Numerical Methods in Engineering*, 37(23):3961–3976, 1994.