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Representing fluid dynamics as a many-body dynamics problem: a vehicle fording analysis test case

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Abstract

In this report we present the outcome of a study that attempted to answer the following question: for complex fluid-solid interaction problems, can the fluid be represented as a large number of rigid spheres that mutually interact via contact forces? We set out to answer this question by first describing the approach used in Chrono::FSI to analyze coupled fluid-solid interaction problems. This framework can be leveraged to investigate applications that include (i) rigid bodies; (ii) bilateral constraints (joints); (iii) unilateral constraints associated with impact and contact phenomena; (iv) and friction/cohesion. Subsequently, a river fording scenario is simulated using two different approaches. The first relies on a Lagrangian method for the fluid dynamics; i.e., the smoothed particle hydrodynamics (SPH) method. The second approach uses frictionless rigid spheres which interact mutually through contact and mimic the motion of the fluid. The two approaches, which fall back on entirely different physics and rely on completely different numerical algorithms, are producing surprisingly similar results.

Keywords: Fluid-Solid Interaction (FSI), Chrono, Vehicle Fording, Smoothed Particle Hydrodynamics (SPH), Differential Variational Inequalities, Equations of Motion, Many-body Dynamics
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1 Introduction

The requirement that military, combat, and tactical use vehicles have an inherent river-crossing ability has existed for some years [21]. River fording operations have to do with the ability of vehicles to function in and near waterways. River fording includes the regions on both river banks, the water in the river and the land under it [4]. If a vehicle does not have the capability to ford a river and there is no ford or bridge available it is necessary to build an improvised bridge, attach a fording kit to the vehicle, or by-pass the river completely [18]. Although most vehicles can handle rivers, it is desired to design faster and more efficient vehicles. Speed is elusive because of the severe drag and wave making properties implicit in box-shaped bodies. More and more power does not assure more speed, except in worthlessly small increments, but rather serves to make bigger and bigger waves [4]. Additionally, vehicles operating in water have the potential to become unstable. Without control the vehicle might float downstream in a whirling path and miss the target exit completely, or arrive at the exit still striving for directional precision [4]. Chrono has the ability to approximate fluid using low friction contact between particles, providing valuable insights into a vehicles ability to cross a river. The goal of this work was to simulate a river fording operation where two different techniques would be compared. The first technique uses a smoothed particle hydrodynamics (SPH) based method and the second uses a set of frictionless three degree of freedom rigid spheres. Using the results from these simulations, understand how well they match each other and better understand the benefits and pitfalls associated with each of them.

In terms of frictional and frictionless contact in multibody dynamics, most of the existing methods can be grouped into two categories. First, and most common, are approaches that use a penalty, or regularization, approach in handling contact between bodies. Second, there are approaches built around a methodology that extends the concept of equations of motion to include differential inclusions [9]. For frictional contact, this methodology leads to differential variational inequality (DVI) problems, which upon discretization assume the form of mathematical programs with equilibrium and complementarity constraints. In this work we leverage a DVI framework for handling frictional contact constraints between rigid bodies as well as the frictionless contact between the spheres making up the fluid phase and the boundary constraints between phases.

The fluid dynamics problem is formulated using the Navier-Stokes equations for fluid momentum and continuity. These equations are spatially discretized using a constraint based smoothed particle hydrodynamics (SPH) Lagrangian method called Constraint Fluids. This method relies on moving markers to store field properties associated with the fluid phase. Space dependent variables such as density and pressure are computed locally via a smoothing function which interpolates contributions from nearby markers. The equations of motion posed in the Constraint Fluids [5] framework can be easily integrated into an existing constraint based frictional contact solver. Boundary conditions are posed as additional non-penetration constraints between fluid markers and rigid bodies.

The solution strategy outlined has been implemented in Chrono::Parallel. The vehicle is
modeled in Chrono::Vehicle, which is a Chrono toolkit that provides a template based approach to rapid vehicle prototyping. In Chrono::Vehicle, templates define the basic modeling elements (bodies, joints, force elements), impose the subsystem topology, prescribe the design parameters, and implement the common functionality for a given type of subsystem (e.g. suspension) particularized to a specific template (e.g. double wishbone). The full set of equations of motion is posed as a quadratic optimization problem and solved using an optimal first-order method [14]. The state of the system is advanced using a semi-implicit time integration scheme.

