EFFICIENT SIMULATION OF FLEXIBLE BODY SYSTEMS WITH FRICTIONAL CONTACT/IMPACT

By

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A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy
(Mechanical Engineering)

at the

UNIVERSITY OF WISCONSIN-MADISON

2015

Date of final oral Examination: 11/24/2014

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ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Dan Negrut, for his support and guidance. I would also like to thank the committee members, Professor Darryl Thelen, Professor Michael Zinn, Professor Robert Rowlands, and Professor Peter Qian for sharing their expertise and for their time. I owe a large debt of gratitude to my friends and colleagues, especially Arman Pazouki and Praveen Yadav, who have been generous with their technical expertise and friendship during the past six years.

Finally I would like to thank my parents, my brothers, and my wife for believing in me and keeping me motivated. This thesis would not have been possible without their unconditional love and support.
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ABSTRACT

This thesis mainly contributes to three areas in the field of multibody dynamics simulation (MBS): (i) numerical integration methods, (ii) modeling of flexible bodies that undergo large rotation and large deformation, and (iii) modeling of frictional contact/impact between flexible bodies. Specifically, six different implicit low order numerical integration methods to solve the differential algebraic equations (DAEs) are discussed herein and their performance is compared based on various metrics such as order of convergence, energy preservation, constraint satisfaction, and efficiency.

In this thesis, the Absolute Nodal Coordinate Formulation (ANCF) is thoroughly explored to model the geometrically nonlinear elastic bodies in the flexible multibody systems. To avoid the locking problems of original ANCF, gradient deficient ANCF is used in index-3 DAE framework for thin beam applications. The formulation of gradient deficient ANCF beam element for straight as well as initially curved beam is investigated. Several numerical experiments are performed to study the convergence behavior of gradient deficient ANCF and to validate the results.

Moreover, two widely used approaches for handling frictional contact of rigid bodies, namely the penalty method and the Differential Variational Inequality (DVI) method, are extended to solve the frictional contact/impact problem of ANCF flexible bodies. This thesis focuses on developing, implementing, and validating these contact modeling approaches within the ANCF framework.

In the first approach, the penalty-based Hertzian contact model and regularized Coulomb friction model is used to model contact/impact between ANCF beams. A novel approach of the spherical
decomposition of beams is introduced, which simplifies the beam contact detection problem. The beam-to-beam and beam-to-rigid contact/impact cases are considered in the numerical experiments. The results are validated using the commercial software, ABAQUS and the research software, FEAP.

In the second approach, the traditional DVI method is extended to model frictional contact/impact between ANCF beams. A detailed derivation of this approach for both frictionless and frictional contact/impact is provided along with implementation details. The complementarity condition in the traditional DVI method is only valid for purely inelastic impact. A new complementarity condition is introduced, which includes the coefficient of restitution to model a generalized impact problem. The numerical experiments confirm that total energy of the system is conserved for the frictionless impact case due to introduction of new complementarily condition. Moreover, the results of DVI approach are compared with those obtained using the penalty approach for the beam-to-beam and beam-to-rigid contact/impact cases.

Finally, the capabilities of ANCF beams are demonstrated through the simulation of complex flexible multibody systems. The models considered for demonstration purpose include a satellite model with self deployable solar panels and an elevator traveling cable model.
1 Introduction

With recent advancements in the computational hardware capabilities, computer simulations have been augmenting, if not replacing, the physical prototyping of mechanical systems. The use of CAE tools has become critical to engineering design for nearly every manufacturing company.

In the last few decades, multibody dynamics simulation (MBS) has emerged as a CAE tool which is widely used in product design process in the automotive and aerospace industry. Finite element analysis is the most popular simulation tool but it tends to break down in situations where many components are in motion and interacting with each other. On the other hand, MBS provides tools for quickly modeling and simulating complex assemblies with many components. MBS provides better prediction of loads earlier in the design process, making it possible to accurately predict life prior to the prototype phase. As a result, MBS makes it possible to simulate the performance of mechanical system to a much higher level of fidelity. Moreover, MBS improves engineering efficiency and reduces product development costs by enabling early system-level design validation. Engineers can evaluate and manage the complex interactions between disciplines including motion, structures, actuation, and controls to better optimize product designs for performance, safety, and comfort.

1.1 Motivation

MBS was originally developed for modeling rigid multibody systems with simple tree-like topologies, but now it has evolved to the point, where it can handle flexible multibody systems with arbitrary topologies. Greater emphasis has been placed on high-speed, lightweight,
precise mechanical systems. Often these systems will contain one or more structural components where deformation effects are paramount for design analyses and the rigid body assumption is no longer valid. Flexible multibody dynamics provides the technology to correctly include a component’s flexibility even in the presence of large overall motion and complex interaction with other modeling elements. Currently, in the flexible multibody systems the deformable bodies are generally modeled as linearly elastic bodies in which the elastic displacement field is approximated using modal expansion techniques. However, for systems in which the elastic bodies undergo large rotation and/or large deformation: e.g. belts, cables, beams, plates/shells applications, the linear elastic representation is not sufficient, and inclusion of nonlinear elasticity in the flexible multibody system is necessary.

In the recent past, several methods such as geometrically exact beam formulation[1, 2] , incremental finite element approach or so called the co-rotational formulation [3, 4], etc. have been developed to model nonlinear elastic bodies. In this work, the Absolute Nodal Coordinate Formulation (ANCF) [5] is thoroughly explored to model the geometrically nonlinear elastic bodies in the flexible multibody systems. The original or fully parameterized ANCF suffers from several locking problems. To alleviate these issues several improvements in the original ANCF are proposed in the literature [6-8]. In this work the gradient deficient ANCF beam formulation [8] is used to avoid the locking problems. Implementation of the gradient deficient ANCF beam formulation within the framework of widely used implicit and explicit numerical integrators in the general purpose MBS software is thoroughly discussed.

Another active research area in the field of MBS is contact/impact problems. Numerical methods for simulating systems that feature frictional contact/impact are critical in many fields of applied mechanics. Real-life problems may include cases with few contacts: e.g. gear
mechanism, leaf springs, robotic applications, etc., as well as many-body dynamics applications involving millions of contact: e.g. earth-moving machines operating on granular soils, simulation of granular material, etc. Currently, there are two popular approaches for handling frictional contact in the MBS. The first approach with continuous contact/impact model [9, 10] or regularized contact model [11] is a penalty method, where interaction force is computed based on the parameters of fictitious spring-damper system placed between the colliding bodies. The other approach, based on differential variational inequality (DVI) [12], enforces non-penetration of rigid bodies via a constraint-based formulation. In DVI approach, a cone-constrained quadratic optimization problem must be solved at each time step of the simulation, where the unknowns are the normal and frictional contact forces between interacting bodies.

While studying these contact modeling approaches in the literature, the researcher noticed that they are mostly used for rigid-rigid contact cases. There is a need to generalize these approaches so that they also support rigid-flexible and flexible-flexible contact cases. In the modern MBS software, the general purpose contact modeling approaches should be able to accurately simulate the contact phenomenon between nonlinear elastic bodies. In this work both penalty and DVI methods are extended to support rigid-flexible and flexible-flexible contact cases. This research work focuses on developing, implementing, and validating these contact modeling approaches within the ANCF framework.

1.2 Thesis Overview

The thesis document is organized as follows: Chapter 2 provides a review of multibody dynamics in which different techniques for the solution of differential-algebraic equations
(DAEs) of multibody dynamics are discussed. Moreover, a literature review of linear and nonlinear flexible multibody dynamics is provided followed by a discussion on locking issues of fully parameterized ANCF. Chapter 3 provides a discussion on six different implicit low order numerical integration methods, which are widely used to solve the DAEs in multibody dynamics. The performance of these integration methods is compared in terms of order of convergence, energy preservation, constraint satisfaction and efficiency.

The gradient deficient ANCF beam element is introduced in Chapter 4. Theoretical and implementation details of this locking-free beam element within the framework of explicit and implicit numerical integration methods are provided. Several numerical experiments are performed to study the convergence behavior and to validate the results.

Chapter 5 discusses the penalty-based Hertzian contact model used to model frictional contact/impact between ANCF beams. A novel approach of spherical decomposition of beams is introduced, which simplifies the beam contact detection problem. The beam-to-beam and beam-to-rigid contact/impact cases are considered in the numerical experiments. The results are validated using the commercial software ABAQUS and the research software FEAP.

In Chapter 6, the traditional DVI approach is extended to model frictional contact/impact between ANCF beams. A detailed derivation of this approach for both frictionless and frictional contact/impact is provided along with implementation details. The complementarity condition in the traditional DVI approach is only valid for purely inelastic impact. A new complementarity condition is introduced which includes the coefficient of restitution to model a generalized impact problem. The results are compared with those obtained using the penalty approach.
Chapter 7 provides a demonstration of technology where some real life applications of ANCF beams are discussed. Finally some concluding remarks and the directions for future research are discussed in Chapter 8.

1.3 **Specific Contributions**

The specific contributions of the author are summarized as follows:

a. **Numerical Integration Methods**
   - The numerical algorithms of low order implicit integration methods (Newmark, HHT-I3, NSTIFF, NSTIFF-SI2, HHT-ADD, and HHT-SI2) to solve DAEs are discussed and implemented in Matlab.
   - The implementation of new integration methods like HHT-ADD and HHT-SI2, and their comparison with the existing numerical integration methods based on various metrics gives insight to correctly choose the integration method, depending on the nature of a multibody model.

b. **ANCF**
   - The formulation of original ANCF, gradient deficient ANCF and initially curved ANCF beam is discussed and implemented in Matlab.
   - The implementation of ANCF within the index-3 DAE framework is shown.
   - The locking problems of original ANCF, the convergence analysis of gradient deficient ANCF, and the validation of ANCF using ABAQUS and FEAP are shown through various numerical experiments.

c. **ANCF + Contact/Impact using penalty method**
- The formulation of a continuous contact force model based on Hertzian contact theory and the regularized Coulomb friction model is discussed and implemented to solve the frictional contact/impact problem of ANCF flexible bodies.

- A novel approach of spherical decomposition is introduced for the contact discretization of the ANCF flexible body. The spherical decomposition expedites contact detection and allows using reduced number of finite elements compared to the conventional contact discretization schemes.

- The numerical experiments with beam-to-beam and beam-to-rigid frictional contact/impact cases are carried out and the results of the proposed method are validated against ABAQUS and FEAP results.

d. ANCF + Contact/Impact using DVI

- The DVI algorithm is extended to model the frictional contact/impact problem of ANCF flexible bodies.

- The DVI algorithm is modified to handle the impact problems with nonzero coefficient of restitution. Note that the original DVI algorithm is limited to the contact or purely inelastic impact problems.

- A spherical decomposition approach is used for the contact discretization of ANCF in the contact kinematics of DVI formulation.

- The complementarity conditions are derived for both contact and impact cases.

- The cone complementarity problem (CCP) formulation for frictional contact between ANCF flexible bodies is derived and the projected Gauss-Jacobi algorithm is implemented to solve the CCP.
- The energy analysis is done for the beam-to-beam and beam-to-rigid frictional contact/impact cases. The results of these test cases are compared with the penalty method results.
2 Literature Review

2.1 Multibody Dynamics

Multibody dynamics can be defined as the discipline whose focus is the development of modeling techniques and solution methods that are instrumental in describing the time evolution of mechanical systems consisting of interconnected bodies subjected to the action of externally applied forces. Over the last two decades multibody dynamics simulation (MBS) has become an integral part of the product development cycle in industry, playing an increasingly important role in engineering design. Through MBS, the dynamics of a mechanical system is understood by solving the Differential Algebraic Equations (DAEs) that govern the time evolution of the system.

The generalized coordinates of a body in the multibody system are chosen herein to be its position and orientation with respect to the global reference frame. For body $i$, they are the Cartesian coordinates for position, and Euler parameters for orientation of the rigid body centroidal reference frame; i.e., $r_i = \hat{X}_i, Y_i, Z_i \hat{T}_i$, and $p_i = \hat{e}_{i\sigma}, e_{i\tau}, e_{i\zeta} \hat{T}_i$, respectively. It should be noted that the use of quaternions, i.e. Euler parameters, is better suited for computational application since the use of Euler or Bryant angles may lead to singularity problems.

The state of the system containing $n_b$ rigid bodies is denoted by the generalized positions

$$\mathbf{q} = \hat{e}_1, \ldots, \hat{e}_{n_b}, p_1, \ldots, p_{n_b} \hat{T}_i \hat{\mathbf{I}} \hat{\mathbf{I}} \hat{\mathbf{n}}, \quad n = 7n_b,$$

where the Euler parameters $p_i$ satisfy the normalization constraint [13]
The array of generalized velocities \( \mathbf{q} = \left[ \dot{q}_1^T, \ldots, \dot{q}_n^T \right]^T \) captures the velocity state of the system. The kinematic joints connecting bodies in a multibody system restrict their relative motion and impose constraints on the position and velocity generalized coordinates. These kinematic constraints are formulated as algebraic expressions involving generalized coordinates. In this analysis, without loss of generality, these constraints are considered to be bilateral and holonomic:

\[
F(q, t) = [F_1(q, t), \ldots, F_m(q, t)]^T
\]

where \( m \) is total number of constraints present in the mechanical system. Differentiating Eq. (2) with respect to time yields the velocity kinematic constraint equation

\[
F_q(F, q) \dot{\mathbf{q}} + F_t = 0
\]

where the over-dot denotes differentiation with respect to time and the subscript denotes partial differentiation,

\[
F_q = \frac{\partial F}{\partial q_i} \quad \text{for} \quad 1 < i < m, \quad 1 < j < n.
\]

The acceleration kinematic constraint equation is obtained by differentiating Eq. (3) with respect to time:

\[
F_q(q, t) \ddot{q} + \left( (F_q(q, t) q \ddot{q}) + 2F_{q_t}(q, t) q \dot{\mathbf{q}} + F_{t_t}(q, t) \right) \cdot \tau(q, q) = 0
\]

The generalized coordinates of the system change in time under the effect of applied forces such that these constraint equations are satisfied at all times. The time evolution of the system is governed by the Lagrange multiplier form of the constrained equations of motion [13]

\[
M(q) \ddot{q} + F_q^T(q) \lambda = Q(q, q, t)
\]
where $\mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}^n$ is the generalized mass, $\lambda \dot{\mathbf{q}}^m$ is the Lagrange multiplier associated with the kinematic constraints, and $\mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}}, t) \dot{\mathbf{q}}^n$ is the applied force on the generalized coordinates $\mathbf{q} \dot{\mathbf{q}}^n$. The acceleration kinematic constraint equations are associated with the equations of motion to obtain a linear system in the generalized accelerations and Lagrange multipliers [14]

$$
\begin{align*}
\mathbf{M} & \ddot{\mathbf{q}} \\
\mathbf{F} & = \mathbf{Q} \dot{\mathbf{q}}
\end{align*}
$$

(6)

2.2 Numerical Solution Methods for DAEs

Equations (2) through (5) represent a system of Differential Algebraic Equations (DAEs) of index 3 [15]. Mathematically, DAEs are not ODEs [16], since DAEs are basically differential equations defined on, or constrained to, submanifolds of $\mathbf{q}^n$. The index of a DAE is defined as the number of derivatives required to transform the DAE into an ODE [17]. The task of solving the DAE problem is more difficult and prone to intense numerical computation than that of solving an ODE [18]. In this context, since Eqs. (2) and (5) that govern the time evolution of the system are solved numerically, Eqs. (3) and (4) are not guaranteed to be satisfied. When this is the case, numerical solution at velocity and acceleration levels can drift away from the analytical solution. Consequently, future position configurations of the system will be wrong, since the numerical integration stage embedded in the numerical algorithm uses corrupted derivative information [19].
There are specialized numerical methods dedicated to the solutions of the DAE of multibody dynamics such as stabilization methods, projection methods, and state-space methods. A detailed review of this topic is provided in [19]. More recently the direct methods belonging to the generalized-alpha family are developed in [20, 21]. A brief review of these methods is presented next.

2.2.1 Stabilization Methods

A first algorithm for numerical integration of DAEs based on the constraint stabilization technique was provided by Baumgarte [22]. In this method, the problem is reduced to Eq.(6) and is directly integrated. Since after direct integration the constraints fail to be satisfied, at the next time step, the right side of the acceleration kinematic constraint equation is modified to take into account the constraint violation. The form of the right side of the acceleration kinematic constraint equation is altered to

\[ \tau_{\text{new}}(\mathbf{q}, t) = \tau(\mathbf{q}, t) - 2a\mathbf{F}(\mathbf{q}, t) - b\mathbf{F}(\mathbf{q}, t) \]  

(7)

where last two terms compensate for errors in satisfying constraint equations at the position and velocity levels. Ostermeyer [23] has discussed the criteria for optimally choosing the values of \(a\) and \(b\). Asher et al. [24] improved Baumgarte’s approach by suggesting more sophisticated techniques which use a compensation coefficient matrix [25]. This approach is more reliable for nonstiff and highly oscillatory problems and it is extended to the simulation of mechanical systems with many closed loops using sequential regularization methods in [26].

2.2.2 Projection methods

Another approach for numerical solution of the DAEs of multibody dynamics is based on so called projection techniques [27]. Additional multipliers are introduced to account for the
requirement that the solution satisfy constraint equations at position, velocity, and for some formulations, acceleration level. Gear et al. [28] reduce the DAE to an analytically equivalent index 2 problem, in which projection is only performed at the velocity level. An extra multiplier, \( \eta \), is introduced to ensure that the velocity kinematic constraint equation of Eq. (3) is also satisfied. The algorithm uses a backward differentiation formula (BDF) to discretize the following form of the equations of motion:

\[
\begin{align*}
\mathbf{v} &= \dot{\mathbf{q}} + F^T_q(q, t) h \\
M(q)\ddot{\mathbf{q}} &= Q - F^T_q(q, t) \lambda \\
F(q, t) &= 0 \\
F_q(q, t) v + F_i(q, t) &= 0.
\end{align*}
\] (8)

In a similar approach, proposed by Fuhrer and Leimkuhler [29], the DAE is reduced to index 1 and an additional multiplier \( h \) is introduced, along with the requirement that the acceleration kinematic equation of Eq. (4) is satisfied. The methods proposed by Fuhrer and Leimkuhler and Gear et al. belong to the class of so called derivative projection methods; i.e., expressions for derivatives are modified by additional multipliers that ensure constraint satisfaction.

