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Incompressible Implicit SPH

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Abstract

This technical report summarized the Implicit Incompressible SPH (IISPH) formulation for the simulation of the incompressible fluid dynamics. In a meshfree, marker-based method, IISPH formulates a direct iterative method relying on the density update equation to impose constant density condition.

Keywords: Implicit, Incompressible, Smoothed Particle Hydrodynamics.
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1 Introduction

SPH is a mesh-free method to simulate fluid flow using Lagrangian markers. In a weakly compressible SPH (WSPH) [4], the fluid pressure is tied to the density variations through an equation of state (EOS). Therefore, the fluid incompressibility is only relative. Moreover, the coupling between velocities and pressure is obtained through density (density is obtained with the velocity information, pressure is updated with EOS, and velocity is updated through Navier-Stokes (NS) with the density and pressure information). The stiff EOS leads to instability of the solver and forces the solver to use very small time steps in order to keep the solution procedure stable. Therefore, coupling the pressure and velocities without a stiff EOS is believed to increase the accuracy/stability of SPH solvers. In this technical report, the IISPH method developed by Ihmsen et al. [2] is explained and some numerical experiments are carried out to assess its potential.

2 Numerical Method

The IISPH method discretizes the left hand side of the continuity equation, \( \frac{D\rho}{Dt} = \rho \nabla \cdot \mathbf{v} \), using first-order Euler method and the right hand side using SPH discretization. Therefore, the continuity equation is written as:

\[
\frac{\rho_i(t + \Delta t) - \rho_i(t)}{\Delta t} = \sum_{j \in S(i)} m_j \mathbf{v}_{ij}(t + \Delta t) \nabla W_{ij}
\]

(1)

where \( S(i) \) represents the support domain associated with marker \( i \), \( \rho_i \) and \( m_j \) are respectively the density and the mass associated with marker \( i \) and \( j \). The 2D cubic spline interpolation kernel, \( W \), is defined as:

\[
W(q, h) = \frac{5}{14\pi h^2} \times \begin{cases} (2 - q)^3 - 4(1 - q)^3, & 0 \leq q < 1 \\ (2 - q)^3, & 1 \leq q < 2 \\ 0, & q \geq 2 \end{cases}
\]

(2)

where \( h \) is the kernel function’s characteristic length and \( q \equiv |\mathbf{r}|/h \). The radius of the support domain, \( \kappa h \), is proportional to the characteristic length \( h \) through the parameter \( \kappa \) which is equal to 2 for the cubic spline kernel. \( W_{ij} \equiv W_{|r=x_{ij}} \); \( \nabla \) is the gradient with respect to \( \mathbf{x}_i \), i.e. \( \partial/\partial \mathbf{x}_i \), which is written everywhere in this manuscript as \( \nabla W_{ij} \) for convenient. Moreover, in all the following equations \( \mathbf{X}_{ij} \) is translated into \( \mathbf{X}_i - \mathbf{X}_j \), for instance \( \mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j \).

In Eq. (1) both \( \mathbf{v}_{ij}(t + \Delta t) \) and \( \rho_i(t + \Delta t) \) are unknowns. However, in order for the continuity equation to satisfy the incompressibility, the \( \rho_i(t + \Delta t) = \rho_0 \) condition should be met. This equation is used later to derive an equation for pressure.

Newton’s second law is expressed in a form that explicitly states the pressure and non-pressure components of the force acting at the location of marker \( i \). In order to do so, the
RHS of $F_i = m_i \frac{Dv_i}{Dt}$ is discretized using first-order Euler method, and $F_i$ is divided into pressure part, and non-pressure part as follows:

$$F^p(t) + F^{np}(t) = m_i \frac{v_i(t + \Delta t) - v_i(t)}{\Delta t}$$

where $F^p$ and $F^{np}$ represent the pressure forces and non-pressure forces respectively. By manipulating the above equation the new velocities are obtained as:

$$v_i(t + \Delta t) = v_i(t) + \frac{F^p_i(t) + F^{np}_i}{m_i} \Delta t$$

Velocities are divided into two components based on the pressure contribution of forces as follows:

$$v_{i,l+1} = \left( v_{i,l} + \frac{F^{np}_{i,l}}{m_i} \Delta t \right) + \frac{F^p_{i,l}}{m_i} \Delta t = v^{np}_{i,l} + v^p_{i,l+1}$$