2 Background

Herein, the set of generalized coordinates used to position and orient a rigid body $j$ in the 3D Euclidean space are $r_j \in \mathbb{R}^3$ and $\epsilon_j \in \mathbb{R}^4$ [11]. The former provides the absolute position of the center of mass of body $j$, while the latter represents a set of Euler parameters (quaternions) that characterize body orientation in a global reference frame. The set of generalized coordinates for a system of $n_b$ bodies works out to be $q = [r_1^T, \epsilon_1^T, \ldots, r_{n_b}^T, \epsilon_{n_b}^T]^T \in \mathbb{R}^{7n_b}$ and their time derivatives $\dot{q} = [\dot{r}_1^T, \dot{\epsilon}_1^T, \ldots, \dot{r}_{n_b}^T, \dot{\epsilon}_{n_b}^T]^T \in \mathbb{R}^{7n_b}$. Rather than using $\dot{q}$ to pose the Newton-Euler equations of motion, the array $v = [\dot{r}_1^T, \dot{\omega}_1^T, \ldots, \dot{r}_{n_b}^T, \dot{\omega}_{n_b}^T]^T \in \mathbb{R}^{6n_b}$ is used since it leads to: (i) a smaller problem; and (ii) a constant, symmetric and positive definite mass matrix. There is a simple linear transformation that for each body relates its angular velocity expressed in the body-fixed reference frame, $\bar{\omega}_B$, to the time derivatives of the Euler parameters $\dot{\epsilon}_B$. Specifically, $\bar{\omega}_B = 2G(\epsilon_B)\dot{\epsilon}_B$, where the entries in the matrix $G \in \mathbb{R}^{3x4}$ depend linearly on the Euler parameters $\epsilon_B$ [11]. Defining the block diagonal matrix $L(q) \equiv \text{diag}[I_{3x3}, \frac{1}{2}G^T(\epsilon_1), \ldots, I_{3x3}, \frac{1}{2}G^T(\epsilon_{n_b})] \in \mathbb{R}^{7n_b \times 6n_b}$, where $I_{3x3}$ is the identity matrix, yields $\dot{q} = L(q)v$. For the fluid phase in a system of $n_f$ objects each fluid marker or rigid body $k$ consists of a position $r \in \mathbb{R}^3$ and a velocity $\dot{r} \in \mathbb{R}^3$. The fluid phase only has translational degrees of freedom and does not rotate, in this document objects that have six degrees of freedom will be denoted as 6dof and objects with three degrees of freedom as 3dof.

2.1 Rigid Body Modeling Aspects

Consider the contact between two bodies $A$ and $B$ represented in Fig. 1. Assuming that the body geometries are regular at the contact point, the contact point along with the shared tangent plane are used to define two local reference frames, one for each body. For body $A$, the normal $n_{i,A}$ at contact point $i$ is chosen to be perpendicular on the tangent plane at the shared contact point and to point towards the exterior of body $A$. Two mutually perpendicular unit vectors $u_{i,A}$ and $w_{i,A}$ are chosen to define a right-hand local reference frame associated with contact $i$ on body $A$. A similar sequence of steps is followed to define a local reference frame for body $B$ based on $n_{i,B}$, $u_{i,B}$, $w_{i,B} \in \mathbb{R}^3$. The Lagrange multiplier $\hat{\gamma}$ associated with contact $i$ is used to pose a complementarity condition in relation to the gap.
(distance) $\Phi$ between bodies $A$ and $B$: $0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(q) \geq 0$. When the bodies in contact have a smooth convex geometry, producing the gap function is straightforward. For complex and/or nonconvex geometries defining $\Phi_i(q)$ might pose difficulties [2,10].