A second projection technique is based on the coordinate projection approach. The derivatives are no longer modified, and integration is carried out to obtain a solution of the index 1, or for some formulations index 2 DAE. Since all variables are integrated, they do not satisfy the constraint equations, and some form of coordinate projection technique is employed to bring the ODE solution to the constraint manifold. This is the approach followed by Eich [30] and Shampine [30, 31]. The code MEXX (Lubich et al. [32]) for integration of multibody systems is
based on coordinate projection and uses relatively expensive but very accurate extrapolation methods for the numerical integration of the ODE.

2.2.3 **State Space Methods**

State-space methods first reduce the DAEs to a smaller dimension ODE problem, thus benefiting from the extensive body of knowledge associated with ODE solvers. Specifically, the DAEs induce differential equations on the constraint manifold [33], which can be reduced on a subspace of the $n$-dimensional Euclidean space. The resulting state-space ODEs (SSODEs) are integrated using classical numerical integration formulas. The one-to-one local mapping from the manifold to the subspace of independent coordinates is then used to determine the point on the manifold corresponding to the solution of the SSODEs. This framework formalizes the theory of numerical solution of DAEs using the language of differential manifolds [34]. Practical approaches in this class of methods are due to Wehage and Haug [35], Liang and Lace [36], Potra and Rheinboldt [33], and Yen [37]. The main factor that differentiates these approaches is the choice of manifold parameterization. State-space methods have been subject to criticism in two aspects. First, the choice of parameterization generally is not global. Second, poor choices of the projection space result in SSODEs that are numerically demanding, mainly at the expense of overall efficiency and robustness of the algorithm [38]. Although the theoretical framework for these methods was outlined several years ago [33, 39], it was only recently that implicit numerical integration methods for DAEs have been proposed in the context of SSODEs for multibody dynamics analysis [40, 41]. The major intrinsic drawback associated with state-space methods remains the expensive DAE to ODE reduction process that is further exacerbated in the context of implicit integration, which is the norm in industry applications.
2.2.4 Other Direct Methods

When dealing with systems that include flexible substructures and bodies, numerical methods have been sought that are capable of introducing controllable numerical dissipation to damp out spurious high frequencies, which are an artifact of the spatial discretization, without affecting the low frequencies of the system and the accuracy of the method [42, 43]. Several methods have been proposed for structural dynamic simulation, such as the HHT method (also called α-method, [44]) and the generalized α-method [45]. These are order two methods proposed in conjunction with ODE problems associated with structural dynamics. Recently, theoretical and implementation aspects of the HHT-based numerical integrator for the simulation of large mechanical systems with flexible bodies have been discussed by Negrut et al. [46] and Arnold et al. [47]. Several low-order numerical integration methods such as NEWMARK, HHT-I3, HHT-ADD, HHT-SI2, NSTIFF, and NSTIFF-SI2 [21, 48] are thoroughly investigated in Chapter 3.

2.3 Flexible Multibody Dynamics

Structural flexibility plays a fundamental role in the dynamic behavior of mechanical systems with slender components. The need to accurately model the flexibility of mechanical systems has led to the development of the field of flexible multibody dynamics, which considers the effect of deformation of the system components when accounting for the overall motion of the system.

Several papers have been written about flexible multibody dynamics with due consideration to using different coordinate systems and coupling of the rigid motion with elastic

Several basic methods used in flexible multibody dynamics are reviewed in [52]. Among these methods, the floating frame of reference formulation, the finite element incremental methods, and large rotation vector formulations are the more relevant approaches employed in modeling. In the floating frame of reference formulation, the equations of motion are expressed in terms of a coupled set of reference coordinates (representing rigid body motion) and elastic coordinates (representing the deformation of the bodies). The deformation of the bodies can be approximated by the Ritz method as the superposition of known shape functions scaled by unknown time dependant coefficients [23]. The shape functions obtained using finite element models of flexible bodies in multibody systems result in a detailed system representation and a high number of system equations. The number of system equations of such a nodal approach can be reduced considerably by using a modal representation of the deformation. Generally Component Mode Synthesis is used for this purpose. This is further discussed in [53], where the authors applied substructuring techniques to mechanical systems that are composed of rigid and flexible bodies, in which coupling between gross motion and elastic deformation is considered.

2.3.1 Floating Frame of Reference Formulation

In the floating frame of reference formulation, two sets of coordinate are used, namely reference and elastic. Reference coordinates define the location and orientation of a selected
body coordinate system, while elastic coordinates define the deformation of the body relative to the chosen coordinate system as shown in Figure 1. The global position vector of any arbitrary point on the deformable body $i$ can be written as

$$ r^i = R^i + A^i(u_o^i + u'_r) , $$

(9)

where $u_o^i$ represents the position of point $P$ in the undeformed state and $u'_r$ is the deformation vector. However, it can be seen that there is no separation between the rigid body motion and the elastic deformation. In this formulation, the deformation of bodies can be described using the finite element method or component mode technique as follows.

$$ u'_r = S'q'_r , $$

(10)

where $S'$ is shape matrix and $q'_r$ represents the elastic generalized coordinates of body $i$. The equation of motion of a deformable body $i$ can be written in general form as [54]
where $\mathbf{M}'$ and $\mathbf{K}'$ are, respectively, the symmetric mass and stiffness matrices, $\mathbf{Q}^e_\mathbf{f}$ is the vector of externally applied forces, $\mathbf{Q}_\mathbf{v}$ is vector of Coriolis and centrifugal forces, and $\mathbf{Q}^e_\mathbf{c}$ is the vector of constraint forces. The vector of generalized coordinates of the multibody system can be partitioned as $\mathbf{q} = \hat{\mathbf{q}}_r^T, \mathbf{q}^T \mathbf{g}$ where subscripts $r$ and $f$ refer to reference and elastic coordinates. This coordinate partitioning is used in the equations of motion of the flexible multibody system as follows [54]:

$$
\begin{bmatrix}
\hat{\mathbf{M}}_r & \mathbf{M}_r \\
\mathbf{M}_r^T & \mathbf{K}_r
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{q}}_r \\
\mathbf{q}_r
\end{bmatrix}
+ \mathbf{F}_q
= \begin{bmatrix}
\mathbf{Q}_r \\
\mathbf{Q}_f
\end{bmatrix}
+ \begin{bmatrix}
\dot{\mathbf{Q}}_r \\
\dot{\mathbf{Q}}_f
\end{bmatrix}
$$

where $\mathbf{F}_q$ is constraint Jacobian matrix, and $\mathbf{Q}_r$ and $\mathbf{Q}_f$ represent the generalized forces associated with rigid body reference and modal (elastic) coordinates respectively. It should be noted that the stiffness matrix is the same as the one used in structural dynamics since the elastic coordinates are defined in the body coordinate reference frame. The mass matrix in Eq. (12) depends on elastic deformation of a body and hence is a function of time. This approach is widely used for the dynamic formulation of flexible bodies that undergo large translational and rotational reference displacements and small elastic deformation.

2.3.2 Nonlinear Finite Element Approach

2.3.2.1 Geometrically Exact Beam Theory

The pioneering work of Simo and Vu-Quoc in the 1980s [1, 2, 55] provided a finite element procedure for a beam under arbitrary deformations and motions, which is usually
referred to as the geometrically exact beam formulation (GEBF). This formulation is based on the classical nonlinear rod theories of Reissner [56] and Antman [57]. The GEBF is an example of a large rotation vector formulation in which finite rotations coordinates, rather than infinitesimal rotations are used which allows the exact description of arbitrary rigid body motions. In this formulation, two independent fields (nodal coordinates) are used: the rotations of the cross-section and the global displacements (position vector) of the centerline of an element. The displacement and rotation fields are interpolated independently based on their nodal values. Although the two fields are interpolated independently, both are coupled through elastic forces [58].

The representation of the three-dimensional rotation of a moving frame (cross-section frame) with respect to the inertial frame is not linear, while the finite rotations are, in general, not commutative. The interpolation of the rotations along the beam length becomes, thus, nontrivial and must be handled carefully [59]. Rational techniques for the interpolation of finite rotations have been proposed by Jelenić and Crisfield in [60, 61].

In GEBF, the cross sections are assumed to be rigid during deformation, but they may not be perpendicular to the tangent directions of the centroid lines as the shear deformation of the cross sections is also included. However, some numerical difficulties such as shear locking may appear, if the shape functions are not well chosen. This problem can be alleviated using a reduced numerical integration procedure [62].

2.3.2.2 Corotational Formulation

One of the most widely used computational procedures for non-isoparametric or structural elements i.e. beam, plate, and shell elements that undergo large rotation is the
incremental finite element approach. Several incremental procedures have been developed in which infinitesimal rotations are used as nodal coordinates. The corotational formulation is used for arbitrarily large rotation but small strain problem. The basic idea in the corotational formulation is the decomposition of the global displacement field into rigid body motion and strain producing deformation components. This is done in finite element formulations by attaching a local co-rotating coordinate frame to each element in the structure. With respect to the local co-rotating frame, the deformations are small and hence strains and subsequently stresses are calculated based on the local element displacements. The equations are solved for the local displacement increments that are then used to update the global displacement field of the element. It should be noted that in this approach the nonlinear kinematics of the finite element is defined in terms of a large reference motion plus a small deformation. This implies that in one time step there should not be large variation in: 1) the deformation within each element, and 2) the large reference motion. Consequently, the most important parameter that governs this procedure is the time/load step which must remain small as shown in [63].

The corotational formulation is also referred as ‘Convected Coordinate System’ by Belytschko and Hsieh [3] and as ‘Natural Approach’ by Argyris et al.[64]. In [3], a convected coordinate system is assigned to each finite element and the element internal forces are first defined in the element convected coordinate system and then transformed to the global coordinate system. Rankin and Brogan [4] introduced a concept of element independent corotational formulation which has been implemented in several general purpose finite element codes such as ANSYS [63]. Corotational formulation has a big advantage for geometrically nonlinear small strain problems. One can reuse the same stiffness, mass, and damping matrices that are developed for linear problem and can transform them so that they are valid for large
rotation, small strain case. Geometric nonlinearities due to large displacements and rotations are taken into account without the requirement of a finite strain formulation and alternative stress definitions. Consequently this is a method of choice for the legacy codes.

2.3.2.3 Absolute Nodal Coordinate Formulation (ANCF)

For almost a decade the Absolute Nodal Coordinate formulation (ANCF) has been widely used to carry out the dynamics analysis of flexible bodies that undergo large rotation and large deformation. This formulation is consistent with the nonlinear theory of continuum mechanics and easy to implement. Also, it leads to constant mass matrix which makes it computationally more efficient compared to other nonlinear finite element formulations.

The fully parameterized ANCF beam element was originally introduced in [5]. In this beam element, no infinitesimal rotations are used as nodal coordinates; instead, the position vector gradient field is used. This quantity is obtained as the partial derivative of the position vector field with respect to the spatial coordinates of the finite element. The global position vector, \( \mathbf{r} \) of an arbitrary point on the beam element can be expressed as

\[
\mathbf{r}(\mathbf{x}, \mathbf{e}) = S(\mathbf{x}) \mathbf{e},
\]

where \( S \) is a shape function matrix that depends on element spatial coordinates \( \mathbf{x} = \hat{\mathbf{x}} \mathbf{y} \mathbf{z} \hat{\mathbf{u}} \), and \( \mathbf{e} \) is the vector of nodal coordinates of the element. The nodal coordinates at node \( k \) are defined using the global translational coordinates \( \mathbf{r}^k \) and the displacement gradients:

\[
\mathbf{e}^k = \begin{bmatrix}
\dot{\mathbf{r}}^k \\
\mathbf{e}^k
\end{bmatrix} = \begin{bmatrix}
\mathbf{e}^k \\
\mathbf{e}^k
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \mathbf{r}^k}{\partial \mathbf{x}} \\
\frac{\partial \mathbf{r}^k}{\partial \mathbf{y}} \\
\frac{\partial \mathbf{r}^k}{\partial \mathbf{z}} \\
\frac{\partial \mathbf{r}^k}{\partial \hat{\mathbf{u}}}
\end{bmatrix}.
\]

Thus each node has 12 coordinates: three components of global position vector of the node and nine components of the position vector gradient at the node as shown in Figure 2.
For a 2-node beam element as shown in Figure 2, the vector of nodal coordinates is

$$
e = \begin{bmatrix} e^1 & e^2 & e^3 & e^4 \end{bmatrix}^T, \quad e^i = \frac{\partial r^i}{\partial x}, \frac{\partial r^i}{\partial y}, \frac{\partial r^i}{\partial z}, (r^2)^T, \frac{\partial r^2}{\partial x}, \frac{\partial r^2}{\partial y}, \frac{\partial r^2}{\partial z}, \frac{\partial r^2}{\partial \theta} \end{bmatrix}^T, \quad i = 1, \ldots, 8 \quad (15)$$

The shape function matrix for this element is defined as

$$S = [S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6 \quad S_7 \quad S_8] \hat{u} \quad i = 1, \ldots, 8 \quad (15)$$

where $I$ is the $3 \times 3$ identity matrix and the shape functions $S_j, j = 1, \ldots, 8$ are defined as

$$
\begin{align*}
S_1 &= 1 - 3x^2 + 2x^3, \quad S_2 = \lambda(x - 2x^2 + x^3) \\
S_3 &= \lambda(h - x\lambda), \quad S_4 = \lambda(z - xz) \\
S_5 &= 3x^2 - 2x^3, \quad S_6 = \lambda(-x^2 + x^3) \\
S_7 &= lx\lambda, \quad S_8 = lxz
\end{align*}
$$

where $x = x/l, h = y/l, z = z/l$ and $l$ is the length of the element. The interpolation function is cubic in $x$ and linear in $y$ and $z$ direction.
Using the principle of virtual work for the continuum, the element equation of motion is obtained as

\[ M \delta Q_e = Q_e, \]  

(16)

where \( Q_e \) is the vector of generalized element elastic forces, \( Q_e \) is the vector of generalized element external forces, and \( M \) is the symmetric consistent mass matrix of element defined as

\[ M = \int r S^T S dV, \]  

(17)

where \( r \) and \( u \) are, respectively, the element mass density and volume. The expression for the mass matrix given in Eq. (17) is derived using the virtual work of the inertia forces. It should be noted that the element mass matrix is not a function of the time-dependent nodal coordinates.

The generalized element external force vector (\( Q_e \hat{\imath} i^{24} \)) due to gravity can be obtained as

\[ Q_e = \int S^T f_g dV, \]  

(18)

where \( f_g = [0, -rg, 0]^T \) is the gravity force vector considering \( Y \) as the vertical axis. If a concentrated/point force is applied to an element at some point, the generalized element external force vector (\( Q_e \hat{\imath} i^{24} \)) in this case is obtained using the principle of virtual work as

\[ Q_e = S^T f, \]  

(19)

where \( f \) is an external point force and \( S \) is the shape function matrix defined at the point of application of the force.

The elastic force vector \( Q_e \) is computed using the general continuum mechanics approach. The virtual work due to beam elastic forces can be written as
\[ dW_s = -\oint s^T \partial \sigma dv. \] (20)

Here \( e \) is the Green-Lagrange strain vector and \( \sigma \) is the second Piola-Kirchhoff stress vector defined as

\[ s = E e, \] (21)

where \( E \) is the matrix of elastic coefficients. Using the principle of virtual work and using Eqs. (20) and (21), the vector of the generalized elastic forces \( Q_s \) is obtained as

\[ Q_s = \oint \epsilon \epsilon^T E e dv. \] (22)

The strain vector \( e \) is derived from the corresponding Green-Lagrange strain tensor defined as

\[ e_m = \frac{1}{2} (F^T F - I) , \] (23)

where \( F \) is the matrix of the position vector gradients, and \( I \overset{\text{sym}}{\in} R^{n \times n} \) is the identity matrix. Let the location of any arbitrary point on the beam element in the reference configuration (initial configuration at time = 0) is defined by the vector \( \text{vecl} \). This is a global position vector of an arbitrary point on the beam element at time = 0 (i.e. \( r_0 \)). The gradient tensor \( F \) is defined as

\[ F = \frac{\partial r}{\partial x} = \frac{\partial \xi}{\partial x} \mathbf{u} \mathbf{u}^{-1}. \]

We have \( \text{vecl} = S(\text{vecl}) e \) and \( x = r_0 = S(\text{vecl}) e_0 \), where \( e_0 \) the vector of nodal coordinates of the element at time \( = 0 \). The beam is assumed to be initially horizontal so the transformation matrix becomes \( \mathbf{I} \overset{\text{sym}}{\in} R^{n \times n} \) identity matrix in this case and the expression for \( F \) simplifies to
From this expression the Green-Lagrange strain tensor $e_m$ is obtained using Eq. (23). It should be noted that $e_m$ is a $3 \times 3$ matrix, from which the strain vector $e \hat{\mathbf{i}}^6$ is obtained. Finally the expression for $e$ is used to compute $e^\mathcal{G} \hat{\mathbf{i}}^6$, and the vector of the generalized elastic forces $Q_s$ is obtained using Eq. (22).