where $v^{np}_{i,l} = v_{i,l} + \frac{F^{np}_{i,l}}{m_i} \Delta t$ and $v^p_{i,l} = \frac{F^p_{i,l}}{m_i} \Delta t$, and subscripts $l$ and $l + 1$ are selected to indicate the discretized values at time $t$ and $t + \Delta t$, respectively. The IISPH formulation resembles the Chorin’s two step projection algorithm where i) intermediate velocity $u^*$ correlates with viscous forces and velocity of previous time step, and ii) new velocity correlates with the information of the pressure at new time step and the intermediate velocity. In this work, $v^{np}_{i,l}$ resembles the intermediate velocity, $u^*$, in Chorin’s projection method and $v^p_{i,l+1}$ incorporates the contribution of the pressure gradient term in N-S equations. In Chorin’s projection method, pressure equation is obtained by imposing the divergence free condition on the velocity of the new time step. However, in this work, the pressure equation is obtained by imposing the incompressibility condition on the density of the new time step as follows.

Following Eq. (1), density is also divided into the pressure and non pressure parts. The intermediate density resulting from the intermediate velocity $v^{np}_{ij,l}$ can be obtained using

$$\rho^{np}_{i,l} = \rho_{i,l} + \Delta t \sum_{j \in S(i)} m_j v^{np}_{ij,l} \nabla W_{ij}.$$  

By subtracting the above equation from

$$\rho_{i,l+1} = \rho_{i,l} + \Delta t \sum_{j \in S(i)} m_j v_{ij,l+1} \nabla W_{ij},$$

it can be concluded that:

$$\rho_{i,l+1} - \rho^{np}_{i,l} = \Delta t \sum_{j \in S(i)} m_j (v_{ij,l+1} - v^{np}_{ij,l}) \nabla W_{ij},$$

where $\rho_{i,l} = \sum_j m_j W_{ij}$. In order to preserve incompressibility, $\rho_{i,l+1} = \rho_0$ must be satisfied within the margin of numerical accuracy. Moreover, using Eq. (5) and the definition of
the above equation is further simplified to the following form, which implicitly gives the pressure equation:

\[
\rho_0 - \rho_i = \Delta t \sum_{j \in S(i)} m_j v_{i,j,l+1} \nabla W_{ij} = \Delta t \sum_{j \in S(i)} m_j \left( \frac{F_{j,i,l+1}^p}{m_j} \nabla W_{ij} \right)
\]

\[
= \Delta t^2 \sum_{j \in S(i)} m_j \left( \frac{F_{j,i,l+1}^p - F_{j,i,l+1}^p}{m_i} \right) \nabla W_{ij}.
\]

The pressure will appear in this equation upon relating the forces to pressures as will be discussed later. By substituting \(\rho_i^np\) from Eq. (6), Eq. (9) is written as,

\[
\rho_0 - \rho_i = \Delta t \sum_{j \in S(i)} m_j v_{i,j}^np \nabla W_{ij} = \Delta t \sum_{j \in S(i)} m_j \left( \frac{F_{j,i,l+1}^p}{m_i} - \frac{F_{j,i,l+1}^p}{m_j} \right) \nabla W_{ij}.
\]

(10)

where \(v_{i,j}^np = v_{i,j} + \frac{F_{i,j,l}^{np}}{m_i} \Delta t\) is used to simplify the LHS. The momentum-preserving pressure forces, \(F_{i,l+1}^p\), are calculated as [4]:

\[
F_{i,l+1}^p = -m_i \sum_{j \in S(i)} m_j \left( \frac{p_{i,l+1}}{\rho_{i,l}^2} + \frac{p_{j,l+1}}{\rho_{j,l}^2} \right) \nabla W_{ij},
\]

(12)

and \(F_{i,l}^{np} = m_i \sum_{j \in S(i)} m_j \Pi_{ij}\) with

\[
\Pi_{ij} = -\left( \mu_i + \mu_j \right) x_{ij} \nabla W_{ij} \rho_{ij}^2 \left( x_{ij}^2 + \epsilon h_{ij} \right) v_{i,j,l}.
\]

(13)

where \(h_{ij} = (h_i + h_j)/2\) is defined based on kernel function’s characteristic length, and \(\rho_{ij} = (\rho_i + \rho_j)/2\).