In what follows, by convention, $\alpha_i \equiv \alpha_{i,A}$, for $\alpha \in \{n, u, w\}$. The force associated with contact $i$ can then be decomposed into the normal component, $F_{i,N} = \hat{\gamma}_{i,n}n_i$, and the tangential component, $F_{i,T} = \hat{\gamma}_{i,u}u_i + \hat{\gamma}_{i,w}w_i$, where the multipliers $\hat{\gamma}_{i,n} > 0$, $\hat{\gamma}_{i,u}$, and $\hat{\gamma}_{i,w}$ represent the magnitude of the force in each direction. The friction forces are assumed to satisfy the Coulomb dry-friction model, which can be expressed as [19,20]

$$\sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \leq \mu_i \hat{\gamma}_{i,n}$$

$$\|v_{i,T}\| \left(\sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} - \mu_i \hat{\gamma}_{i,n}\right) = 0,$$

$$\langle F_{i,T}, v_{i,T} \rangle = -\|F_{i,T}\|\|v_{i,T}\|,$$

where $v_{i,T}$ represents the relative tangential velocity between bodies $A$ and $B$ at the point of contact. These equations represent the first order Karush-Kuhn-Tucker optimality condition for the following optimization problem in two dummy variables $y, z \in \mathbb{R}$:

$$(\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \arg \min_{\sqrt{y^2 + z^2} \leq \mu_i \hat{\gamma}_{i,n}} v_{i,T}^T (yu_i + zw_i).$$

The force at the $i^{th}$ contact point can be expressed as $F_i = F_{i,N} + F_{i,T} = \hat{\gamma}_{i,n}n_i + \hat{\gamma}_{i,u}u_i + \hat{\gamma}_{i,w}w_i$. 

Figure 1: Contact $i$ between two bodies $A, B \in \{1, 2, \ldots, n_b\}$. 

\[\text{Figure 1: Contact } i \text{ between two bodies } A, B \in \{1, 2, \ldots, n_b\}.\]
\( \gamma_{i,u} w_i \in \Upsilon_i \), where \( \Upsilon_i \) is a 3D cone of slope \( \tan^{-1} \mu_i \), i.e., \( \Upsilon_i = \{ [x, y, z]^T \in \mathbb{R}^3 \mid \sqrt{y^2 + z^2} \leq \mu_i x \} \), oriented along \( n_i \) and with its tip at the contact point.

The Newton-Euler equations of motion [20] then assume the following expression:

Generators of positions
\[
\begin{align*}
\mathbf{q}^{(l+1)} &= \mathbf{q}^{(l)} + \frac{h}{2} \mathbf{L}_c \mathbf{v}^{(l+1)} \\
\end{align*}
\]

Velocity transformation matrix

Generators of speeds
\[
\begin{align*}
\mathbf{M}(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) &= h f(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) - \mathbf{G}(\mathbf{q}^{(l)}, t) \lambda + \sum_{i \in \mathcal{A}(q^{(l)}, \delta)} \left( \gamma_{i,n} \mathbf{D}_{i,n} + \gamma_{i,u} \mathbf{D}_{i,u} + \gamma_{i,w} \mathbf{D}_{i,w} \right) \\
\end{align*}
\]

Reaction impulse
\[
\begin{align*}
\mathbf{0} &= \frac{1}{h} \mathbf{g}(\mathbf{q}^{(l)}, t) + \mathbf{G}^T \mathbf{v}^{(l+1)} + \frac{1}{h} \mathbf{g}_l \\
\end{align*}
\]

Stabilization term

\[
\begin{align*}
\mathbf{D}_i = [\mathbf{D}_{i,n}, \mathbf{D}_{i,u}, \mathbf{D}_{i,w}] \in \mathbb{R}^{6n_b \times 3} \text{ are defined as} \\
\mathbf{D}_i^T &= \begin{bmatrix} 0 & \ldots & -\mathbf{A}_{i,p}^T \mathbf{A}_{i,p} \mathbf{A}_{i,n} \tilde{s}_{i,A} & 0 & \ldots & 0 \\
\end{bmatrix} \\
&= \begin{bmatrix} 0 & \ldots & -\mathbf{A}_{i,p}^T \mathbf{A}_{i,p} \tilde{s}_{i,B} & \ldots & 0 \\
\end{bmatrix}, \tag{4}
\end{align*}
\]