**Use of Gauss Quadrature Formula**

The space-wise (over a volume) numerical integration of the finite element matrices and vectors i.e. element mass matrix, external force vector, elastic force vector defined in Eqs. (17), (18), and (22) is carried out using the Gauss Quadrature formula. In three dimensions this formula assumes the form

$$
I = \int \int \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{g}(\mathbf{x}, h, z) dxdhdz = \sum_{i}^{3} \sum_{j}^{3} \sum_{k}^{3} \mathbf{w}_i \mathbf{w}_j \mathbf{w}_k \mathbf{g}(\mathbf{x}_i, h_j, z_k),
$$

(24)

where $(\mathbf{x}_i, h_j, z_k)$ represent the Gauss points and $\mathbf{w}_i$, $\mathbf{w}_j$, $\mathbf{w}_k$ are the corresponding weight factors. With $n$ integration points, a polynomial with highest monomial $x^{2n-1}$ can be integrated exactly. Depending on the highest power of a variable in function $\mathbf{g}(\mathbf{x}, h, z)$, the number of Gauss points $(i, j, k)$ in that direction is selected so that the polynomial function is integrated exactly (full integration). There is no advantage in using a number of Gauss points larger than the number that gives exact evaluation of the integral.
Locking problems of Fully Parameterized ANCF

Locking is commonly used to describe the phenomenon that arises when the numerically calculated stiffness of an element is much larger than the exact value. Unlike the classical beam formulation, the continuum mechanics based approach used in fully parameterized ANCF allows the cross-section of beam to deform. In this formulation, the Poisson ratio couples axial strains and the transverse normal strains in the stress–strain relation. Thus, an axial stretch of the unconstrained beam leads to a cross-section deformation. This effect induces Poisson locking, which deteriorates the element performance [59]. The locking leads to highly coupled deformation modes that result from the deformation of the cross-sections and the bending of the beam. These coupled deformation modes, known as ANCF coupled deformation modes [65, 66], can have significantly high frequencies, which can be a source of numerical and/or convergence problems in the case of very stiff and thin structures. In order to eliminate the coupled deformation modes associated with the cross-section deformation, the strain components are defined along the beam centerline, whereas the curvature/torsion matrix [67] is used to define the bending and torsion. This approach is known as the elastic line approach [8, 68]. In this approach the beam element does not include coupling between the cross-section and bending deformations and therefore the ANCF-coupled deformation modes are systematically eliminated. The Green-Lagrange strain expressions obtained using the elastic line approach can be found in [8].

Note that the interpolation function used in fully parameterized ANCF is cubic in $x$ and linear in $y$ and $z$ direction. The incompatibility in the interpolation polynomials causes a shear locking, which leads to an artificial numerically induced bending stiffness of the beam element [6]. This results in slow numerical convergence of these elements in bending dominated problems. To alleviate these shear locking problems, use of a higher order ANCF beam element
is proposed in [6], and a selective reduced integration technique is used in [69]. Also, the Hellinger–Reissner principle [68] and the Hu–Washizu principle [68] can be effectively used in the elastic line approach to circumvent the shear locking problems.
3 Low Order Implicit Integration Methods

Several approaches used to solve DAEs in the context of MBS are reviewed in Chapter 2. Real-life mechanical systems limit the use of high order integration formulas due to the presence in the associated models of friction and contact/impact elements. High-order numerical integration methods are impractical in these cases, and producing a numerical solution relies on the use of low order integration formulas. In this chapter, six different low order integration methods for solving the DAE problem are thoroughly discussed. These integration formulas were originally designed to solve stiff ODE problems. The derivation of theoretical formulations and the order of convergence proofs of these integration methods are beyond the scope of this thesis. The emphasis is given on the implementation details of these integration methods in the context of MBS and their performance is compared based on various metrics.

The low order numerical integration formulas include Newmark, HHT-I3, HHT-SI2, HHT-ADD, NSTIFF, and NSTIFF-SI2, which are used to solve index-3 and stabilized index-2 DAE in conjunction with four multibody models ranging from very simple to complex including flexible bodies. The performance of these integration methods is compared in terms of order of convergence, energy preservation, constraint satisfaction and efficiency. Among these integration methods the HHT-ADD and HHT-SI2 are relatively new methods [70], so their performance comparison with other widely used integration methods is the main objective of this chapter.
3.1 Low Order Integration Algorithms

The first integration method considered in this study is essentially the BDF method of order two proposed in [71], and it serves the purpose of providing a reference when comparing the performance of the other algorithms. The second order BDF formula is cast into a form suitable for direct numerical integration of second order differential equations:

\[
q_{n+1} = \frac{4}{3} q_n - \frac{1}{3} q_{n-1} + h \left( \frac{8}{9} \dot{q}_n - \frac{2}{9} \dot{q}_{n-1} \right) + \frac{4}{9} h^2 \ddot{q}_{n+1} \\
\dot{q}_{n+1} = \frac{4}{3} \dot{q}_n - \frac{1}{3} \dot{q}_{n-1} + \frac{2}{3} h \ddot{q}_{n-1}
\]  

(1)

These formulas, used in conjunction with the equations of motion and position kinematic constraint equations, lead to a second order method herein called NSTIFF:

\[
M(q_{n+1})\ddot{q}_{n+1} + (\Phi_q^T \lambda)_{n+1} - Q_{n+1} = 0 \\
\frac{9}{4h^2} \Phi(q_{n+1}, t_{n+1}) = 0
\]  

(2)

As suggested in [46] and recently analyzed in [72], the scaling of the kinematic constraint equations by the inverse of the integration step-size \( h^2 \) is done in order to prevent an ill-conditioning of the Jacobian \( J_{\text{NSTIFF}} \) associated with the Newton-type method employed to solve Eq. (2), which is regarded as a nonlinear system in \( \dot{q}_{n+1} \) and \( \lambda_{n+1} \):

\[
J_{\text{NSTIFF}} = \begin{bmatrix}
M + P & \Phi_q^T \\
\Phi_q & 0
\end{bmatrix}
\]

where \( P = \frac{4}{9} h^2 (M(q)\ddot{q} + (\Phi_q^T \lambda) - Q)_q - \frac{2}{3} hQ_q \). Note that when \( h \to 0 \), the condition number of \( J_{\text{NSTIFF}} \) remains bounded. The scaling of the position constraint equation by \( \frac{9}{4h^2} \) leads to a
bounded value. To see this, first note that for all the numerical integration formulas considered herein, locally, \( \| \mathbf{q}_{n+1} - \mathbf{\tilde{q}}_{n+1} \| = O(h^2) \), where \( \mathbf{\tilde{q}}_{n+1} \) is the exact solution and \( \mathbf{q}_{n+1} \) is an approximation obtained after taking one integration step. Then,

\[
\Phi(\mathbf{q}, t) = \Phi(\mathbf{\tilde{q}}, t) + \Phi_{\mathbf{q}}(\mathbf{\tilde{q}}, t)(\mathbf{q} - \mathbf{\tilde{q}}) + ... \\
= \Phi_{\mathbf{q}}(\mathbf{\tilde{q}}, t)(\mathbf{q} - \mathbf{\tilde{q}}) + ...
\]

where the subscript \( n+1 \) on \( \mathbf{q} \), \( \mathbf{\tilde{q}} \), and \( t \) was dropped for convenience. It follows that

\[
\frac{9}{4h^2} \Phi(\mathbf{q}_{n+1}, t_{n+1}) = O(h^0),
\]

which justifies the scaling proposed in Eq. (2).

The second numerical integration method considered uses the Newmark formulas [73]. It requires the selection of two parameters \( \gamma \geq \frac{1}{2}, \beta \geq \frac{(\gamma + 1/2)^2}{4} \) based on which, given the acceleration \( \mathbf{\ddot{q}}_{n+1} \) at the new time step \( t_{n+1} \), the new position and velocity are obtained as

\[
\mathbf{q}_{n+1} = \mathbf{q}_n + h \mathbf{\ddot{q}}_n + \frac{h^2}{2} \left[ (1 - 2\beta) \mathbf{\dddot{q}}_n + 2\beta \mathbf{\ddot{q}}_{n+1} \right] \\
\mathbf{\dot{q}}_{n+1} = \mathbf{\dot{q}}_n + h \left[ (1 - \gamma) \mathbf{\dddot{q}}_n + \gamma \mathbf{\ddot{q}}_{n+1} \right].
\]

Given an integration step-size \( h \), the discretization scheme operates on the equations of motion and position kinematic constraint equations to lead to the nonlinear system

\[
(M\mathbf{\ddot{q}})_{n+1} + (\Phi_{\mathbf{q}}^T \lambda)_{n+1} - \mathbf{Q}_{n+1} = \mathbf{0} \\
\frac{1}{\beta h^2} \Phi(\mathbf{q}_{n+1}, t_{n+1}) = \mathbf{0}.
\]

The method, called hereafter NEWMARK, is order one unless \( \gamma = 1/2 \) and \( \beta = 1/4 \). This choice leads to the trapezoidal method, which is known in the literature to have stability problems when used in conjunction with index-3 DAEs [74]. Note that the Jacobian \( J_{\text{NEWMARK}} \) is identical to
\( J_{\text{NStiff}} \), except that the matrix \( P \) is replaced by a matrix \( \hat{P} \) obtained by replacing \( \frac{4}{9} \) with \( \beta \) and \( \frac{2}{3} \) with \( \gamma \).

Referred to as HHT-I3, the third method considered in this study relies on the HHT method [44], widely used in the structural dynamics community and first considered in the context of multibody dynamics analysis in [74]. HHT-I3 is defined as follows (note that the discretized equations of motion have been scaled by \( \frac{1}{1+\alpha} \)):

\[
\begin{align*}
q_{n+1} &= q_n + h \dot{q}_n + \frac{h^2}{2} \left[ (1 - 2\beta) a_n + 2\beta a_{n+1} \right] \\
\dot{q}_{n+1} &= \dot{q}_n + h \left[ (1 - \gamma) a_n + \gamma a_{n+1} \right]
\end{align*}
\tag{5}
\]

The notation used in Eq. (5) is meant to emphasize that there is a distinction between \( \ddot{q}_{n+1} \) and \( a_{n+1} \) (compare with Eq. (3)). Concretely, \( a_{n+1} \) is an approximation of \( \ddot{q}(t_n + (1 + \alpha)h) \). This raises some difficulties in choosing \( a_0 \), an attribute that is associated with the use of HHT in general and is not specific to HHT-I3. In [36] it is recommended to take \( a_0 = \dot{q}_0 \) and in spite of this approximation the same convergence results hold for the global behavior of the method. For more accurate results, an implicit and therefore slightly more involved way of computing \( a_0 \) is suggested in [75]. Finally, note that the last two equations in Eq. (5) lead to a nonlinear system that is solved with a Newton-like method for \( a_{n+1} \) and \( \dot{\lambda}_{n+1} \). The associated Jacobian
J_{HHT3} = \begin{bmatrix} \frac{1}{1+\alpha} & -M & \Phi_q^T \\ \Phi_q & M & \Phi_q^T \\ 0 & \Phi_q & 0 \end{bmatrix},

does not become ill conditioned when \( h \to 0 \). Taking the limit, \( \hat{P} \to 0 \), and \( J_{HHT3} \) is nonsingular as long as the kinematic constraints are independent and the symmetric mass matrix is nonsingular, which are both mild assumptions.

The last three numerical integration methods considered herein take into account the velocity kinematic constraint equations. The salient attribute of these methods is a resulting set of consistent generalized velocities, an aspect relevant in frictional contact and controls applications. The method referred to as NSTIFF-SI2 is an implementation of the stabilized index 2 formulation reported in [28] that uses second order BDF formulas [71]:

\[ q_{n+1} = \frac{4}{3} q_n - \frac{1}{3} q_{n-1} + \frac{2}{3} h \dot{q}_{n+1}, \]

\[ v_{n+1} = \frac{4}{3} v_n - \frac{1}{3} v_{n-1} + \frac{2}{3} h \dot{v}_{n+1}. \]

NSTIFF-SI2 explicitly accounts for the velocity kinematic constraint equations and relies on an extra set of Lagrange multipliers \( \mu \) to enforce these constraints. The unknowns are \( \Phi, \Phi_q, \lambda, \) and \( \mu \). The new configuration at \( t_{n+1} \) is the solution of the following system of nonlinear equations:

\[ M(q_{n+\frac{1}{2}}) \ddot{q} + \Phi_q^T(q_{n+\frac{1}{2}}) \dot{q}_{n+\frac{1}{2}} = 0 \]

\[ \dot{q}_{n+\frac{1}{2}} - \dot{q}_{n+1} + \Phi_q^T(q_{n+1}) \mu_{n+1} = 0 \]

\[ \frac{3}{2h} \Phi(q_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) = 0 \]

\[ \frac{3}{2h} \Phi_q(q_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) \dot{v}_{n+\frac{1}{2}} + \frac{3}{2h} \Phi_t(q_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) = 0. \]
When using a Newton-type method, the associated Jacobian assumes the form

$$
J_{\text{NSTIFF-SI2}} = \begin{bmatrix}
M & \frac{2h}{3}(M\ddot{v} + \Phi_q^T\lambda - Q)_q & \Phi_q^T & 0 \\
\frac{2h}{3}I & -I - \frac{4h}{3}(\Phi_q^T\mu)_q & 0 & \Phi_q^T \\
\Phi_q & 0 & 0 & 0 \\
(\Phi_q v)_q + \Phi_q' & \Phi_q & 0 & 0 \\
\end{bmatrix}
$$

Under mild conditions (symmetric nonsingular mass matrix and independent set of kinematic constraints) it can be easily shown that $J_{\text{NSTIFF-SI2}}$ remains nonsingular when $h \to 0$. Also note that in the absence of discretization errors, $\mu$ would be identically zero.

The fifth method considered in this study introduces a correction into the Newmark formulas based on the constraint accelerations and was shown to have global convergence order two [20, 75]. Given a configuration $(q_n, \dot{q}_n, a_n)$, and defining $f(t, q, \dot{q}) := M^{-1}(q)Q(t, q, \dot{q})$ and $r(q, \dot{q}) := M^{-1}(q)\Phi_q^T$, the unknowns $q_{n+1}$, $\dot{q}_{n+1}$, $\mathbf{a}_{n+1}$, $\Psi_f$, and $\Psi_{ll}$ are found as the solution of the following nonlinear system:

$$
q_{n+1} = q_n + h\dot{q}_n + \frac{h^2}{2}(1 - 2\beta)a_n + 2\beta a_{n+1} + \frac{h^2}{2}(1 - b)R_f + bR_{ll} \\
\dot{q}_{n+1} = \dot{q}_n + h((1 - \gamma)\ddot{q}_n + \gamma \ddot{q}_{n+1}) + \frac{h^2}{2}(R_f + R_{ll}) \\
0 = \Phi(q_{n+1}, \dot{q}_{n+1}) \\
0 = \Phi_q(q_{n+1}, t_{n+1})\dot{q}_{n+1} + \Phi_\dot{q}(q_{n+1}, t_{n+1}) \\
a_{n+1} = (1 + \alpha)f(t_{n+1}, q_{n+1}, \dot{q}_{n+1}) - \alpha f(t, q_n, \dot{q}_n)
$$

(8)
where \( b \neq 1/2 \) is a free coefficient, \( \mathbf{R}_i \psi \mathbf{r}(t_n, \mathbf{q}_n, \dot{\mathbf{q}}_n) \), and \( \mathbf{R}_{i,i} \psi \mathbf{r}(t_{n+1}, \mathbf{q}_{n+1}, \dot{\mathbf{q}}_{n+1}) \). This method is referred to as HHT-ADD and is discussed at length in [20, 75], where local and global error analysis results are provided along with an investigation of stability properties. In addition to displaying attractive numerical damping controlled through the parameter \( \alpha \in [-0.3, 0] \), the method is shown to be order 2. The major drawback of this method is the multiplication by the inverse of the mass matrix. Specifically, this becomes a major concern in the inexact-Newton step when dealing with flexible body problems where, due to the coupling in the deformation modes, the mass matrix can have large dense blocks. The Jacobian \( \mathbf{J}_{\text{HHT-ADD}} \) is provided in Ref. [20].