The values of the parameters on the LHS of Eq. (11) are known. The RHS can be further simplified according to the simplified version of Eq. (12):

\[
\Delta t^2 \frac{F_{i,l+1}^p}{m_i} = -\Delta t^2 \sum_{k \in S(i)} m_k \left( \frac{p_{i,l+1}}{\rho_{i,l}^2} + \frac{p_{k,l+1}}{\rho_{k,l}^2} \right) \nabla W_{ik}
\]

\[
= \left( -\Delta t^2 \sum_{k} \frac{m_k}{\rho_{i,k}^2} \nabla W_{ik} \right) p_{i,l+1} + \left( \sum_{k \in S(i)} -\Delta t^2 \frac{p_{k,l+1}}{\rho_{k,l}^2} \nabla W_{ik} \right)
\]

\[
= d_{ii} p_{i,l+1} + \sum_{k \in S(i)} d_{ik} p_{k,l+1},
\]

(14)
where \( d_{ik} = -\Delta t^2 \frac{m_k}{\rho_k i} \nabla W_{ik} \). A similar equation can be written for \( \Delta t^2 \frac{F_{i,l+1}^{np}}{m_j} \). By making use of Eq. (14) the RHS of the equation (11) can be written as

\[
RHS = \sum_{j \in S(i)} m_j \left( (d_{ii} p_{i,l+1} + \sum_{k \in S(i)} d_{ik} p_{k,l+1}) - (d_{jj} p_{j,l+1} + \sum_{k \in S(j)} d_{jk} p_{k,l+1}) \right) \nabla W_{ij},
\]

(15)

The contribution of the marker \( i \) can be extracted from the last term of Eq. 15 as

\[
\sum_{k \in S(j)} d_{jk} p_{k,l+1} + d_{ji} p_{i,l+1},
\]

(16)

which is necessary for the derivation of the Jacobi iterative scheme. Therefore, the RHS of the Eq. (11) is written as:

\[
RHS = \sum_{j \in S(i)} m_j \left( (d_{ii} p_{i,l+1} + \sum_{k \in S(i)} d_{ik} p_{k,l+1}) - (d_{jj} p_{j,l+1} + \sum_{k \in S(j), k \neq i} d_{jk} p_{k,l+1}) \right) \nabla W_{ij}
\]

\[= \sum_{j \in S(i)} p_{i,l+1}(d_{ii} - d_{ji}) m_j \nabla W_{ij} \]

\[+ \sum_{j \in S(i)} m_j \left( \sum_{k \in S(i)} d_{ik} p_{k,l+1} - d_{jj} p_{j,l+1} - \sum_{k \in S(j), k \neq i} d_{jk} p_{k,l+1} \right) \nabla W_{ij} \]

\[= a_{ii} p_i + \sum_{k \in S(i)} m_j \left( \sum_{k} d_{ik} p_{k,l+1} - d_{jj} p_{j,l+1} - \sum_{k \in S(j), k \neq i} d_{jk} p_{k,l+1} \right) \nabla W_{ij}
\]

(17)

where \( a_{ii} = \sum_{j \in S(i)} (d_{ii} - d_{ji}) m_j \nabla W_{ij} \), \( d_{ii} = -\Delta t^2 \sum_{k \in S(i)} \frac{m_k}{\rho_k i} \nabla W_{ik} \), and \( d_{ik} = \sum_{k \in S(i)} -\Delta t^2 \frac{m_k}{\rho_k i} \nabla W_{ik} \).

Ultimately, by substituting Eq. (17) in Eq. (11), the iterative rule for updating the pressure of makers is obtained as:

\[
p_{i}^{(r)} = \frac{\rho_0 - \rho_{i}^{np} - \sum_{j \in S(i)} m_j \left( \sum_{k \in S(i)} d_{ik} p_{k,l+1}^{(r-1)} - d_{jj} p_{j,l+1}^{(r-1)} - \sum_{k \in S(j), k \neq i} d_{jk} p_{k,l+1}^{(r-1)} \right) \nabla W_{ij}}{a_{ii}}
\]

(18)

where superscripts \((r - 1)\) and \((r)\) are respectively referring to values at previous and new iterations, and \( \rho_{i}^{np} = \rho_{i,l} + \Delta t \sum_{j \in S(i)} m_j (v_{ij,l} + \frac{F_{i,l}^{np}}{m_i} - \frac{F_{j,l}^{np}}{m_j}) \Delta t) \nabla W_{ij} \).