where \( \mathbf{A}_{i,p} = [\mathbf{n}_i, \mathbf{u}_i, \mathbf{w}_i] \in \mathbb{R}^{3 \times 3} \) is the orientation matrix associated with contact \( i \); \( \mathbf{A}_A = \mathbf{A}(\mathbf{A}_A) \) and \( \mathbf{A}_B = \mathbf{A}(\mathbf{A}_B) \) are the rotation matrices of bodies \( A \) and \( B \) respectively; and the vectors \( \tilde{s}_{i,A} \) and \( \tilde{s}_{i,B} \) represent the contact point positions in body-relative coordinates as illustrated in Fig. 1. Finally, the set of active and potential unilateral constraints is denoted by \( \mathcal{A}(q^{(l)}, \delta) \) and is defined based on the bodies that are mutually less than a distance \( \delta \) apart.

For interaction between a 6dof rigid body and a frictionless 3dof rigid body the equations can be greatly simplified. Tangential friction forces do not need to be computed and as a result the size of the problem is reduced from three constraints per contact to one constraint. A 3dof does not experience torque and the contact position only needs to be computed in body-relative coordinates for the 6dof rigid body. This means that for a contact between a 6dof object and a 3dof frictionless object \( \mathbf{D}_i \) simplified to \( \mathbf{D}_i = [\mathbf{D}_{i,n}, \mathbf{D}_{i,u}, \mathbf{D}_{i,w}] \in \mathbb{R}^{6n_b \times 1} \) for the 6dof body and \( \mathbf{D}_i = [\mathbf{D}_{i,n}, \mathbf{D}_{i,u}, \mathbf{D}_{i,w}] \in \mathbb{R}^{3n_b \times 1} \) for the 3dof body. For a contact between two 3dof bodies each body contributes a jacobian of size \( \mathbf{D}_i \in \mathbb{R}^{3n_b \times 1} \) for a contact.

### 2.2 Fluid Modeling Aspects

SPH is a meshless Lagrangian method that can be used to approximatively solve the Navier-Stokes equations for a viscous, Newtonian fluid.
\[
\frac{Dv}{Dt} = \frac{1}{\rho} - \nabla p + \mu \nabla^2 v + \rho g
\]  

(5)

where \( v \) is the velocity, \( \rho \) is the density, \( p \) is the pressure, \( \mu \) is the viscosity and \( g \) is the gravitational force acting upon the fluid. Generally these equations require the solution of partial differential equations (PDEs) and SPH uses volumetric integrals to approximate the field variables like velocity and density. The domain is discretized into a collection of markers which hold the field properties of interest, in order to compute the property at a certain location the property is interpolated based on values at neighboring markers. Interpolation is performed though so called “kernel approximation” over a finite support domain \( \Omega \)

\[
f(x) = \int_{\Omega} f(x') \delta(x - x') dV
\]  

(6)

where \( f(x) \) is a function of the position \( x \). The Dirac delta function \( \delta \) in Eq. 6 can be replaced with a smoothing function \( W \), also known as a kernel function, to yield:

\[
f(x) = \int_{\Omega} f(x') W(x - x', h) dV
\]  

(7)

The kernel approximation in Eq. 7 can be further approximated by using the markers which discretize the domain. The integral is replaced by a sum over the markers neighboring a location of interest \( x \) and can also be used to represent derivative of a given field property.

\[
f(x) = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) W(x - x_j, h)
\]  

(8)

\[
\nabla f(x) = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) \nabla W(x - x_j, h)
\]