The last integration method investigated, HHT-SI2, represents a new algorithm that is analyzed theoretically in [70]. It represents a variation on the HHT-ADD algorithm that avoids multiplication by the inverse of the mass matrix. As such, it is amenable to handling mechanical systems with flexible bodies in which the formulation relies on the floating frame of reference approach [54]. For HHT-SI2 the Newmark integration formulas are modified slightly by introducing a correction \( \overline{\mathbf{a}} \):

\[
\mathbf{q}_{n+1} = \mathbf{q}_n + h \dot{\mathbf{q}}_n + \frac{h^2}{2} \left[(1-2\beta)\mathbf{a}_n + 2\beta(\mathbf{a}_{n+1} + \overline{\mathbf{a}})\right] \tag{9}
\]

\[
\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + h \left[(1-\gamma)\mathbf{a}_n + \gamma \mathbf{a}_{n+1}\right]. \tag{10}
\]

In advancing the integration from a given configuration at time \( t_n \) to \( t_{n+1} \), the unknowns \( \mathbf{a}_{n+1}, \overline{\mathbf{a}}, \lambda_{n+1}, \) and \( \mu \) are found as the solution of the nonlinear system of equations:
\[
\frac{1}{1 + \alpha} \bar{M}_{n+1} \dot{\lambda}_n Q + \left( \Phi_q^T - \right)_{n+1} \dot{\lambda}_n = \frac{Q}{1 + \alpha} \left( \Phi_q^T - \right)_{n+1} = \\
\bar{M}_{n+1} \bar{\lambda} - \Phi_q^T (t_{n+1}, q_{n+1}) \mu = 0 \\
\frac{1}{h^2} \Phi_q (q_{n+1}, t_{n+1}) \dot{q}_{n+1} + \frac{1}{h} \Phi_q (q_{n+1}, t_{n+1}) = 0
\]

where \(\bar{M}_{n+1} := M (t_n + h(1 + \alpha), q_n + h(1 + \alpha) \dot{q}_n)\). Here \(\bar{\lambda}\) and \(\mu\) are auxiliary variables local to the current time step. Introducing the notation \(R = \beta h^2 (\Phi_q^T \mu)_{q} \), \(Q = \beta h^2 (\Phi_q^T \lambda - Q)_{q} - \gamma h Q_q\), \(F = \beta h^2 (\Phi_q^T \lambda - Q)_{q}\), and \(V = h (\Phi_q (q, t) \dot{q} + \Phi_l (q, t))_{q}\), the Jacobian associated with the discretized problem assumes the expression

\[
J_{\text{HHT-SI2}}^{\text{h}} = 
\begin{bmatrix}
\frac{1}{1 + \alpha} \bar{M}_{n+1} + Q & F & \Phi_q^T & 0 \\
-R & \bar{M}_{n+1} - R & 0 & -\Phi_q^T \\
\beta \Phi_q & \beta \Phi_q & 0 & 0 \\
\gamma \Phi_q + \beta V & \beta V & 0 & 0
\end{bmatrix}
\]

If \(J_{\text{HHT-SI2}}^{0} = \lim_{h \to 0} J_{\text{HHT-SI2}}\), then

\[
J_{\text{HHT-SI2}}^{0} = 
\begin{bmatrix}
\frac{1}{1 + \alpha} \bar{M}_{n+1} & 0 & \Phi_q^T & 0 \\
0 & \bar{M}_{n+1} & 0 & -\Phi_q^T \\
\beta \Phi_q & 0 & \frac{1}{2} \Phi_q & 0 \\
\gamma \Phi_q & 0 & 0 & 0
\end{bmatrix}
\]

and under mild assumptions (symmetric nonsingular mass matrix and independent set of kinematic constraints) the matrix \(J_{\text{HHT-SI2}}^{0}\) turns out to be nonsingular. This guarantees acceptable behavior at small values of the step-size, a situation typically encountered in penalty-based frictional contact problems. The main result regarding the convergence of the new method
HHT-SI2 is stated as follows. Suppose that the initial configuration at time $t_0$ is such that

$$0 = M(t_0, q_0)a_0 + \Phi_q^r \lambda_0 - Q(t_0, q_0, \dot{q}_0, \lambda_0)$$
$$0 = \Phi(t_0, q_0)$$
$$0 = \Phi_s(t_0, q_0) + \Phi_q(t_0, q_0)\dot{q}_0$$
$$0 = \Phi_d(t_0, q_0) + 2\Phi_q(t_0, q_0)\dot{q}_0 + \left[\Phi_q(t_0, q_0)\dot{q}_0\right]_q \dot{q}_0 + \Phi_q(t_0, q_0)a_0 ,$$

and $a_\alpha - a(t_0 + \alpha h) = O(h)$. Then the numerical approximation $(q_n, \dot{q}_n, a_{n+\alpha}, \lambda_n)$ produced by the HHT-SI2 method in Eqs. (9) - (11) satisfies

$$q_n - q(t_n) = O(h^2)$$
$$\dot{q}_n - \dot{q}(t_n) = O(h^2)$$
$$a_{n+\alpha} - a(t_n + \alpha h) = O(h^2)$$
$$\lambda_n - \lambda(t_n) = O(h^2) ,$$

for $0 < h \leq h_{\text{max}}$ and $t_n - t_0 = nh \leq \text{const}$, where $h_{\text{max}}$ is suitably chosen. Here $q(t_n)$, $\dot{q}(t_n)$, $a(t_n + \alpha h)$, and $\lambda(t_n)$ denote the exact value of the respective unknown quantities at the times indicated in parentheses. A formal proof of this is provided in [70].

### 3.1.1 Implementation Details

The computational flow associated with any of the six integration methods discussed can be abstracted in the following way. A set of unknowns $w_{n+1}$ is computed as the solution of a nonlinear system $Y(w_{n+1}) = 0$. In turn, the position and velocity at the new configuration $t_{n+1}$ is evaluated based on a set of integration formulas: $\dot{q}_{n+1} = l_1(w_{n+1})$, and $q_{n+1} = l_2(w_{n+1})$. Illustrating this abstraction for HHT-SI2, the expression of $Y$ is obtained from Eq. (11), $l_1$ is provided by Eq. (10), and $l_2$ is provided by Eq. (9). Regardless of the method used, advancing the solution from $t_n$ to $t_{n+1}$ follows a simple recipe:
Certain variations of this algorithm can improve its efficiency. For instance, rather than evaluating it at each time step, the Jacobian can be evaluated less frequently. While a costly proposition in itself, each Jacobian evaluation is necessarily followed by a factorization step, which is also costly. Note that although the convergence test relies exclusively on the correction norm at line \(L8\) of the pseudocode, the test could also include the norm of residual, i.e. the right side of the linear system in \(L7\).

### 3.2 Models Considered

The models considered for testing and comparison of algorithm performance are a double pendulum, a slider crank, and a slider crank with flexible connecting rod and a seven body mechanism [76]. The model parameters and the initial conditions used are summarized below.

---

\[
t_{n+1} = t_n + h
\]

\[
w_{n+1}^{(0)} = w_n
\]

**Do**

\[
d_{n+1}^{(k)} = l_1(w_{n+1}^{(k)})
\]

\[
d_{n+1}^{(k)} = l_2(w_{n+1}^{(k)})
\]

Evaluate Jacobian \(J\)

Solve linear system \(J \Delta w^{(k)} = Y(w_{n+1}^{(k)})\)

If \(\|\Delta w^{(k)}\| \leq \delta\) then break

Apply correction: \(w_{n+1}^{(k+1)} = w_{n+1}^{(k)} - \Delta w^{(k)}\)

**Enddo**

\[
w_{n+1} = w_{n+1}^{(k)}
\]
Figure 3 shows the schematic of a double pendulum. Torsional springs and dampers are included in the model at the pin joints. The parameter values used in this model are \( m_1 = 3 \, \text{kg}, \, L_1 = 1\, \text{m}, \, k_1 = 400\, \text{N/m}, \, c_1 = 15 \, \text{Ns/m}, \, m_2 = 0.3 \, \text{kg}, \, L_2 = 1.5 \, \text{m}, \, k_2 = 300000 \, \text{N/m} \) and \( c_2 = 5000 \, \text{Ns/m} \). The initial conditions were \( q(0) = 2p, \, \dot{q}(0) = 0, \, \ddot{q}(0) = c \) and \( \dddot{q}(0) = 10 \).

b. Slider Crank

The schematic of a slider crank model including a spring-damper element is shown in Figure 4. The parameters associated with the model are \( m_1 = 3 \, \text{kg}, \, L_1 = 0.3 \, \text{m}, \, m_2 = 0.9 \, \text{kg}, \, L_2 = 0.6 \, \text{m}, \, k = 100 \, \text{N/m} \) and \( c = 5 \, \text{Ns/m} \). Both links are symmetric and homogeneous, and the center of mass is at the midpoint. The initial conditions used for simulation of motion were \( q(0) = 3p/2, \, \dot{q}(0) = c \, \text{rad/s} \).

![Figure 3: Double pendulum](image-url)
c. Flexible Slider Crank

This model is similar to the rigid slider crank shown in Figure 4, except that the spring and damper are not included and the connecting rod (link 2) is flexible. The parameter values used in this model are $m_1 = 3$ kg, $L_1 = 0.3$ m, $m_2 = 0.9$ kg and $L_2 = 0.6$ m, cross-section area $A = 5.74\times10^{-6}$ m$^2$, moment of inertia $I = 2.765\times10^{-8}$ m$^4$, and Young’s modulus $E = 200$ GPa. Both links are symmetric and homogeneous, and the center of mass is at the midpoint. The initial conditions are $\mathbf{q}(0) = \frac{3\mathbf{p}}{2}$, $\dot{\mathbf{q}}(0) = 1$ rad/s. The equations of motion are formulated using the floating frame of reference formulation [54].

d. Seven Body Mechanism

The model is presented in Figure 5. For this set of numerical experiments, the value of the damping $c$ was set to zero. An account of the geometry of the mechanism, along with inertia properties and initial conditions is provided in [76, 77]. The mechanism moves due to a torque applied to crank 1. All bodies in the model are rigid.
3.3 Numerical Experiments and Results

The numerical algorithms NSTIFF, NEWMARK, HHT-I3, NSTIFF-SI2, HHT-ADD, and HHT-SI2 were implemented in MATLAB and used in conjunction with the above models. Several experiments were run to evaluate the algorithms' performance and compare them in relation to the order of global convergence, energy preservation, constraint satisfaction, and efficiency.

3.3.1 Global Convergence Analysis

The goal of the first set of numerical experiments is to assess how the global integration error decreases with the integration step-size, i.e. to carry out a convergence analysis. To investigate the convergence order of each numerical method for the rigid slider crank, a reference solution was first determined by deriving a set of second order ODEs that govern the time evolution of the system. This ODE problem is subsequently solved using a fourth order
Runge-Kutta method (see, for instance, [76]) with a step-size of $h = 10^{-6}s$. The convergence behavior is shown in Figure 6, which displays the crank angle absolute error at time $T = 2s$ obtained with each method over a set of integration step-sizes. Ideally, these slopes should be two, except for NEWMARK, which should display first order convergence and therefore a slope of one. Indeed, the numerical results confirm that all methods behave as predicted by theory as shown in Figure 7.

Since the equations of motion were too involved, for the flexible slider crank model and the seven body mechanism they were not reduced first to a set of ODEs. Rather, the reference solution was obtained with HHT-ADD with a step-size of $h = 10^{-6}s$. The flexible slider crank was simulated for $2s$ and the numerical solution was compared to the reference solution at the final time. The results suggest that NSTIFF, HHT-I3, HHT-SI2, HHT-ADD, NSTIFF-SI2 exhibit order two convergence, in line with theoretical results established in conjunction with these algorithms. Furthermore, NEWMARK shows global convergence of order one for all models. The convergence orders hold both for the generalized coordinates and their time derivative, that is, both for positions and velocities. Figure 8 displays the convergence and order for the flexible slider crank; the results reported concern the angular velocity of the crank. Finally, for body 5 of the seven body mechanism (see Figure 5) the convergence plot for its orientation is displayed in Figure 9.
Figure 6: Convergence, slider crank

Figure 7: Convergence order, slider crank
3.3.2 Energy Preservation

In the HHT method, high-frequency oscillations that are of no interest as well as parasitic high-frequency oscillations that are a byproduct of the finite element discretization are damped.
with the parameter $\alpha$. The choice of $\alpha$ is based on the desired level of damping for a particular model. The smaller the value of $\alpha$, the more damping is induced in the numerical solution. The effect of this damping can be seen from energy preservation plots shown in Figure 10 and Figure 11. These energy plots are for the slider-crank model from which the translational damper was removed. The system is conservative, and for the particular reference system employed, the total energy should be constant and equal to zero.

Figure 10: Energy dissipation at $\alpha = -0.3$

Figure 11: Energy dissipation at $\alpha = -0.05$
As anticipated, when \( \alpha = -0.3 \) the numerical damping-induced dissipation is more pronounced than the \( \alpha = -0.05 \) case. Even more relevant is an investigation of how the numerical energy dissipation changes with the step-size. Results in Figure 11 indicate a highly oscillatory pattern. To capture the degree to which a numerical scheme dissipates energy, an average energy dissipation over an interval \([0, T]\) is computed as

\[
\epsilon(T) = \frac{1}{T} \int_0^T |E_{\text{tot}}(t)| dt
\]

If no numerical dissipation was present in the system then \( \epsilon(T) = 0, \quad T > 0 \). On a log-log scale, Figure 12 shows \( \epsilon(10) \) for the rigid slider crank model with no physical damping. Surprisingly, this average total energy error converges to zero like \( O(h^q) \), where \( q \) is the order of the method. In other words, for NEWMARK it converges to zero like \( O(h) \), while for all the other methods the average energy error converges like \( O(h^2) \) as seen in Figure 13.

Figure 12: Energy dissipation characteristics – slider crank
3.3.3 Kinematic Constraint Drift

The rationale behind stabilizing the numerical solution of the index 3 DAE of multibody dynamics using the velocity kinematic constraint equations is to prevent drift in satisfying this set of algebraic constraints. Three of the six methods analyzed in this study, namely HHT-ADD, HHT-SI2, and NSTIFF-SI2, enforce these equations. As such, no velocity constraint drift is expected in the numerical solution.

This is confirmed by the plots in Figure 14-Figure 16. These figures display a plot of the velocity constraint violation in the X direction against the velocity constraint violation in the Y direction for the rigid slider-crank mechanism. Data was plotted at each time step and, as anticipated, confirms that the velocity kinematic constraint equations are satisfied within machine precision.
Figure 14: Velocity constraint drift - HHT-ADD

Figure 15: Velocity constraint drift - HHT-SI2

Figure 16: Velocity constraint drift - NSTIFF-SI2
Figure 17: Velocity constraint drift – NEWMARK

Figure 18: Velocity constraint drift - HHT-I3

Figure 19: Velocity constraint drift - NSTIFF
Figure 17-Figure 19 show the same information for the rigid slider crank with no damping; the plots report data obtained during a 10 second long simulation, with a step-size $h = 2^{-10}$ s. The most remarkable thing is that NEWMARK, HHT-I3, and NSTIFF display the same error behavior. Moreover, as the step-size decreases, the box that bounds the plot shrinks but the shape of the curves remains the same for all three integration methods. The root cause of this behavior remains to be investigated but these results suggest that this limit cycle behavior is a characteristic of the direct index-3 methodology rather than that of the algorithm used for the numerical solution.

### 3.3.4 Runtime Comparison

The integration methods’ performance is also compared in relation to their efficiency. For this purpose, the average CPU time of each algorithm is measured. The goal is to compare the amount of work per time step required to produce an approximation of the solution. In this undertaking, the integration step-size was identical for all algorithms, although it was different for different models. Also, specific to each model was the simulation end time. In order to allow a unified perspective on the efficiency issue, the CPU times required to complete the analyses were reported in Figure 20, after being normalized to the time it took the HHT-I3 method to finish the simulation. In other words, for each of the four models, the HHT-I3 provides the reference. The results in Figure 20 suggest that having the kinematic velocity constraint equations enforced usually leads to an approximate simulation slowdown of 30%, unless the model is heavily constrained, as is the case with the seven body mechanism. For the seven body mechanism, the number of second order differential equations was 21 and the number of constraints 20, in which case relying on the velocity kinematic constraint equations for
stabilization purposes slows down the overall simulation due to a rather significant increase in the dimension of the problem: from 41 nonlinear differential algebraic equations for HHT-I3, to 61 for HHT-SI2 and NSTIFF-SI2. Finally, as expected, the HHT-ADD is very costly for the flexible body model given that the mass matrix ceases to be constant. This trend gets exacerbated as the dimension of the problem increases, as is the case with the seven body mechanism, effectively making HHT-ADD an algorithm that is robust but of limited practical interest. Note that the mass matrix associated with the HHT-SI2 algorithm being evaluated somewhere midstep and then kept constant led to improved performance when compared to the NSTIFF-SI2 alternative. In other words, for large models it is anticipated that the newly proposed algorithm HHT-SI2 will be attractive both on grounds of efficiency and variable damping characteristics. It should also be pointed out that the timing results reported herein are only qualitative as there are a multitude of factors that ultimately dictate the efficiency of an
algorithm: memory access, step-size selection, Newton-convergence issues, predictor, etc. The impact of these factors is highlighted in [46], where it is reported that these implementation issues actually rendered HHT-I3 two times faster than NSTIFF.

3.4 Concluding Remarks

This Chapter investigates six low-order numerical integration formulas for determining the time evolution of constrained multibody systems. The motivation for this effort was twofold. First, the vast majority of large real-life models contain high stiffness, discontinuities, friction, and contacts that effectively make low-order integration formulas the only viable alternative for numerical simulation. The comparison of these commonly used integration formulas shed light on some advantages and disadvantages associated with each method. Second, the comparison served as the vehicle that introduced a new integration method, HHT-SI2, and placed it in the wider family of index 3 and stabilized index 2 methods for the numerical solution of the DAEs of multibody dynamics.

Compared to higher-order implicit formulas, the numerical methods investigated are robust and straightforward to implement. The algorithms discussed do not have ill-conditioning issues associated with small integration step-sizes due to the suggested scaling, are backed up by sound theoretical results, and come in two flavors: index-3 and stabilized index-2. Based on the convergence order and timing results presented, for problems where accurately satisfying the velocity kinematic constraint equations is not a priority, HHT-I3 represents a good choice. It is a second order method that has the ability to change the amount of numerical damping that enters the solution process. The NSTIFF method is the next best alternative. However, the method is
plagued by a somewhat more intense numerical damping that cannot be controlled like in HHT-I3. For a slower but more robust approach, one can select either HHT-SI2 or NSTIFF-SI2 methods. They are comparable in terms of efficiency, yet HHT-SI2 has an edge due to (i) its ability to adjust the value of numerical damping introduced in the solution, and (ii) the handling of the mass matrix, which is bound to lead to efficiency gains for large models. Relative to the simulation times associated with the straight I3 methods, preliminary results indicate that satisfying both the position and velocity kinematic constraint equations comes at a price of about a 30% increase in simulation time.
4 Absolute Nodal Coordinate Formulation

Several nonlinear finite element formulations for structural elements, i.e. beams, plates, and shells, that undergo large rotation and large deformation are reviewed in Chapter 2. The absolute nodal coordinate formulation has some unique features that distinguish it from other existing nonlinear finite element formulations. The global position slope vectors are used as nodal coordinates which leads to a constant mass matrix and the quadratic velocity dependant centrifugal and Coriolis forces are zero. It automatically satisfies the principles of mechanics and can correctly describe an arbitrary rigid-body motion including finite rotations [8]. Moreover, the nodal coordinates of the element are defined in a fixed inertial coordinate system, and consequently no transformation is required for the element coordinates.