This equation needs to be solved iteratively until the residuals of \( p_{i,l+1}^{(r)} \) are small enough. A Successive Over Relaxation (SOR) method may be used to increase the speed of iterative scheme as

\[
p_{i}^{(r)} = (1 - \omega)p_{i}^{(r-1)} + \omega p_{i}^{(r)},
\]

(19)

where \( \omega \) in the above equation is to be found experimentally. After reaching a convergence, the pressure values obtained from iterative scheme in Eq. (18) are used in Eq. (14) to
Furthermore, $F_{i,l}^{np}$ is constructed from Eq. (13). These two terms will then be used in Eq. (3) to obtain the updated velocities. The markers position is updated afterwards relying on the value of the velocities at the new time step using first-order Euler scheme. Since the markers’ position are updated only once during a time step, only one proximity computation is required per time step. This is in contrast with implicit SPH method [?] where two proximity computations is needed per step: one for the predicted state (velocity/positions) and another one for the correct state.

2.1 Boundary Conditions

2.1.1 Walls

For SPH markers close to solid boundaries pressure equation (18) captures the contribution of the fluid markers. The contribution of solid objects is calculated via Boundary Condition Enforcing (BCE) markers placed on and close to the solid’s surface as shown in Figure 1 [?]. Two approaches are considered herein to implement the boundary conditions. In the first approach, the pressure of the BCE markers, similar to fluid markers, is calculated according to equation (18) to repulse the inner flow markers. In this approach, pressure of the BCE markers depend on the pressure of the fluid markers and vice versa [5]. We noticed that the convergence of the iterative procedure of Eq. (18) is slower when the first approach is used for boundary handling.

Figure 1: BCE and fluid markers, key for the coupling between fluid and solid, are represented by black and white circles, respectively. A section of the rigid body is shown herein as the gray area. The BCE markers positioned in the interior of the body (markers $g$ and $f$ in the figure) are placed at a depth less than or equal to the size of the compact support associated with the kernel function $W$. 
In the second approach, also known as the generalized wall boundary condition, the velocity of a BCE marker, \( \mathbf{v}_a \), is calculated based on [1].

\[
\mathbf{v}_a = 2\mathbf{v}^p_a - \mathbf{\tilde{v}}_a,
\]

where \( a \) denotes a BCE marker, \( \mathbf{v}^p_a \) is the prescribed wall velocity, and \( \mathbf{\tilde{v}}_a \) is an extrapolation of the smoothed velocity field of the fluid phase to the BCE markers,

\[
\mathbf{\tilde{v}}_a = \frac{\sum_{b \in F} \mathbf{v}_b W_{ab}}{\sum_{b \in F} W_{ab}}.
\]

In equation (21), \( F \) denotes the set of fluid markers overlapping the location of marker \( a \). The pressure of a BCE marker is calculated according to force balance at the wall interface, which is calculated as

\[
p_a = \frac{\sum_{b \in F} p_b W_{ab} + (\mathbf{g} - \mathbf{a}_w) \cdot \sum_{b \in F} \rho_b \mathbf{r}_{ab} W_{ab}}{\sum_{b \in F} W_{ab}},
\]

where \( \mathbf{g} \) and \( \mathbf{a}_w \) are the gravity and wall accelerations.

Convergence of the pressure equation (18) is faster in the second approach in comparison to the first approach since the pressure of the BCE markers are calculated more efficiently in (22) than equation (18). Moreover, the generalized BCE method given by equations (20) and (22) can be easily applied to arbitrary geometries.

2.1.2 Free Surfaces

The boundary condition of the free surface is an essential boundary condition for pressure (\( p = 0 \)). Density of the fluid markers is used to capture the free surface. Due to the insufficient number of particles in the support domain of the free surface markers, the density of the fluid markers drops at the free surface. Having essential BC for pressure is necessary in the solution of IISPH formulation since the pressure is solved in an iterative manner and the absence of a fixed pressure in the domain causes divergence in pressure calculation.

3 Validation and Results

In order to validate the IISPH implementation discussed above, two standard problems are chosen to show the capabilities of this method.

3.1 Hydrostatic Tank

The following presents the 2D simulation of cross-section of a rectangular tank with the height of \( H = 0.6 \) and width of \( L = 0.5 \). At \( t = 0 \) the markers are placed on Cartesian lattice,
and pressure and density are initialized with zero and reference density, respectively. Gravity acceleration is applied to the markers slowly according to \( g = \frac{g_0}{2} (1 - \cos \left( \frac{t}{t_{\text{max}}} \right) ) \) equation for \( t < t_{\text{max}} \) and \( g = g_0 \) for \( t > t_{\text{max}} \) as suggested in [1]. This provides the necessary damping that prevents markers from unsteadily oscillating and brings the marker in static equilibrium. Figure 2 demonstrates the results obtained using IISPH and the first approach for boundary handling.