The choice of \( W \) is problem dependent and can be tailored for performance and accuracy needs but should satisfy several properties \[12\]. For instance, the kernel function should be monotonically decreasing as the distance between \( x \) and a neighboring point \( x' \) increases, more distant markers should have less influence than closer ones. Kernel functions have a compact support within an influence radius of \( x \) to avoid a quadratic computational complexity and improve computational performance. Linear kernel functions are simple to compute but interpolate poorly when compared to quadratic or cubic kernel functions which can require larger support volumes but provide smoother interpolation. For results presented in this document the poly6 kernel \[15\] is used for \( W \) and and spiky kernel is used for \( \nabla W \) \[8\].

\[
W_{\text{poly6}}(x_j, h) = \frac{315}{64\pi h^9} (h^2 - x_{ij}^2)^3
\]

\[
\nabla W_{\text{spiky}}(x_j, h) = -\frac{45}{\pi h^6} (h - |x_{ij}|)^2 x_{ij}
\]  

(9)
The density of a marker $\rho_j$ can be computed by summing the mass contribution of the markers in a support region.

$$\rho_i = \sum_j^N m_j W_{ij}$$  \hspace{1cm} (10)

It is quite common in SPH for markers to have an insufficient number of neighbors resulting in a poor approximation of the density at a location. To correct for this the density of a marker can be normalized to improve the approximation.

$$\rho_i = \frac{\sum_{j=1}^N m_j W_{ij}}{\sum_{j=1}^N \frac{m_j}{\rho_j} W_{ij}}$$  \hspace{1cm} (11)

### 2.2.1 Constraint Fluids

Constraint fluids [5] is an approach that leverages many of the concepts introduced in SPH and poses the Navier-Stokes equations as a constrained minimization problem which enforces a many-body density constraint between a single marker and its neighbors. Viscous dissipation can be modeled by adding additional viscosity constraints [17] which dissipate energy in the material due to friction and shear. The concept of using constrains has also been extended to elastoplasticity by enforcing a strain based constraint, rather than density, on each marker [16] The main difference between traditional SPH based fluid simulations and constraint fluids is that the density is enforced using a many-body constraint rather than a force. Depending on the discretization a corrective impulse [5] or a position correction [13] is applied to each marker to enforce the density constraint. This holonomic constraint tries to match the current density of a fluid marker $\rho_i$ with the target density $\rho_0$ which results in the following indicator function.

$$d_i = \frac{\rho_i - \rho_0}{\rho_0}$$  \hspace{1cm} (12)

The jacobian of this constraint can be found by differentiating $d$ with respect to time and applying the chain rule

$$\dot{d}_i = \frac{d}{dt} \left(\frac{\rho_i}{\rho_0} - 1\right)$$

$$\dot{d}_i = \sum_j \frac{m_j}{\rho_j} \frac{dW_{ij}}{dt}$$

$$\dot{d}_i = \sum_j \frac{m_j}{\rho_j} \frac{dW_{ij}}{dx_{ij}}$$

$$\dot{d}_i = \sum_j \frac{m_j}{\rho_j} \frac{dW_{ij}}{dx_{ij}} x_{ij} (v_i - v_j)$$

$$\dot{d}_i = \sum_j \frac{m_j}{\rho_j} \frac{dW_{ij}}{dx_{ij}} x_{ij} v_i - \sum_j \frac{m_j}{\rho_j} \frac{dW_{ij}}{dx_{ij}} x_{ij} v_j$$
Taking the non-velocity terms in Eq. 13 we can form the diagonal $D_{ii}$ and off diagonal $D_{ij}$ terms of the Jacobian matrix $D$. Note that the diagonal terms are the sum of the off diagonal ones for each marker.

$$D_{ii} = \frac{1}{\rho_0} \sum_{k \neq i} m_k \frac{dW_{ij}}{dx_{ij}} x^T_{ik}$$

$$D_{ij} = -\frac{m_j}{\rho_0} \frac{dW_{ij}}{dx_{ij}} x^T_{ik}$$

(14)

Each constraint contributes a single row in the Jacobian matrix with a length of $3n$ where $n$ is the number of fluid markers. Each row has three values at the columns corresponding to the current marker $i$ and three values for each neighboring marker $j$. The Jacobian has a constant size, depending on the number of fluid markers in the simulation, and the density of this matrix changes depending on the number of neighbors a marker has.