The frictional contact/impact problem between several slender beams is addressed in this work. This scenario is relevant in a wide spectrum of applications, from hair and cloth modeling, to polymer simulation and to DNA coiling analysis. The original ANCF beam elements were not a good choice for this problem because they displayed locking problems. Instead, the gradient deficient ANCF 3D beam elements, also referred to as low order cable elements in [6, 8], are used to model the slender beams. It should be noted that the nonlinear material models are not included here. The discussion of ANCF is limited to isotropic elastic material model.

The theoretical formulation of the gradient deficient (GD) ANCF beam elements is presented in this chapter. The implementation of GD ANCF beam elements within the multibody dynamics framework is thoroughly discussed. Several numerical experiments are performed to study the convergence behavior and to validate the results. The GD ANCF results are also
compared with the original ANCF results to demonstrate the effects of the locking problems. The details of gradient deficient plate/shell formulation are given in the Appendix.

4.1 Gradient Deficient ANCF beam

In the case of thin beams, it can be assumed that the displacement field will be just like the Euler-Bernoulli beams, in which the cross-section remains perpendicular to the beam axis. The GD ANCF beam element is derived from the Euler-Bernoulli beam theory where the shear deformation is neglected. It should be noted that the gradient deficient ANCF beam element does not describe a rotation of beam about its own axis so the torsional effects cannot be modeled [6]. This formulation shows no shear locking problems for thin and stiff beams and it is computationally efficient compared to the original ANCF due to reduced nodal coordinates.

The GD ANCF beam elements shown in Figure 21 are two node elements where one position vector and only one gradient vector are used as nodal coordinates \( \mathbf{e}_i = [\mathbf{r}^T \ \mathbf{r}_i^T]^T \). Thus each node has 6 coordinates: three components of global position vector of the node and three components of position vector gradient at the node. The global position vector of an arbitrary point on the beam centerline is given by

\[
\mathbf{r}(\mathbf{x}, \mathbf{e}) = S(\mathbf{x})\mathbf{e},
\]

where \( \mathbf{e} = [e_1^T \ e_2^T]^T \in \mathbb{R}^{12} \) is the vector of element nodal coordinates and \( \mathbf{x} \) is spatial coordinate in the beam reference configuration as shown in Figure 21. The shape function matrix for this element is defined as \( S = [S_1, S_2, S_3, S_4] \in \mathbb{R}^{3 \times 12} \) where \( \mathbf{I} \) is the \( 3 \times 3 \) identity matrix and the shape functions \( S_j, j = 1, ..., 4 \) are defined as [8]
Figure 21: Gradient deficient (GD) ANCF beam element

\[ S_1 = 1 - 3\xi^2 + 2\xi^3, \quad S_2 = l(\xi - 2\xi^2 + \xi^3) \]
\[ S_3 = 3\xi^2 - 2\xi^3, \quad S_4 = l(-\xi^2 + \xi^3) \]  \hspace{1cm} (2)

Here \( \xi = x/l \), and \( l \) is the element length. Using the principle of virtual work for the continuum, the element equation of motion is obtained as:

\[ \mathbf{M}\ddot{\mathbf{e}} + \mathbf{Q}_s = \mathbf{Q}_e , \]  \hspace{1cm} (3)

where \( \mathbf{Q}_s \) is the vector of generalized element elastic forces, \( \mathbf{Q}_e \) is the vector of generalized element external forces, and \( \mathbf{M} \) is the symmetric consistent element mass matrix defined as

\[ \mathbf{M} = A\int_0^l \rho S^T S \, dx. \]  \hspace{1cm} (4)
Here $\rho$ and $A$ are the element mass density and cross-section area, respectively. The expression for the mass matrix given in (4) is derived using the virtual work of the inertia forces. Note that the element mass matrix is not a function of the time-dependent nodal coordinates.

The generalized element external force vector ($Q_e \in \mathbb{R}^{12}$) due to gravity can be obtained as

$$Q_e = A^\top \int_0^L S^T f_g \, dx,$$

where $f_g = [0, -\rho g, 0]^T$ is the gravity force vector considering $Y$ as the vertical axis. If a concentrated/point force is applied to an element at some point, the generalized element external force vector ($Q_e \in \mathbb{R}^{12}$) in this case is obtained using the principle of virtual work as

$$Q_e = S^T f,$$

where $f$ is an external point force and $S$ is the shape function matrix defined at the point of application of the force.

In the Euler-Bernoulli beam theory, the total strain energy is split into two parts: a part due to bending based on the curvature of the deformed beam centerline, and a part due to the axial strain. The strain energy expression for the GD ANCF beam element is

$$U = \frac{1}{2} \int_0^L EA \varepsilon_{11}^2 \, dx + \frac{1}{2} \int_0^L EI \kappa^2 \, dx,$$

where $\varepsilon_{11} = \frac{1}{2} \left( r_x^T r_x - 1 \right)$ is the axial strain, $I$ is the area moment of inertia, and the magnitude of curvature vector $\kappa$ is given as [8]

$$\kappa = \frac{|r_x \times r_x|}{|r_x|^3}.$$
Equal flexural rigidity i.e. $EI_{zz} = EI_{yy}$ is assumed in this investigation [6]. The vector of the element elastic forces $Q_e \in \mathbb{R}^{12}$ is determined from the strain energy expression as

$$Q_e = \frac{\partial U}{\partial e} = \int_0^l EA(\epsilon_{11}) \left( \frac{\partial \epsilon_{11}}{\partial e} \right)^T dx + \int_0^l EI(\kappa) \left( \frac{\partial \kappa}{\partial e} \right)^T dx. \quad (9)$$

For the GD ANCF beam element, the equation of motion, and the expressions for element mass matrix and element external force are the same as in the case of a fully parameterized ANCF beam element. However, computing the element elastic force is much easier in the gradient deficient case. Since only one spatial coordinate ($\xi$) is used in shape functions, the numerical integration is carried out using the Gauss-quadrature formula in one dimension only. The generalized external force vector $Q_{\text{ext}}$, generalized elastic force vector $Q_{\text{int}}$, and the mass matrix for the beam are assembled using the element level vectors/matrices through element connectivity.

4.1.1 A Curved GD ANCF Beam

The GD ANCF formulation can be easily generalized to model the initially curved beams [67]. For the curved beam it is assumed that no stresses are present when the beam is in its initially curved configuration. Thus, the strains at the initial curved configuration are eliminated in the beam analysis. A curved GD ANCF beam element is shown in Figure 22. Note that the length of a curved beam element is the arc length in its initial/reference configuration.

The initial axial strain, i.e., the one dimensional Green strain can be expressed as

$$\epsilon_{11}^0 = \frac{1}{2} \left( r_{\xi}^T r_{\xi} - 1 \right),$$

where $r_{\xi}$ is the gradient vector in the initial curved configuration at a point
Figure 22: Gradient deficient (GD) ANCF beam element (initially curved)

defined by spatial coordinate \( \mathbf{x} \). Similarly the magnitude of initial curvature vector \( \kappa^0 \) can be obtained as

\[
\kappa^0 = \frac{\| \mathbf{r}_{0z} \times \mathbf{r}_{0xx} \|}{\| \mathbf{r}_{0z} \|^3}.
\]

The strain energy expression for the curved GD ANCF beam element becomes

\[
U_{\text{curved}} = \frac{1}{2} \int_0^l E A (\varepsilon_{11} - \varepsilon_{11}^0)^2 \, dx + \frac{1}{2} \int_0^l E I (\kappa - \kappa^0)^2 \, dx.
\]  \tag{10}

Note that the effects of axial strain and bending strain in the initial curved configuration are eliminated from the strain energy expression. Similar to the regular GD ANCF beam, the vector of the element elastic forces (\( \mathbf{Q}_s \in \mathbb{R}^{12} \)) is determined from the strain energy expression as

\[
\mathbf{Q}_{s,\text{curved}} = \frac{\partial U_{\text{curved}}}{\partial \mathbf{e}} = \int_0^l E A (\varepsilon_{11} - \varepsilon_{11}^0) \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right)^T \, dx + \int_0^l E I (\kappa - \kappa^0) \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right)^T \, dx.
\]  \tag{11}
For a curved beam element, the element mass matrix and the vector of generalized element external forces mass matrix have the same expressions as defined in Eqs. (4) and (5), respectively.

4.2 GD ANCF with Bilateral Constraints

The kinematic joints impose the constraints on relative motion of the bodies in a mechanical system. Without loss of generality, the constraints here are considered to be bilateral and holonomic. These constraints can be expressed as algebraic equations of the form

$$ F(q, t) = [F_1(q, t) ... F_m(q, t)]^T, $$

(12)

where $m$ is the total number of independent constraint equations that must be satisfied by the generalized coordinates $q = [q_1^T ... q_n^T]^T \in \mathbb{R}^p$. Here $n$ is the total number of bodies and $p$ is the total number of degrees of freedom present in the system. If each body in the system is assumed to be the beam then the generalized coordinates of each body (beam) are defined as $q_b = [e_1^T ... e_{ele}^T]^T$; $e_1, ..., e_{ele} \in \mathbb{R}^{12}$ where $ele$ is number of ANCF beam elements used in beam $b$.

Several types of mechanical joints can be easily modeled in ANCF. Some of these joints and their associated constraints are as follows: The spherical joint between two nodes of any two bodies will require the position vector of each node to be identical. The revolute joint will have additional two constraints to the spherical joint constraints. Specifically, the gradient vectors of two nodes will remain in a plane perpendicular to the axis of the revolute joint. The fixed joint will require all the nodal coordinates of two nodes to be identical. Note that alternatively, the element connectivity can be modeled using fixed joints between the nodes. Here, the common
node between two elements can be treated as two different nodes attached to each other through a fixed joint.

The generalized coordinates of the system change in time under the effect of applied forces such that these constraint equations are satisfied at all times. The time evolution of the system is governed by the Lagrange multiplier form of the constrained equations of motion

\[ M\ddot{q} + F_q^T(q)\lambda + Q_{\text{int}}(q) = Q_{\text{ext}}(\dot{q}, q, t), \tag{13} \]

where \( M \in \mathbb{R}^{p \times p} \) is the generalized mass, a constraint Jacobian matrix is \( F_q = \begin{bmatrix} \frac{\partial F_i}{\partial q_j} \end{bmatrix} \), for \( 1 < i < m, 1 < j < p \) and \( Q_{\text{ext}}(\dot{q}, q, t) \in \mathbb{R}^p \) is the applied force on the generalized coordinates \( q \in \mathbb{R}^p \) and \( Q_{\text{int}}(\dot{q}, q, t) \in \mathbb{R}^p \) is the vector of generalized elastic forces. The solution of these equations \( q(t) \) must also satisfy the constraint equations (e.g. Eq. (12)). These constraint equations lead in Eq. (13) to the presence of the reaction force \( F_q^T(q)\lambda \), where \( \lambda \in \mathbb{R}^m \) is the Lagrange multiplier associated with the kinematic constraints.

Eq. (12) and Eq. (13) together form a system of index-3 DAE as shown in Chapter 2. Here we will consider the NEWMARK and HHT-I3 integration methods to show how the GD ANCF equations are motion are solved in the MBS framework. Both NEWMARK and HHT-I3 were originally used in the structural dynamics community for the numerical integration of a linear set of second order ODEs. In the NEWMARK formulation, the discretization of the equation of motion yields

\[ (M\ddot{q})_{n+1} + (F_q^T\lambda)_{n+1} + (Q_{\text{int}} - Q_{\text{ext}})_{n+1} = 0, \tag{14} \]

whereas the discretized equation of motion in HHT-I3 becomes
Given the acceleration \( \ddot{q}_{n+1} \) at the new time step \( t_{n+1} \), the new position and velocity are obtained as

\[
q_{n+1} = q_n + h \dot{q}_n + \frac{h^2}{2} \left[ (1-2\beta) \ddot{q}_n + 2\beta \ddot{q}_{n+1} \right],
\]

\[
\dot{q}_{n+1} = \dot{q}_n + h \left[ (1-\gamma) \ddot{q}_n + \gamma \ddot{q}_{n+1} \right],
\]

where \( h \) is the integration step size and \( \gamma \geq \frac{1}{2}, \beta \geq \frac{(\gamma + 1/2)^2}{4} \) in the NEWMARK method. In the HHT-I3 method, \( g = \frac{1-2a}{2}, b = \frac{(1-a)^2}{4} \), where a parameter \( a \) controls the numerical damping.

The discretization of the constraint equations gives

\[
F(q_{n+1}, t_{n+1}) = 0.
\]

It should be noted that in the NEWMARK and HHT method, \( q \) and \( \dot{q} \) are expressed as a function of \( \ddot{q} \) using the integration formulas given by Eq. (16) and Eq. (17).

A Newton’s method can be used to solve the system of nonlinear equations defined by Eq. (14) and Eq. (18) for the set of unknowns \( \ddot{q} \) and \( \lambda \). The iterative algorithm of Newton’s method requires at each iteration \( (k) \), the solution of the linear system

\[
\begin{bmatrix}
\ddot{M} & F_q^T \\
F_q & 0
\end{bmatrix}
\begin{bmatrix}
\Delta \ddot{q}^{(k)} \\
\Delta \lambda
\end{bmatrix} =
\begin{bmatrix}
-e_{1}^{(k)} \\
-e_{2}
\end{bmatrix},
\]

where \( e_{i} \) are the residuals in satisfying the set of the discretized equation of motion and constraint equations, with the latter scaled such that
Similarly, the residuals \( \mathbf{e}_i \) in the HHT-I3 method are defined as

\[
\begin{align*}
\mathbf{e}_1^{\text{HHT}} &= \frac{1}{1 + \alpha} (\mathbf{M} \dot{\mathbf{q}})_n + (\mathbf{F}_q^T)_{n+1} \mathbf{Q} -(\mathbf{Q}^\alpha)_{n+1} \mathbf{Q} \frac{\alpha}{1 + \alpha} (\mathbf{F}_q^T)_{n+1} \\
\mathbf{e}_2^{\text{HHT}} &= \frac{1}{\beta h^2} \mathbf{F} (\mathbf{q}_{n+1}, t_{n+1}).
\end{align*}
\]

This scaling is done in order to improve the conditioning of the Jacobian Matrix for Newton’s method [48]. The matrix \( \hat{\mathbf{M}} \) in Eq. (19) in the NEWMARK and HHT-I3 method is defined as follows

\[
\hat{\mathbf{M}}^{\text{NEWMARK}} = \frac{\partial \mathbf{e}_1}{\partial \mathbf{q}} = \mathbf{M} + \beta h^2 \left[ \mathbf{1 - \alpha} (\mathbf{M} \dot{\mathbf{q}})_q + (\mathbf{F}_q^T)_{q} \mathbf{Q} + \frac{\partial Q^\alpha_{\text{int}}}{\partial \mathbf{q}} - \frac{\partial Q^\alpha_{\text{ext}}}{\partial \mathbf{q}} \right] - \gamma h^2 \left[ \mathbf{\partial Q^\alpha_{\text{ext}}} / \partial \mathbf{q} \right], \tag{22}
\]

\[
\hat{\mathbf{M}}^{\text{HHT}} = \frac{\partial \mathbf{e}_1}{\partial \mathbf{q}} = \frac{1}{1 + \alpha} \mathbf{M} + \beta h^2 \left[ \frac{1}{1 + \alpha} (\mathbf{M} \dot{\mathbf{q}})_q + (\mathbf{F}_q^T)_{q} \mathbf{Q} + \frac{\partial Q^\alpha_{\text{int}}}{\partial \mathbf{q}} - \frac{\partial Q^\alpha_{\text{ext}}}{\partial \mathbf{q}} \right] - \gamma h^2 \left[ \mathbf{\partial Q^\alpha_{\text{ext}}} / \partial \mathbf{q} \right]. \tag{23}
\]

It should be noted that the first term becomes zero as shown above since the ANCF mass matrix is constant. For most of the kinematic joints (e.g. spherical, revolute, fixed, etc.) the term \((\Phi^T_q \dot{\mathbf{q}})_q\) becomes zero. If the external force is constant (e.g. gravity), the terms \(\frac{\partial Q^\alpha_{\text{int}}}{\partial \mathbf{q}}, \frac{\partial Q^\alpha_{\text{ext}}}{\partial \mathbf{q}}\) become zero. Here \(\frac{\partial Q^\alpha_{\text{int}}}{\partial \mathbf{q}}\) is the most compute-intensive term which represents the tangent stiffness matrix associated with nonlinear ANCF formulation.
The tangent stiffness matrix for the ANCF beam element $K_s \in \mathbb{R}^{12 \times 12}$ is derived from the element elastic force as $K_s = \frac{\partial Q_s}{\partial \mathbf{e}}$, which becomes

$$K_s = \int_0^l EA(\varepsilon_{11}) \frac{\partial}{\partial \mathbf{e}} \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right)^T d\mathbf{x} + \int_0^l EA \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right)^T \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right) d\mathbf{x}$$

$$+ \int_0^l EI(\kappa) \frac{\partial}{\partial \mathbf{e}} \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right)^T d\mathbf{x} + \int_0^l EI \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right)^T \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right) d\mathbf{x}.$$  \hspace{1cm} (24)

Similarly for the curved beam element, the tangent stiffness matrix $K_{s_{\text{curved}}} \in \mathbb{R}^{12 \times 12}$ can be derived as

$$K_{s_{\text{curved}}} = \int_0^l EA(\varepsilon_{11} - \varepsilon_{11}^0) \frac{\partial}{\partial \mathbf{e}} \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right)^T d\mathbf{x} + \int_0^l EA \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right)^T \left( \frac{\partial \varepsilon_{11}}{\partial \mathbf{e}} \right) d\mathbf{x}$$

$$+ \int_0^l EI(\kappa - \kappa^0) \frac{\partial}{\partial \mathbf{e}} \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right)^T d\mathbf{x} + \int_0^l EI \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right)^T \left( \frac{\partial \kappa}{\partial \mathbf{e}} \right) d\mathbf{x}.$$  \hspace{1cm} (25)

Here each integral can be evaluated using Gauss quadrature formula. Though only NEWMARK and HHT-I3 integration methods are considered here, one can see that the GD ANCF beam formulation can be easily combined with any implicit or explicit numerical integration method used in the MBS.