![Figure 2](image)

(a) Pressure distribution at \( t = 0 \)  
(b) Pressure distribution at \( t = 1.5s \)

Figure 2: Pressure distribution in the hydrostatic tank using the first approach for BCE markers

It can be seen that pressure field shows an oscillating pattern especially away from the free surface. This is due to the fact that solution of Navier-Stokes equation will result in an oscillating pattern in pressure unless special treatment is made. Using staggered grids for structured grids, or addition of the dissipation terms for collocated grids are examples of such treatment in the Eulerian approach.

Figure 3 demonstrates the results obtained using IISPH and the generalized wall boundary condition (approach 2) for boundary handling.
The ordered particle distribution for $\frac{t}{t_{max}} > 1.5$ and linear pressure distribution of the markers are comparable to results obtained in [1]. It can be seen that the boundary condition of equation (22) which extrapolates the pressure field near the fluid markers, alleviates the pressure fluctuation that appears in the first approach to a great extend.

### 3.2 Dam Break

Dam break simulation is a classical model to show capability of Lagrangian approaches in modeling fluid dynamics. In what follows, a 2D simulation of break of a dam with the height of 1 m and the width of 1 m is presented. Fluid flow is modeled as inviscid flow with 1600 fluid markers with the reference density of $\rho = 1$ kg/m$^3$, particles are placed in a uniform Cartesian lattice at $t = 0$ s and the gravity of $g = 1$ m/s$^2$ is applied in the negative y direction as discussed in reference [1]. Water-front propagation study is shown in Figure 4.

It can be seen from Figure 4 that the water-front speed is over-estimated in the present study. The reason behind this discrepancy could be attributed to several factors; The fact that inviscid flow assumption is applied is the most relevant reason why the flow speed is faster in the present work.

Moreover, snapshots of fluid markers at different simulation times $T$ are shown in Figure 5 and are color-coded with velocity magnitude.
Figure 4: Comparison of water-front propagation between IISPH with generalized wall boundary conditions and experimental results of Martin and Moyce [3].
Figure 5: Snapshots of fluid markers at different times color-coded with velocity magnitude.
Appendix A  Momentum-preserving pressure forces

In this appendix the derivation of equation (12) is explained using the fundamental concept of SPH. For any function $f(x)$, the gradient of the function can be written as follows [4]:

$$\nabla f(x) = \rho [\nabla \left( \frac{f(x)}{\rho} \right) + \frac{f(x)}{\rho^2} \nabla \rho], \quad (23)$$

which incorporates the density. Moreover, based on fundamental SPH relationships, the approximation of the gradient of $f(x)$ can be written as:

$$< \nabla f(x_i) > = \sum_{j \in S(i)} \frac{m_j}{\rho_j} f(x_j) \nabla W(x - x_j, h), \quad (24)$$

where $<>$ represents the SPH approximation. By making use of the SPH approximation of equation (24) for the RHS of equation (23), it is concluded that:

$$< \nabla f(x_i) > = \rho_i \left[ \sum_{j \in S(i)} \frac{m_j}{\rho_j} \left( \frac{f(x_j)}{\rho_j} \nabla W(x_i - x_j, h) + f(x_i) / \rho_i \right) \nabla W(x_i - x_j, h) \right]$$

$$= \rho_i \left[ \sum_{j \in S(i)} m_j \left( \frac{f(x_j)}{\rho_j^2} + \frac{f(x_i)}{\rho_i^2} \right) \nabla W(x_i - x_j, h) \right] \quad (25)$$

where expressions outside of the gradient terms, $f(x)/\rho^2$ in equation (23) for instance, were evaluated at the particle $i$ itself. Now if $f(x_i)$ is replaced by $p(x_i)$, the pressure gradient term in the Navier-Stokes equations can be written as:

$$f^p_i = -\frac{1}{\rho_i} \nabla p(x_i) = -\sum_{j \in S(i)} m_j \left( \frac{p(x_i)}{\rho_i^2} + \frac{p(x_j)}{\rho_j^2} \right) \nabla W(x_i - x_j, h), \quad (26)$$

which is the pressure contribution to the external forces exerted to the marker $i$.

References