One of the problems with SPH like methods is that markers that lack neighbors tend to pull their neighbors in closer causing clumping to occur. One solution is to use an extra ‘near pressure’ term that provides a repulsive impulse [7, 13]. This artificial pressure term $p_{corr}$ is used to correct the computed pressure impulses at the end of the step. Typical values used for constants are $|\Delta q| = 0.1 h t o 0.3 h$, $k = 0.1$ and $n = 4$.

$$p_{corr} = \sum k W_{ij}(x_{ij}, h)^n \frac{W(\Delta q, h)}{W(\Delta q, h)}$$

(15)

When dealing with rigid boundaries, fluid markers are treated as rigid spheres and similar to a 3dof rigid body, a non-penetration constraint is created for enforce incompressible contact. Coupling in this manner is straightforward and allows interaction between fluid and rigid bodies in a physically accurate manner, see Fig. . Note that frictional contact between rigid boundaries and the fluid can be used to capture energy losses at the surface.

## 2.3 Numerical Solution Methodology

The numerical solution methodology for the aforementioned DVI problem is built around the following two decisions: (D1) following the approach proposed in [20], a symplectic half implicit Euler methods is used to discretize the dynamics; and (D2) a zero gap non-penetration condition between bodies in mutual contact is enforced at the new time step $t^{(l+1)}$. Given a consistent position $q^{(l)}$ and velocity $v^{(l)}$ at time $t^{(l)}$, the numerical solution at $t^{(l+1)} = t^{(l)} + h$ is obtained by solving the following mathematical programming problem with complementarity and equilibrium constraints:
\[
\begin{align*}
\text{Generalized positions} & \quad \dot{\mathbf{q}}^{(l+1)} = \mathbf{q}^{(l)} + \Delta t \mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)} \\
\text{Step size} & \quad \Delta t \mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)} \\
\text{Velocity transformation matrix} & \quad \mathbf{M}(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) = \Delta t \mathbf{f}(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) - \mathbf{G}(\mathbf{q}^{(l)}, t)\lambda + \mathbf{D}_c(\mathbf{q}^{(l)}, t)\gamma_c - \mathbf{D}_d(\mathbf{q}^{(l)}, t)\gamma_d - \mathbf{D}_b(\mathbf{q}^{(l)}, t)\gamma_b \\
\text{Joint impulse} & \quad \mathbf{M}(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) = \Delta t \mathbf{f}(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) - \mathbf{G}(\mathbf{q}^{(l)}, t)\lambda + \mathbf{D}_c(\mathbf{q}^{(l)}, t)\gamma_c - \mathbf{D}_d(\mathbf{q}^{(l)}, t)\gamma_d - \mathbf{D}_b(\mathbf{q}^{(l)}, t)\gamma_b \\
\text{Applied impulse} & \quad 0 = \frac{1}{\Delta t} \mathbf{g}(\mathbf{q}^{(l)}, t) + \mathbf{G}^\top \mathbf{v}^{(l+1)} + \frac{1}{\Delta t} \mathbf{g}_t \\
\text{Density Impulse} & \quad -\mathbf{D}_d^T \mathbf{v}^{(l+1)} = \mathbf{g}_d(\mathbf{q}^{(l)}, t) \\
\text{Stabilization term} & \quad 0 = \frac{1}{\Delta t} \mathbf{g}(\mathbf{q}^{(l)}, t) + \mathbf{G}^\top \mathbf{v}^{(l+1)} + \frac{1}{\Delta t} \mathbf{g}_t \\
\text{Stabilization term} & \quad -\mathbf{D}_d^T \mathbf{v}^{(l+1)} = \mathbf{g}_d(\mathbf{q}^{(l)}, t) \\
\text{Stabilization term} & \quad j \in \mathcal{A}(\mathbf{q}^{(l)}, \delta) : 0 \leq \frac{1}{\Delta t} \Phi_{b,j}(\mathbf{q}^{(l)}) + \mathbf{D}_{b,j,n}^T \mathbf{v}^{(l+1)} \perp \gamma_{b,j,n} \geq 0 \\
\text{Stabilization term} & \quad i \in \mathcal{A}(\mathbf{q}^{(l)}, \delta) : 0 \leq \frac{1}{\Delta t} \Phi_{c,i}(\mathbf{q}^{(l)}) + \mathbf{D}_{c,i,n}^T \mathbf{v}^{(l+1)} \perp \gamma_{c,i,n} \geq 0 \\
\text{Stabilization term} & \quad (\gamma_{c,i,u}, \gamma_{c,i,w}) = \arg\min_{\sqrt{\gamma_{c,i,u}^2 + \gamma_{c,i,w}^2} \leq \mu_i \gamma_{c,i,n}} \mathbf{v}^\top(\gamma_{c,i,u} \mathbf{D}_{c,i,u} + \gamma_{c,i,w} \mathbf{D}_{c,i,w}) \\
\end{align*}
\]