4.3 Numerical Experiments and Results

Several numerical experiments are carried out in order to (a) Validate the GD ANCF against FEAP [78] and ABAQUS; (b) Compare the GD ANCF beam elements against fully parameterized ANCF beam elements; (c) Carry out a convergence analysis of the gradient deficient ANCF beam elements; and (d) Verify the results of GD ANCF using the benchmark tests commonly used in the literature. The model details used in the first set of numerical
experiments are given in Table 1. In all numerical experiments the flexible pendulums/beams are assumed to be in horizontal configuration initially with no initial velocity as shown in Figure 23.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (m)</td>
<td>1</td>
</tr>
<tr>
<td>Cross-section Area(m$^2$)</td>
<td>0.02 x 0.02</td>
</tr>
<tr>
<td>Material Density (kg/ m$^3$)</td>
<td>7200</td>
</tr>
<tr>
<td>Modulus of elasticity (Pa)</td>
<td>2.0E7</td>
</tr>
<tr>
<td>Second moment of inertia (m$^4$)</td>
<td>1.33e-8</td>
</tr>
<tr>
<td>Boundary Condition</td>
<td>Pinned at one end</td>
</tr>
<tr>
<td>External Force</td>
<td>Gravity</td>
</tr>
</tbody>
</table>

Figure 23: A flexible beam pendulum pinned at one end

4.3.1 Validation of the Gradient Deficient ANCF Elements

Firstly an energy analysis of GD ANCF beam is done using Model-1. Since only gravity force is acting, the total energy should remain constant. The implicit numerical integration methods such as NEWMARK and HHT-I3 induce numerical damping. To avoid any numerical damping the symplectic integration method is used in this experiment. The symplectic
integrators are known for their energy conserving characteristics. Here the first order semi-implicit Euler (symplectic Euler) method is used:

\[
q_{n+1} = q_n + \mathbf{q}_n \\
q_{n+1} = q_n + \mathbf{q}_{n+1}.
\]  

(26)

Using this integration method a 10 sec simulation of Model-1 is run using the constant step size of 1.E-4. Figure 24 shows that the total energy using GD ANCF elements remains constant as the system is conservative. It also shows that the symplectic integration scheme used in this simulation remains stable.

![ANCF Energy Analysis](image)

Figure 24: Energy analysis – GD ANCF (SE: strain energy, KE: kinetic energy, PE: potential energy, TE: total energy)

In the next set of numerical experiments a generalized 3D motion of the ANCF beam with no contact is studied. Model-1 is used to study 3D motion of a highly deformable pendulum (beam pinned at one end) under the effect of gravity and externally applied force. The pendulum
is modeled using 10 ANCF beam elements, 10 BEAM33 elements in ABAQUS, and 20 brick elements in FEAP. To generate a 3D motion, a constant force of 1N in Z direction is applied at the end node for 2 seconds. Figure 25, Figure 26, and Figure 27 show the displacements of the pendulum-tip. It can be seen that the ANCF results match well with the ABAQUS and FEAP results. The GD ANCF elements do not suffer from shear locking problems. These results show that a generalized 3D motion with no torsion along the beam axis can be modeled correctly using the GD ANCF approach.

Figure 25: X displacement of a pendulum-tip (ANCF, ABAQUS and FEAP comparison)
Figure 26: Y displacement of a pendulum-tip (ANCF, ABAQUS and FEAP comparison)

Figure 27: Z displacement of a pendulum-tip (ANCF, ABAQUS and FEAP comparison)
4.3.2 Comparison with Original ANCF Beam

In this numerical experiment the beam is modeled using the GD ANCF beam elements and the fully parameterized ANCF beam elements. The tip displacements of the flexible pendulum are compared with those obtained using fully parameterized ANCF beam elements. Figure 28 shows considerable difference between two results, which is attributed to the locking problem of original ANCF beam elements. It can be seen that the locking problems of original ANCF beam elements can lead to wrong results in case of thin beams.

![Figure 28: X and Y displacement of a pendulum-tip (ANCF and GD ANCF comparison)](image)

4.3.3 Convergence Analysis

In order to investigate the convergence of GD ANCF elements, a Model-1 beam with a tip-mass of 1 kg is used in this analysis. The beam is modeled using various numbers of elements and the pendulum tip positions are compared for a 2 sec long simulation. The NEWMARK integration method with constant step size of 1.E-4 sec is used to run the simulation. Figure 29
shows that as the number of elements is increased, the results converge to a single value. The gradient deficient ANCF beam elements exhibit good convergence characteristics. Thus the large deformation of slender beams can be effectively modeled with very few gradient deficient ANCF beam elements.

![Figure 29: X and Y coordinate of a pendulum-tip](image)

**4.3.4 Benchmark Tests**

In order to verify the implementation of GD ANCF with bilateral constraints, the spin-up of a rotating cantilever beam model is considered. The beam spin-up maneuver, as shown in Figure 30, has been used in the past as a benchmark problem for evaluating the geometric stiffening effect of beam elements [6, 67]. The parameters of the model are rectangular cross-section with \( A = 7.299\text{E-5} \text{ m}^2 \), area moment of inertia \( I_{yy} = I_{zz} = 8.215\text{E-9} \text{ m}^4 \), length \( L = 8 \text{ m} \), Young’s modulus \( E = 6.895\text{E10} \text{ N/m}^2 \), and density \( \rho = 2766.67 \text{ kg/m}^3 \). The angular motion of the hub is prescribed as
Figure 30: Spin-up maneuver of a straight beam

\[
\theta(t) = \begin{cases} 
\frac{\omega_s}{T_s} \left[ \frac{t^2}{2} + \left( \frac{T_s}{2\pi} \right)^2 \left( \cos \frac{2\pi t}{T_s} - 1 \right) \right] & 0 \leq t \leq T_s \\
\omega_s \left( t - \frac{T_s}{2} \right) & t > T_s
\end{cases}
\] (27)

where \( T_s = 15 \text{s} \) and \( \omega_s = 4 \text{ rad/s} \). The in-plane tip displacements are measured with respect to a co-rotational tangent frame rigidly attached to the hub, as shown in Figure 30. A 20 sec long simulation is run with constant step size of 1E-4 sec using HHT-I3 integration method. The longitudinal and transverse deflections of the beam tip with respect to a co-rotational frame are shown in Figure 31 and Figure 32, respectively. The results obtained using 10 GD ANCF elements are in perfect agreement with those obtained from 10 elements of ADAMS/Geometrically-Exact Beams [79].
Figure 31: Longitudinal deflection at the tip of spin-up maneuver (straight beam)

Figure 32: Transverse deflection at the tip of spin-up maneuver (straight beam)
To find out the applicability of the GD ANCF approach for the curved beams, a curved beam with an initial curvature of 45° in the XY plane is considered as shown in Figure 33. A curved beam has length $L = 8$ m with initial curvature radius $R = 4L/\pi$. All other model parameters, including the prescribed angular velocity of the hub, are assumed to be the same as the previous example. The longitudinal and transverse deflections of the beam tip as shown in Figure 34 and Figure 35 are in perfect agreement with those calculated from ADAMS/Geometrically-Exact Beams. It can be seen from Figure 35 that the transverse deflection is larger compared to the straight beam because of the initial curvature of beam. The results of these benchmark problems also match well with those given in the literature [67].
Figure 34: Longitudinal deflection at the tip of spin-up maneuver (curved beam)

Figure 35: Transverse deflection at the tip of spin-up maneuver (curved beam)
Thus the GD ANCF approach can correctly model the effects of geometric stiffening as well as the effects of initial curvature of beam.

4.4 Concluding Remarks

The formulation of GD ANCF beam element for straight as well as initially curved beam is investigated in this chapter. The implementation of GD ANCF in index-3 DAEs is easy and straightforward as all nodal coordinates are defined in the global inertial frame. The only drawback of this formulation is generalized 3D motion of the beam cannot be modeled as torsional motion is not considered.

The GD ANCF beam elements can be used in thin beam and cable applications where torsional effects are not important. For such applications, GD ANCF has an edge over original (fully parameterized) ANCF due to (i) locking problems of original ANCF are eliminated, especially the shear locking of original ANCF produces erroneous results for thin and stiff beam applications; (ii) GD ANCF is more efficient compared to original ANCF as it uses less number of nodal coordinates. Specifically, GD ANCF uses 12 and original ANCF uses 24 nodal coordinates per beam element.

Moreover, GD ANCF shows good convergence characteristics. Therefore very few elements are required to capture the large deformation of beam. The results of various numerical experiments match well with those obtained using commercial softwares, Abaqus and ADAMS, and research software, FEAP.
5 Frictional Contact/Impact Modeling – Penalty Approach

The purpose of this chapter is to introduce a continuous contact force model (CCFM), which is essentially a penalty method to capture contact/impact phenomena between flexible bodies. The CCFM used herein has been previously used in the multibody dynamics community for handling the contact/impact problem between rigid bodies [80]. In this work, it is also considered for the contact/impact problem between flexible bodies using a spherical decomposition of the element geometry. In this context, the objective of this chapter is twofold: a) show how the CCFM approach can be combined with ANCF beams for frictional contact/impact modeling, and b) evaluate and validate the proposed ANCF+CCFM approach. Two test problems are used to validate the ANCF+CCFM approach and its implementation: i) frictional contact between two flexible beams, and ii) frictional contact between a flexible beam and a rigid cylinder. The validation process compares ANCF+CCFM results against results obtained with the commercial finite element software ABAQUS and the research code Finite Element Analysis Program (FEAP) [78]. FEAP is an open source finite element code that is very versatile in relation to formulating and implementing additional constitutive models and solution schemes via user subroutines. The formulations used herein are a mix of FEAP original elements and user routines.

5.1 Introduction and Theoretical Background

Simulating impact/contact phenomena between flexible bodies poses a challenging problem because the time evolution of system is characterized by sudden surges in acceleration values. Moreover, these problems are highly nonlinear in nature due to the presence of one or more of the following: (i) nonlinear boundary conditions (friction at interfaces); (ii) very large
deformations; and (iii) nonlinear material. A standard approach for the solution of contact/impact problems draws on the finite element method. In this approach, nonlinear formulations capable of capturing large deformations and rotations [8, 62] are utilized to model the flexible bodies in conjunction with large deformation contact discretizations [81] to model the contact interface. There are several different formulations that are applied to incorporate the contact and impact phenomena into the variational formulation or the weak form of governing equations [62, 82]. Most standard finite element codes which use either the penalty method or the Lagrange multiplier method [81]. These formulations can be compute intensive as they iteratively solve a nonlinear optimization problem with inequality constraints using an active set strategy, which transforms the original problem into a sequence of problems with equality constraints that converge to the solution. Moreover, these formulations rely on several discretizations (e.g. node-segment, segment-segment, etc.) which are based on nodes on the boundary of the deformable bodies [81, 83, 84]. These discretization schemes often limit the number of flexible bodies in contact that can be handled at a time using the standard finite element codes. Not surprisingly, the complexity of the numerical solution often forces one to compromise on the size of the system simulated in order to keep simulation run time at an acceptable level. This contribution sets out by outlining a CCFM approach CCFM for the analysis of frictional/contact problems between flexible bodies that (i) is more straightforward; (ii) leads to shorter simulation times; and (iii) is easier to map onto emerging parallel computing hardware, a process that can significantly increase the size/complexity of the systems that can be analyzed through physics-based modeling and simulation.


5.2 Continuous Contact Force Model (CCFM)

Most of the solution methods for rigid-body contact/impact problems can be classified as either impulse-momentum-based approaches, also known as discontinuous or hard-body approaches, or compliance-based approaches, also known as continuous or soft-body approaches. The discontinuous approach was used in rigid multibody systems by Haug and Wehage [13], and Khulief and Shabana [85] extended it to flexible systems. The instantaneous impact assumption used in the discontinuous approach limits its application to a multibody system where duration of impact is unknown or large enough to change the system configuration. Compliance-based methods, see for instance [86], are more widely used in multibody dynamic systems since they are straightforward to implement, expeditious, and more versatile. These continuous contact force models (CCFM) assume that the forces and deformations/penetrations vary in a continuous manner and the former are related to the latter through an explicit and relatively simple relationship. Several continuous contact force models developed over the past few decades are based on Hertzian contact theory. Lankarani and Nikravesh [9], and Hunt and Crossly model [10] added a damping term to the Hertz contact model. A linear contact force expression is used in the Kelvin Voigt model [87]. Recently a volumetric-based contact force model has been proposed in [88]. These contact force models are similar to those used in the Discrete Element Method [89], which has been widely employed in rock mechanics, molecular dynamics, and granular dynamics simulation [90-93].

The CCFM adopted herein can be classified as a penalty method, where the force acting between two colliding bodies is computed based on the associated interpenetration. Effectively, CCFM introduces a fictitious spring-damper element that is placed between two colliding bodies.
The spring and damping coefficients can be derived from the continuum theory [94] or calibrated based on experimental data [95]. While the CCFM inherits some of the attractive features of a penalty method being straightforward to implement, expeditious, and versatile, it has some of the inevitable drawbacks associated with this method. The penalty factors must be chosen properly to avoid large, physically inadmissible penetrations. Also, very high spring stiffness may introduce spurious high frequencies which may lead to numerical instability during the solution process.

Herein, the interest is in characterizing the contact between highly flexible bodies. To that end, the approach used relies on the Hertzian compliance contact model [96], which quantifies the force between two spheres of isotropic material and a spherical decomposition of the geometry of the beams that will be discussed shortly. The schematic of CCFM contact is shown in Figure 36. The normal contact force \( F_n \) is evaluated as

\[
F_n = K d^n,
\]

where \( K \) is the generalized stiffness constant and \( d \) is the relative normal penetration between the bodies in contact. The typical value for the compliance exponent, \( n \) is 1.5 for Hertzian model. For the two spheres in contact, the generalized stiffness is a function of the radii of the spheres \( i \) and \( j \), and the material properties as shown in Equations (2) and (3) [97]:

\[
K = \frac{4}{3(s_i + s_j)} \frac{\dot{e}_i R_i R_j}{\dot{\varepsilon}} \frac{u_i^5}{u_j^5}
\]

\[
s_k = \frac{1 - u_k^2}{E_k} ; k = i, j
\]
where \( \nu_k \) is the Poisson’s ratio and \( E_k \) is the Young’s modulus associated with each sphere, respectively. Although overall compliant, the colliding beams are assumed not to experience large local deformation at the point of contact/impact. The contact between two beams can be considered as a rigid-body contact locally in which the Hertzian model provides a good estimate for the contact spring stiffness. Since no damping is involved in the Hertzian model, the contact/impact law does not depend on the coefficient of restitution, and thus a pure elastic contact is achieved between the colliding spheres.

The regularized Coulomb friction model presented in [98] is used where the friction force \( F_t \) is given as

\[
F_t = -m \eta F_n \text{dir}(v_{ct} \cdot u_e),
\]

where \( m \) is the Coulomb friction coefficient, \( v_{ct} \) is the relative tangential velocity of colliding spheres, \( u_e \) is a velocity tolerance, and the term \( \text{dir}(v_{ct} \cdot u_e) \) is defined as...
The CCFM approach can be easily implemented with the explicit numerical integration method; however, it requires a very small integration step-size to maintain the stability and accuracy of the numerical solution. The integration step size depends on size of spheres, material properties associated with the bodies in contact, and relative velocity of the spheres. Because of the rigid body assumptions, stiff springs induced by the CCFM model can lead to high transients in the system dynamics.

5.3 **Spherical Decomposition and Contact Force Computation**

Frictional contact modeling requires the solution of a contact detection problem. Herein it was assumed that the beams have uniform circular cross-section. These beams can be mathematically described as space curves with associated cross-sectional radius (see Eq. (1)). The contact search for two beams can be done by solving a minimum distance problem between the beam centerlines. The optimization problem can be stated as

\[
\min_{x_1, x_2} d = \left\| r(x_1) - r(x_2) \right\| ,
\]

where \( r \) is the global position vector of an arbitrary point on the beam centerline and \( x_1, x_2 \) are along the beam elements. Newton’s method can be used to solve this problem but other approaches that require the gradient and the Hessian matrix of distance \( d \) with respect to the spatial coordinates \( x_1, x_2 \) might be used as well, see for instance [99]. To avoid further
algorithmic complications, the assumption of unique solution of the minimum distance problem is made in [81]. Although this approach provides the precise contact locations, it adds a significant computational overhead to the contact detection problem as the minimum distance problem needs to be solved between all pairs of beam elements at each time step. Herein, an approximate but expeditious way of contact detection was adopted by using the spherical decomposition approach. Specifically, as shown in Figure 37, for collision detection purposes only, the geometry of each flexible beam is represented as a chain of spheres that overlap and are distributed equally along the axis of the beam elements. The mutual contact between two beam elements is assumed to be point-wise, which excludes the beams-in-parallel contact case. In the case of multiple contacts between the beam elements, a pair of spheres with maximum penetration is selected to provide the contact locations along the two beam elements. Thus, the spherical decomposition approach checks the distance between discrete points; i.e., sphere centers, of the colliding beams and defines the collision in terms of the pair of spheres that leads to the smallest distance. As the number of spheres along the beams is increased, the contact location will approach the exact solution of the minimum distance problem given in Eq.(6).