Here, \( \lambda \) represents a constraint impulse associated with a bilateral constraint, \( \gamma_c, \gamma_d, \gamma_b \) are constraint impulses associated with contacts, density constraints and boundary contacts respectively. The superscript \((l+1)\) on \( \gamma_c \) was dropped for notational brevity. All forces acting on the system except the frictional contact, density and boundary constraint forces are evaluated at time \( t^{(l)} \) and denoted by \( \mathbf{f} \equiv f(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) \). The term \( \frac{1}{h} \Phi_i(\mathbf{q}^{(l)}) \) achieves constraint stabilization by eliminating any penetration within one time-step.

For large models with millions of contacts, no effective methods are available for solving the numerical problem in Eqs. (16b)–(16g). This observation motivated a third decision (\( D3 \)), which was to recast the aforementioned numerical problem into a more amenable one by a convexification of the NCP; i.e., by relaxing the complementarity condition in Eq. (16f) [3] to

\[
\begin{align*}
i \in \mathcal{A}(\mathbf{q}^{(l)}, \delta) & : 0 \leq \frac{1}{h} \Phi_i(\mathbf{q}^{(l)}) + \mathbf{D}_{i,n}^T \mathbf{v}^{(l+1)} \\
& -\mu_i \sqrt{(\mathbf{v}^\top \mathbf{D}_{i,u})^2 + (\mathbf{v}^\top \mathbf{D}_{i,w})^2} \perp \gamma_i^u = 0 \, .
\end{align*}
\]

Owing to this relaxation, the resulting set of equations become a cone complementarity problem (CCP). Specifically, solving for \( \mathbf{v}^{(l+1)} \) from Eq. (16b) and plugging its expression in Eq. (17) yields a mathematical programming problem with complementarity constraints.
formulated exclusively in the set of Lagrange multipliers $\gamma$ [1]. To pose the CCP, the following notation is used: the number of contacts in $A(q, \delta)$ is $n_c$; $D \equiv [D_1, \ldots, D_{n_c}] \in \mathbb{R}^{6n_b \times 3n_c}$ is the generalized contact transformation matrix; $D_i \equiv [D_i,n, D_i,v, D_i,w] \in \mathbb{R}^{6n_b \times 3}$ is the contact transformation matrix associated with contact $i \in A(q(l), \delta)$; $r_i \equiv b_i + D_i^T M^{-1} f \in \mathbb{R}^3$ is the generalized contact velocity for contact $i$; $b_i \equiv \begin{bmatrix} 1 \ h \Phi_i(q(l)), 0, 0 \end{bmatrix}^T \in \mathbb{R}^3$ is the unilateral constraint stabilization term; and $N \equiv D^T M^{-1} D \in \mathbb{R}^{3n_c \times 3n_c}$ is the contact associated symmetric positive-semidefinite Schur complement matrix, which is typically very sparse.