The collision detection between the spheres in all beams is performed at each time step. The normal and tangential contact forces between the colliding spheres are calculated using CCFM. These contact forces are treated as externally applied point forces in the nonlinear dynamics analysis with the gradient deficient ANCF. The generalized contact force $Q_{cont} \in \mathbb{R}^{12}$ for the beam element is computed as

$$Q_{cont} = [S(x)]^T F,$$ (7)
where $\mathbf{F} \in \mathbb{R}^3$ is the vector sum of normal and tangential contact force. $\mathbf{S}(x)$ is the shape function matrix defined at the point of contact. Note that the sphere in contact specifies the contact location i.e. spatial coordinate $x$ along the beam axis. For thin beams, the moments of tangential frictional contact forces about the beam centerline are neglected. Note that this approach allows multiple contacts between the colliding beams (with maximum one possible contact between two beam elements) and also self-contact in the case of long, highly deformable beams.

The collision detection problem can be a bottleneck in the simulation of physical systems involving a large number of bodies. For instance, for a polymer simulation problem, one has to consider systems with hundreds of thousands of flexible beams interacting through friction and contact. This collision detection task needs to be performed once at each integration time step. The spherical decomposition approach simplifies the process of collision detection between the slender beams, which then falls back on the problem of collision detection between millions of spheres that together make up the shape of the beams.
5.4 Implementation of CCFM with ANCF beams

A sequential ANCF and CCFM algorithm has been implemented using both MATLAB and C. It can handle multiple beams, and is used for validation and benchmarking the parallel implementation. The details of the parallel implementation of this approach can be found in [100]. There are two main stages to the serial implementation: a preprocessing stage and a time stepping loop.

Preprocessing stage

The preprocessing stage of the ANCF algorithm prepares the data required for solving the degrees of freedom of the beam at each time step in the time loop. To this end, there are five vectors and one matrix initialized in the preprocessing stage. A square mass matrix with dimensions equal to the number of degrees of freedom of the beam is initialized. The mass matrix is constant and does not need to be recomputed at every time step. Following the mass matrix, an external force vector due to gravity is initialized. This part of external force vector contains only forces due to gravity and remains constant over time. Next, the position vector, which holds the degrees of freedom of the beam is created to represent the initial location and orientation of the beam. Lastly, the internal force, acceleration, and velocity vectors, each with a size equal to the number of degrees of freedom of the beam, are initialized to zero.

Time stepping loop

After the preprocessing stage, the contact external force vector, the internal force vector, acceleration vector, velocity vector, and position vector need to be updated at every time step. At each time step the collision detection between the spheres is performed and the contact external force vector is computed using CCFM as explained before. The internal force vector is computed
using Eq. (9), which is by far the most computationally intensive portion of the algorithm. The equation of motion (Eq. (3)) is solved to obtain the acceleration vector, and the position and velocity vector are updated using a symplectic numerical integrator. This position and velocity data is used in the next time step to compute the contact external force vector and the internal force vector; and the cycle is repeated as shown in Figure 38. The symplectic Euler (semi-implicit Euler) integration method discussed in Chapter 4 is used here to advance the numerical simulation.

![Time Stepping Loop](image)

Figure 38: Time looping stage of the ANCF and CCFM algorithm

5.5 **Numerical Experiments and Results**

Several numerical experiments were carried out in order to 

a) Perform a convergence analysis of the GD ANCF beam elements for the contact cases; 

b) Assess the potential of the ANCF and CCFM implementation; and 

c) Validate the ANCF and CCFM implementation against FEAP and ABAQUS. The details of the models used in the numerical experiments and the simulation parameters are given in Table 2. For all the numerical experiments the flexible pendulums/beams are assumed to be in horizontal configuration initially with no initial velocity.
Table 2: Model and simulation parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model -1</th>
<th>Model -2</th>
<th>Model -3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contact type</td>
<td>Flex-Rigid</td>
<td>Flex-Flex</td>
<td>Flex-Rigid</td>
</tr>
<tr>
<td>Length (m)</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Cross-section Area (m^2)</td>
<td>(\pi \times 0.01^2)</td>
<td>(\pi \times 0.01^2)</td>
<td>(\pi \times 0.01^2)</td>
</tr>
<tr>
<td>Material Density (kg/m^3)</td>
<td>7200</td>
<td>7200</td>
<td>7200</td>
</tr>
<tr>
<td>Modulus of elasticity (Pa)</td>
<td>2.0E7</td>
<td>2.0E8, 2.0E9</td>
<td>2.0E7</td>
</tr>
<tr>
<td>Second moment of area (m^4)</td>
<td>7.85E-10</td>
<td>7.85E-10</td>
<td>7.85E-10</td>
</tr>
<tr>
<td>Tip Mass (kg)</td>
<td>1</td>
<td>--</td>
<td>5</td>
</tr>
<tr>
<td>External Force</td>
<td>Gravity + Contact</td>
<td>Gravity + Contact</td>
<td>Gravity + Contact</td>
</tr>
<tr>
<td>Integration Step-Size</td>
<td>1.0E-5</td>
<td>1.0E-5</td>
<td>1.0E-5</td>
</tr>
</tbody>
</table>

Figure 39: Schematic of Model-1
5.5.1 Convergence analysis of ANCF+CCFM

In the first set of numerical experiments the convergence characteristics of ANCF are studied for contact/impact case. For this case the convergence analysis is done with increasing number of ANCF beam elements as well as with increasing number of spheres along the beam centerline. Model-1as shown in Figure 39 is used in this convergence analysis. In this model a beam, pinned at one end, comes into contact with the rigid cylinder fixed at position (0.5, -0.5). From Figure 40 and Figure 41, it can be seen that the pendulum tip displacements tend to converge as the number of elements is increased, which shows that the convergence characteristics of ANCF are not affected when combined with the CCFM approach. It suggests that a modest number of elements are sufficient to capture the dynamics induced by frictional contact while using the CCFM in conjunction with ANCF.

Figure 40: X coordinate of a pendulum-tip (contact case)
Figure 41: Y coordinate of a pendulum-tip (contact case)

Figure 42: X displacement of a pendulum-tip (contact case)
The contact problem between two flexible beams (Model-2) is used for sensitivity analysis of the collision detection with the number of spheres used to define the beam geometries. In this analysis, both beams are modeled using 12 elements and the number of spheres, which are equally distributed along the beam centerline, is varied. Figure 42 and Figure 43 indicate that the pendulum tip displacements converge as the number of spheres is increased. It should be noted that the beams come into contact multiple times. The first contact event happens just after 0.535 sec. For convergence study, the displacements after only the first few contact events are considered and reported at 0.55 sec in Table 3. The solution obtained with a 200-spheres beam model is used as reference solution. Note that the long term dynamic response of this two pendulum model shows chaotic behavior because of the high frequency lateral vibrations induced in the beams due to multiple impacts.
Table 3: Sensitivity of displacement with number of spheres defining the geometry of the beam
(results at 0.55 sec)

<table>
<thead>
<tr>
<th>Spheres</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta X$ (m)</td>
<td>0.004286</td>
<td>0.002805</td>
<td>0.001183</td>
<td>0</td>
</tr>
<tr>
<td>$\Delta Y$ (m)</td>
<td>3.0288e-4</td>
<td>1.3836e-4</td>
<td>7.1951e-5</td>
<td>0</td>
</tr>
</tbody>
</table>

5.5.2 Validation of ANCF+CCFM

Several numerical experiments are carried out in order to assess the potential of the ANCF and CCFM implementation. In these numerical experiments the frictional contact/impact phenomenon between two flexible beams (pinned at one end) is studied using Model-2. A schematic of Model-2 is shown in Figure 44. The Hertzian contact model parameters defined in Eqns. (2) and (3) are used to compute the contact normal force. The Coulomb coefficient in the friction model of Eq. (4) was chosen as $\mu = 0.1$. Snapshots from the numerical simulation of contact between two flexible beams are shown in Figure 45.

![Figure 44: Schematic of Model-2](image-url)
Figure 45: Snapshots from simulation of Model-2

Figure 46: Snapshots from simulation of Model-3
Model-3 is used to simulate the frictional contact between a long flexible beam (pinned at one end) with a tip mass and a rigid cylinder. Figure 46 shows a sequence of three snapshots from the numerical simulation of this model. Note that the ANCF+CCFM approach allows the contact force modeling between rigid bodies and flexible beams. Figure 47 shows the snapshots from the numerical simulation of 8 instances of Model-3. In this parametric study, the number of elements varied in each beam from 1 element to 8 elements. A rigid horizontal cylinder with radius of 0.01 m was fixed at position (1, -1.5) so that the elements would come into contact with the rigid cylinder as they moved. The beams tend to wrap around the rigid cylinder due to inertia of the tip mass. The overall bending stiffness of beam tends to decrease as the number of elements increases and hence different trajectories of the beam tip are observed, as seen in Figure 47. Animations of these simulations are available at [101].

The ANCF+CCFM approach was further validated against ABAQUS and FEAP using the contact problem between two flexible beams. It has been observed that the two beam contact
problem shown in Figure 45 is very sensitive to the contact parameters. In this context, there is no systematic method to choose the parameter values used in the contact force model. Specifically, each of the three implementations considered: ANCF+CCFM, FEAP, and ABAQUS, use different sets of parameters, which were calibrated based on a set of reference data. This data can be experimentally collected or generated by one “trusted” solver, such as ABAQUS. The latter approach was adopted herein. In ABAQUS, the explicit general hard contact algorithm with the default stiffness value is used to compute the normal contact force. In FEAP, mortar surface-to-surface contact elements [102] are used as they are numerically more robust than traditional node-to-surface approaches.

In the numerical experiment, see Figure 45; each beam is modeled using 8 ANCF beam elements and 100 spheres along beam centerline. The Hertzian contact parameters along with Model-2 beams are used in this experiment. In ABAQUS each beam is modeled using 100 BEAM31 (2-node 3D) elements. The penalty method with friction coefficient of 0.1 is used to compute the tangential frictional force. In FEAP 100 2-node 3D beam elements (shear on) are used to model each beam and a surface-to-surface algorithm is used to model the contact [102]. The pendulum-tip displacements for Model-2 beams with $E = 2e8$ (N/m$^2$) are given in Figure 48 and Figure 49. Due to the symmetric nature of the problem, the tip-displacements of second pendulum are just mirror images of those of the first pendulum. Figure 50 and Figure 51 show the pendulum-tip displacements for the Model-2 beams with $E = 2e9$ (N/m$^2$). The results from ANCF + CCFM approach are quite close to the ABAQUS and FEAP results. Since the results are within an acceptable range of accuracy, the combined ANCF + CCFM approach can be a good substitute for the contact algorithms used in commercial FEA softwares.
Figure 48: X displacement of a pendulum-tip (E=2e8)

Figure 49: Y displacement of a pendulum-tip (E=2e8)
Figure 50: X displacement of a pendulum-tip (E=2e9)

Figure 51: Y displacement of a pendulum-tip (E=2e9)
5.5.3 Energy analysis of ANCF+CCFM

It is important to investigate how the total energy of a system is distributed when the contact/impact phenomenon happens. In this set of experiments only frictionless contact/impact problems are considered to analyze the effect of CCFM (Hertzian model) on the total energy. In energy analysis for the contact case, Model-1 with a tip-mass of 1kg is used, which comes into contact with the rigid cylinder fixed at position (0.5, -0.1). The beam remains in contact with the rigid cylinder as it tends to wrap around the rigid cylinder due to the tip-mass inertia. Figure 52 shows the total energy distribution for the flex-rigid contact problem. The total energy of the system is comprised of the potential, kinetic, and strain energy of the ANCF beams. It is evident that the total energy is conserved for this frictionless contact problem. There are momentary drops in the total energy due to strain energy stored in the fictitious springs of the Hertzian model. When the contacting bodies penetrate, some energy is stored in the contact springs, which is not accounted in the total energy. This strain energy is converted back to the kinetic energy when the contacting bodies separate.

Model-2 is used in energy analysis for the impact case. Figure 53 shows the total energy distribution of the flex-flex impact problem. Note that the Hertzian model does not include any damping, so perfectly elastic impact is achieved between the colliding spheres. Moreover, the symplectic integrator used in this analysis does not add any numerical damping. It can be seen from Figure 53 that the total energy of the system is conserved. The total energy during collision is less than the overall total energy due to action of the contact springs, which deform possessing strain energy. As mentioned earlier, this strain energy is not considered in this analysis.
Figure 52: Energy distribution (beam-rigid cylinder contact case)

Figure 53: Energy distribution (beam-beam impact case)
5.6 **Concluding Remarks**

A methodology to combine ANCF with a penalty-based method, CCFM to model frictional contact/impact between flexible beams is outlined in this chapter. The Hertzian contact model along with regularized Coulomb friction model is used to compute the contact forces. A new contact discretization method i.e. spherical decomposition of beams is proposed herein simplifies the contact detection and computation of contact forces.

The CCFM implementation in the ANCF framework is quite straightforward. The Hertzian contact model provides good estimates for penalty parameters (e.g. contact spring stiffness). The only drawback of this method is requirement of small step sizes to maintain the stability of numerical integration method.

The numerical experiments show the convergence characteristics of ANCF are not affected when combined with CCFM. The results of ANCF+CCFM for the flex-flex and flex-rigid contact/impact cases are validated against FEAP and Abaqus. The energy distribution shows that the total energy of system is conserved when the Hertzian contact model is used in the frictionless contact/impact case.
6 Frictional Contact/Impact Modeling – Variational Approach

Two approaches are most often considered when simulating the dynamics of a multibody system with frictional contact/impact. First there is the class of so-called penalty methods drawing on the regularization approach discussed in Chapter 5. A second approach, pursued in this chapter, relies on a non-smooth dynamics principle. The algorithm in this class draws on time-stepping procedures that produce weak solutions of the differential variational inequality (DVI) describing the time evolution of rigid bodies with collision, contact, and friction. In this chapter the DVI algorithm [103] is extended to model frictional contact/impact in ANCF-based flexible multibody dynamics. A detailed formulation and implementation of the extended DVI algorithm is discussed herein. The flexible-flexible and flexible-rigid contact/impact cases were studied and validated using the CCFM approach (penalty method) discussed in Chapter 5.

6.1 Theoretical Background of DVI

Classical approaches based on the time integration of ODEs or DAEs enforce unilateral constraints and friction using the regularization approaches. Compared to these approaches, the DVI approach has recently gained widespread acceptance in the computational dynamics community. Since the pioneering work by J. J. Moreau [104] on measure differential inclusions, many authors have contributed to the topic of nonsmooth dynamics; see, for instance, [12, 105-107]. DVI can be considered as a special case of measure differential inclusion [108]. The theory of DVI provides a generalization of DAEs with discontinuities and a set-valued right hand side.
Therefore, the DVI approach can be used to simulate mechanical models that are subject to nonsmooth events and set-valued constraints, such as those caused by sticking and sliding friction. Additionally, they can deal with nonsmooth phenomena directly, resulting in greater efficiency, better numerical stability, and enhanced robustness [103, 109].

Two main families of solvers spawn from the DVI formulation [103]: The first one that leads to an acceleration-force complementarity problem [110, 111] and the second that generates velocity-impulse, complementarity-based time-stepping methods [107, 112]. The latter case results in schemes convergent to a vector measure differential inclusion, so named because it operates on vector measures or distributions [106]. It has the advantage that it can solve a class of problems with Coulomb friction that would be unsolvable in an acceleration-force context, as the Painlevé paradox [113]. The velocity-impulse, complementarity-based time-stepping method shown in [103] is used herein to solve the DVI problem.

Before going into details of the DVI formulation it is important to briefly review the contact kinematics. The contact problems can be classified into the following two major types [79].

*Impact Problems (Intermittant Contact)* is characterized by contact for short periods of time. It is also known as impulsive contact. Two geometries approach each other, undergo a collision, and separate as a result of the contact. The collision results in the generation of an impulse that affects the momentum of the colliding bodies. Energy loss during the collision is usually modeled as a damping force that is specified with a damping coefficient (penalty method) or a coefficient of restitution.

*Contact Problems (Persistent Contact)* is characterized by contact for relatively long periods of time. External forces acting between the two bodies serve to maintain continuous
contact. Two bodies are said to be in persistent contact when the separation velocity, after a collision event, is close to zero. The bodies, therefore, cannot separate after the contact. This type of problems can be viewed as an inelastic collision of the contacting body with zero coefficient of restitution.

The complementarity conditions [114] in the DVI approach are different for the impact and contact problems; these conditions are discussed in the following subsections.

6.2 **Differential Variational Inequality (DVI)**

6.2.1 **Extension of DVI for Flexible Beams**

To model the frictional contact between slender beams, a spherical decomposition approach introduced in Chapter 5 is used for the contact discretization of the colliding beams. In the spherical decomposition approach, each flexible beam can be considered as a chain of spheres that overlap and are distributed equidistantly along the axis of the gradient deficient ANCF beam elements. At each time step, the collision detection between the spheres in all the beams is performed and a signed distance, $\phi$ and a contact normal, $n$ is determined for each active contact. Note that the contact normal is along a line joining the centers of colliding spheres. The contacts are considered as active, if gap function $\phi \leq 0$. The set of all active contacts will be referred to by $G_A$.