The new quantities introduced – $n_c$, $D$, $D_i$, $r_i$, $b_i$, and $N$ – should be further qualified by a superscript $(l)$ to indicate that they are evaluated in the system configuration corresponding to $t^{(l)}$. For brevity, the superscript was omitted. The CCP then assumes the form

Find $\gamma_i^{(l+1)}$, for $i = 1, \ldots, n_c$

such that $\Upsilon_i \ni \gamma_i^{(l+1)} \perp -(N \gamma_i^{(l+1)} + r_i) \in \Upsilon_i^o$

where $\Upsilon_i = \{ [x, y, z]^T \in \mathbb{R}^3 | \sqrt{y^2 + z^2} \leq \mu_i x \}$

and $\Upsilon_i^o = \{ [x, y, z]^T \in \mathbb{R}^3 | x \leq -\mu_i \sqrt{y^2 + z^2} \}$.

This CCP represents the first order optimality condition of a quadratic optimization problem with conic constraints whose solution provides the set of normal and friction forces associated with the set of contacts in $A(q, \delta)$:

$$\min f(\gamma) = \frac{1}{2} \gamma^T N \gamma + r^T \gamma$$

subject to $\gamma_i \in \Upsilon_i$ for $i = 1, 2, \ldots, n_c$.

The overall approach is summarized as follows: a multibody dynamics frictional contact application is formulated as a DVI problem, which based on decisions $(D1)$ and $(D2)$ morphs into an NCP. The latter is convexified based on $(D3)$ to become a CCP. The CCP is solved by considering an equivalent quadratic optimization problem with conic constraints, whose solution is the desired $\gamma \in \mathbb{R}^{3n_c}$. Equation (16b) is then used to expeditiously compute the velocity $v^{(l+1)}$. The generalized coordinates $q^{(l+1)}$ are recovered using Eq. (16a) and the simulation is advanced at $t^{(l+1)}$.

### 3 Fording Simulation

The simulation setup uses a simplified nine body HMMWV model based on the default template provided in chrono::vehicle. The model consists of a 2086.52 kg chassis connected to 4 rigid lugged wheels via a simplified double wishbone suspension. A four wheel drive powertrain provided better traction and torque distribution compared to a front or rear wheel drive configuration as the vehicle entered the fluid. A PID controller was used for driver model which drove in a straight line while providing throttle and brake input to the vehicle.
to maintain a vehicle speed of 2 m/s. The constants for the PID controller were 4.0, 1.0, 0.0, for the proportional, integral and derivative components respectively. The collision geometry used for this vehicle consists of a convex decomposition of the chassis and the lugged wheel. For the chassis the original visual mesh was simplified by hand to remove extra geometry and was sliced into several sections for processing by v-hacd [6] a robust and controllable convex hull generation algorithm. This algorithm generated a set of convex hulls which were used for collision detection against other rigid objects and the fluid. Fig. shows the original mesh followed by the simplified mesh and the result of the convex decomposition. Similarly, the lugged wheel was decomposed into convex pieces and used for collision detection.

The fording setup, shown in Fig. 4 consists of two end platforms approximately 4.87 m (16 ft) long followed by a slope of length 5.18 m (17 ft) and a bottom slope of length 4.57 m (15 ft). The distance from the end platform to the bottom was 2.43 m (8 ft) and was filled with fluid such that half of the chassis would be under water during the fording operation, see Fig. 5.

3.1 Results

Results showed that over the entire fording operation the engine used approximately 278 530 J of energy to drive through the fluid and 372 112 J to drive through the rigid spheres, a difference of 25%. Rigid spheres are more difficult to move through even when frictionless as spheres interlock when settled and more energy is required to make them flow. This difference in flowability causes the increase in energy required to perform the fording operation. Fig. 8 shows the velocity of the vehicle as it performs the fording operation. Note that while the throttle/brake controller is trying to enforce a velocity of 2 m/s the system is highly dynamic and is often not traveling at exactly 2 m/s. Additionally the controller required more throttle
while traveling through the rigid spheres as shown in Fig. 6 when compared to the fluid due to the larger forces exerted upon the chassis, see Fig. 7.
Figure 4: Fording Setup

Figure 6: Fording Constraint Fluids
Figure 5: Fording Constraint Fluids

Figure 7: Fording Constraint Fluids
Figure 8: Fording Constraint Fluids

References