6.2.2 **Generalized Contact Coordinates**

Consider an active contact, $i \in G_A$, between two spheres which belong to two different beam elements $b_k$ and $b_l$. Let the velocities of these two spheres be $v_1$ and $v_2$. Let $n_i \in \mathbb{R}^3$ be
the normal at the contact pointing toward the exterior of the first sphere. Let $\mathbf{u}_i, \mathbf{w}_i \in \mathbb{R}^3$ be two vectors along the tangential directions in contact plane which are computed using Gram-Schmidt orthonormalization. Here $\mathbf{n}_i, \mathbf{u}_i, \mathbf{w}_i$ are mutually orthogonal vectors of unit length which form a contact coordinate system. Let $\mathbf{\hat{y}}_i = [\mathbf{\hat{y}}_{i,n}, \mathbf{\hat{y}}_{i,u}, \mathbf{\hat{y}}_{i,w}]^T$ be the reaction force in contact coordinates.

A normal relative velocity of two spheres in contact is given by

$$\mathbf{v}_{rel,n} = \mathbf{n}_i^T (\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{n}_i^T \mathbf{S}(x_1) \mathbf{\dot{e}}_{b_i} - \mathbf{n}_i^T \mathbf{S}(x_2) \mathbf{\dot{e}}_{b_i}$$

(1)

where $x_1, x_2$ are the spatial coordinates of colliding spheres, $\mathbf{S}$ is the shape function matrix, and $\mathbf{\dot{e}}_{b_i}, \mathbf{\dot{e}}_{b_i} \in \mathbb{R}^{12}$ are the vectors of nodal velocities of the elements $b_i$ and $b_i$, respectively. Let the vector of generalized coordinates be $\mathbf{q} = [\mathbf{q}_1^T ... \mathbf{q}_n^T]^T \in \mathbb{R}^p$. Here $n$ is the total number of bodies (beams) and $p$ is the total number of degrees of freedom present in the system. Let $\mathbf{\dot{q}} = [\mathbf{\dot{e}}_{b_1}^T ... \mathbf{\dot{e}}_{b_n}^T ...]^T \in \mathbb{R}^p$ be the vector of generalized velocities. Now Eq. (1) can be written as

$$\mathbf{v}_{rel,n} = \mathbf{D}_{i,n}^T \mathbf{\dot{q}}$$

(2)

where $\mathbf{D}_{i,n} = \left[0...0, \mathbf{n}_i^T \mathbf{S}(x_1), 0...0, -\mathbf{n}_i^T \mathbf{S}(x_2), 0...0\right]^T$ the generalized normal vector. Similarly the expressions for the generalized tangent vectors,

$$\mathbf{D}_{i,u} = \left[0...0, \mathbf{u}_i^T \mathbf{S}(x_1), 0...0, -\mathbf{u}_i^T \mathbf{S}(x_2), 0...0\right]^T, \mathbf{D}_{i,w} = \left[0...0, \mathbf{w}_i^T \mathbf{S}(x_1), 0...0, -\mathbf{w}_i^T \mathbf{S}(x_2), 0...0\right]^T$$

can be obtained. The matrix $\mathbf{D}_i = [\mathbf{D}_{i,n} \mathbf{D}_{i,u} \mathbf{D}_{i,w}]$ is a projection matrix which transforms contact coordinate system to the generalized coordinate system. In case of rigid bodies, the projection matrix $\mathbf{D}_i$ has a different expression which is given in [115].
6.2.3 The Complementarity Condition

If the two spheres are already in contact then the nonpenetration condition demands that the gap function $\phi_i(q(t))$ cannot be decreased further, i.e.,

$$\dot{\phi}_i(q(t)) \geq 0 .$$

Equivalently, in the time stepping scheme this condition can be written for the next time step $l + 1$ as

$$\dot{\phi}_i(q^{(l+1)}) = v_{rel,n}^{(l+1)} = D_{i,n}^T q_i^{(l+1)} \geq 0 .$$

If this condition is not satisfied the contact will generate a positive reaction force which will prevent further penetration. It should be noted that small penetrations may occur at contact if a static collision detection algorithm is used with large time steps. The nonpenetration condition so far only prevents further penetration after a collision has been detected; i.e., when the gap function $\phi_i \leq 0$. This restricts our choice of time step sizes as we have to ensure that the discrete time samples do not miss the contacts by too much and the penetrations do not become too large.

In order to allow a larger time step, first, the collision detection algorithms is modified such that the contact becomes active when $\phi_i \leq \varepsilon$ for some positive threshold value $\varepsilon$ [116], and second, instead of demanding the derivative of the gap function to be positive at the next time step $l + 1$, we demand that the gap function itself is positive at the next time step i.e. $\phi_i(q^{(l+1)}) \geq 0$ [117]. The distance at the next time step can be approximated using symplectic Euler integrator as

$$\phi_i(q^{(l+1)}) = \phi_i(q^{(l)}) + hD_{i,n}^T \dot{q}_i^{(l+1)}$$

and using this, the non-interpenetration condition becomes

$$\phi_i(q^{(l)}) + hD_{i,n}^T \dot{q}_i^{(l+1)} \geq 0 .$$
If this condition is satisfied, it means contact is broken and no normal reaction force is applied i.e. \( \hat{y}_{i,n} = 0 \). On the other hand if this condition is not satisfied, a normal reaction force \( \hat{y}_{i,n} \geq 0 \) is applied at the contact to prevent the penetration. The reaction force in normal direction \( \hat{y}_{i,n} \) must be positive as a contact without adhesion should not be able to generate attractive forces. Hence, the complementarity condition becomes

\[
\phi_i(q^{(l)}) + hD_{i,n}^r \dot{q}^{(l+1)} \geq 0, \quad \hat{y}_{i,n} \geq 0, \quad \left( \phi_i(q^{(l)}) + hD_{i,n}^r \dot{q}^{(l+1)} \right) \hat{y}_{i,n} = 0.
\]

After dividing this by the time step \( h \), the velocity-impulse based complementarily condition is obtained as

\[
\frac{\phi_i(q^{(l)})}{h} + D_{i,n}^r \dot{q}^{(l+1)} = 0, \quad y_{i,n} \geq 0, \quad \left( \frac{\phi_i(q^{(l)})}{h} + D_{i,n}^r \dot{q}^{(l+1)} \right) y_{i,n} = 0,
\]

where \( y_{i,n} = h\hat{y}_{i,n} \) is the contact normal reaction impulse. The first term \( \frac{\phi_i(q^{(l)})}{h} \) achieves constraint stabilization [117] and the exclusion of this term leads to the unilateral constraint satisfaction only at velocity level.

The physical consequence of the complementarity condition in Eq. (3) is that when there is an active contact (\( \phi_i \leq \epsilon \)) the normal reaction impulse is applied such that the gap between contacting bodies will be closed at next time step i.e. \( \phi_i(q^{(l+1)}) = 0 \).

Note that the above complementarity condition is valid for the persistent contact problem only i.e. for purely inelastic impact. For the impact problems with nonzero coefficient of restitution (\( c_r \)) the complementarity condition is modified to \(-c_r \phi_i^{(l)} \leq \phi_i^{(l+1)} \perp y_{i,n} \geq 0 \). This condition is also called the impact-level Signorini-Fischera condition [112, 118]. Here, and in
the rest of this work we use the symbol $u \perp v$ to denote the fact that $u^T v = 0$. With $\dot{\phi}_i^{(-)} = D^T_{i,n} \dot{q}^{(-)}$ and $\dot{\phi}_i^{(+)} = D^T_{i,n} \dot{q}^{(+)}$, complementarity condition can be simplified as

$$\left(D^T_{i,n} \dot{q}^{(+)} + c_i D^T_{i,n} \dot{q}^{(-)}\right) \leq 0, \quad \gamma_{i,n} \geq 0, \quad \left(D^T_{i,n} \dot{q}^{(+)} + c_i D^T_{i,n} \dot{q}^{(-)}\right) \gamma_{i,n} = 0 . \quad (4)$$

Here $\dot{q}^{(-)}$ and $\dot{q}^{(+)}$ are the generalized velocities before and after impact, respectively.

### 6.2.4 DVI Formulation for Frictionless Contact

A frictionless contact case is considered first. The reaction force for such a contact will be of the form: $\gamma_i = [\gamma_{i,n}, 0, 0]^T$. In the discrete time stepping scheme the equation of motion of the flexible multibody system (with no bilateral constraints) can be written as

$$M(\ddot{q}^{(i+1)} - \ddot{q}^{(i)}) = h f^{(l)} - h f^{(l)} + \sum_{i} D_i \gamma_i , \quad (5)$$

where $\gamma_i = h \gamma_i$ is the contact reaction impulse. Equations (3) and (5) formulate a discretized DVI problem for the frictionless contact case. Next, we will solve this problem for just one active contact $i$. For this active contact, the transformation matrix from the contact coordinates to the generalized coordinates of a system is defined as $D_i = [D_{k,n} \quad D_{k,n} \quad D_{l,w}]$. Equation (5) becomes

$$M(\ddot{q}^{(i+1)} - \ddot{q}^{(i)}) = h (f^{(l)} - f^{(l)}) + D_{i,n} \gamma_{i,n} . \quad (6)$$

Note that the unknowns for the next time step are the reaction normal impulse $\gamma_{i,n}$ and the generalized velocities $\dot{q}^{(i+1)}$. Rearranging Eq. (6) and multiplying with $D^T_{i,n} M^{-1}$, we get

$$D^T_{i,n} \ddot{q}^{(i+1)} = D^T_{i,n} \left(\ddot{q}^{(i)} + M^{-1} \left[h (f^{(l)} - f^{(l)}) + D_{i,n} \gamma_{i,n}\right]\right) . \quad (7)$$

By adding $\frac{\phi_i(q^{(i)})}{h}$ to Eq. (7), we get
\[
\frac{\phi_i(q^{(i)})}{h} + D_{i,n}^T \dot{q}^{(i+1)} = N \gamma_{i,n} + r_i ,
\]

where

\[
N = D_{i,n}^T M^{-1} D_{i,n} \in \mathbb{R} ,
\]

\[
r_i = \frac{\phi_i(q^{(i)})}{h} + D_{i,n}^T (q^{(i)} + M^{-1} [h(f^{(i)}_{ext} - f^{(i)}_{int})])
\]

Now with only one active contact, the discretized DVI scheme that gives the generalized coordinates and the velocities at the next time becomes

\[
\dot{q}^{(i+1)} = \dot{q}^{(i)} + \dot{M}^{-1} [h(f^{(i)}_{ext} - f^{(i)}_{int}) + D_{i,n} \gamma_{i,n}]
\]

\[
\gamma_{i,n} \geq 0, \ N \gamma_{i,n} + r_i \geq 0, \ (N \gamma_{i,n} + r_i) \gamma_{i,n} = 0
\]

\[
q^{(i+1)} = q^{(i)} + h \dot{q}^{(i+1)}.
\]

Note that \(N \gamma_{i,n} + r_i\) is an affine linear function of \(\gamma_{i,n}\), and the complementarity condition in Eq. (9) is a linear complementarity problem (LCP). The LCP can be solved using standard iterative algorithm like Lemke’s method [119].

### 6.2.5 DVI Formulation for Frictionless Impact

A frictionless impact case with nonzero coefficient of restitution \((c_r)\) is considered here.

The reaction impulse for this impact will be of the form: \(\gamma_i = [\gamma_{i,n}, 0, 0]^T\). Let’s assume that there is just one active contact \(i\) at a given time step \(I\). We consider \(q^{(-)} = q^{(i)}\) as a velocity before impact. The velocity after impact, \(q^{(+)}\) is calculated as

\[
q^{(+)} = q^{(-)} + M^{-1} D_{i,n}^T \gamma_{i,n} .
\]

The complementarity condition given in Eq. (4) can be written as
So for the frictionless impact case with one active contact, we solve the same DVI problem given in Eq.(9); however, the expression for \( r_i \) in this case is: \( r_i = (1 + c_r)D_{i,n}^T \dot{q}^{(i)} \).

6.2.6 DVI Formulation for Frictional Contact

In this section we will consider a frictional contact case. The reaction force for such a contact will be of the form: \( \dot{y}_i = [\dot{y}_{i,n}, \dot{y}_{i,u}, \dot{y}_{i,w}]^T \), which leads to normal component of the force \( F_{i,N} = \dot{y}_{i,n} n_i \), and tangential component of the force \( F_{i,T} = \dot{y}_{i,u} u_i + \dot{y}_{i,w} w_i \). The friction model used here is the Coulomb model, which leads to frictional conic constraints regarded as an extension of complementarity models discussed in [106, 120]. The Coulomb model consists of the following constraints:

\[
\begin{align*}
\dot{y}_{i,n} &\geq 0, \quad \phi_i(q) \geq 0, \quad \phi_i(q)\dot{y}_{i,n} = 0 \\
\mu_i\dot{y}_{i,n} &\geq \sqrt{\dot{y}_{i,u}^2 + \dot{y}_{i,w}^2}, \quad \|v_{i,T}\|\left(\mu_i\dot{y}_{i,n} - \sqrt{\dot{y}_{i,u}^2 + \dot{y}_{i,w}^2}\right) = 0 \\
\langle F_{i,T}, v_{i,T} \rangle &= -\|F_{i,T}\|\|v_{i,T}\|
\end{align*}
\]

where \( v_{i,T} \) is the relative tangential velocity at contact \( i \). The effect of the friction over the dynamics of the system is defined by the friction coefficient \( \mu \in \mathbb{R}^+ \), which typically has a value between 0 and 1 for most materials. The first part of the constraint can be restated as

\[
F_i = F_{i,N} + F_{i,T} = \dot{y}_{i,n} n_i + \dot{y}_{i,u} u_i + \dot{y}_{i,w} w_i \in \kappa,
\]

where \( \kappa \) is a cone in three dimensions, whose slope is \( \arctan(\mu) \). The constraint \( \langle F_{i,T}, v_{i,T} \rangle = -\|F_{i,T}\|\|v_{i,T}\| \) requires that the tangential force to be opposite to the tangential
velocity. An equivalent convenient way of expressing this constraint is by using the maximum dissipation principle 

\[
\left( \gamma_{i,u}, \gamma_{i,w} \right) = \arg \min_{\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \leq \mu \gamma_{i,n}} \left[ (\gamma_{i,u} \mathbf{D}_{i,u} + \gamma_{i,w} \mathbf{D}_{i,w})^T \hat{\mathbf{q}} \right]
\]

[104, 106]. For this minimization problem, the first-order necessary KKT conditions are equivalent to the Coulomb model above [103]. Effectively, the condition in this equation states that the friction force is such that, given a tangential velocity and a normal, the power dissipated is maximized. In the reaction ‘impulse’ format the maximum dissipation principle is given as

\[
\left( \gamma_{i,u}, \gamma_{i,w} \right) = \arg \min_{\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \leq \mu \gamma_{i,n}} \left[ (\gamma_{i,u} \mathbf{D}_{i,u} + \gamma_{i,w} \mathbf{D}_{i,w})^T \hat{\mathbf{q}} \right]
\]

(14)

The last relation is obtained from the identities \( \mathbf{u}_i^T \mathbf{v}_{i,t} = \mathbf{D}_{i,u}^T \hat{\mathbf{q}} \) and \( \mathbf{w}_i^T \mathbf{v}_{i,t} = \mathbf{D}_{i,w}^T \hat{\mathbf{q}} \). Now Eq. (3), Eq. (5) and Eq. (14) formulate a discretized DVI problem for the frictional contact case as

\[
M(\mathbf{q}^{(i+1)} - \mathbf{q}^{(i)}) = h\mathbf{f}_{ext}^{(i)} - h\mathbf{f}_{int}^{(i)} + \sum_{i \in G_d} \left( \gamma_{i,n} \mathbf{D}_{i,n} + \gamma_{i,u} \mathbf{D}_{i,u} + \gamma_{i,w} \mathbf{D}_{i,w} \right)
\]

\[
i \in G_A : 0 \leq \frac{\phi_i(\mathbf{q}^{(i)})}{h} + \mathbf{D}_{i,n}^T \mathbf{q}^{(i+1)} \perp \gamma_{i,n} \geq 0,
\]

\[
\left( \gamma_{i,u}, \gamma_{i,w} \right) = \arg \min_{\sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \leq \mu \gamma_{i,n}} \left[ (\gamma_{i,u} \mathbf{D}_{i,u} + \gamma_{i,w} \mathbf{D}_{i,w})^T \hat{\mathbf{q}}^{(i+1)} \right]
\]

\[
\mathbf{q}^{(i+1)} = \mathbf{q}^{(i)} + h \hat{\mathbf{q}}^{(i+1)}
\]

Several numerical methods can be used to solve Eq. (15). For instance, one can approximate the Coulomb friction cones in 3D as faceted pyramids, thus leading to a LCP that can be solved using pivoting methods or simplex methods. However, these methods usually require a large computational overhead and their complexity class is in the worst case exponential [121]. Therefore, in [122] it was demonstrated that the problem can be cast as a monotone optimization problem by introducing a relaxation to the complementarity constraints in Eq. (3) as
Here \( \mu_i \sqrt{(D_{i,u}^T \dot{q}^{(l+1)})^2 + (D_{i,w}^T \dot{q}^{(l+1)})^2} \) is a convex relaxation term which is equal to \( \mu_i \|v_{i,T}\| \). It is evident that the effect of this term will be negligible in either low friction or low tangential speed case. Interestingly, the combination of this term and a stabilization term \( \frac{\phi}{h} \) has the effect of creating a contact boundary layer between the bodies, of width equal to \( \mu_i \|v_{i,T}\| h \), which compensates for the relaxation term [103]. Additionally, it has been shown in [123] that the solution of the modified scheme in Eq. (16) will still approach the solution of the original problem in Eq. (15) as \( h \to 0 \).

Now we will show how this modified scheme will lead to a Cone Complementarity Problem. By defining a vector \( g_i = \left[ \frac{\phi(q^{(l)})}{h} + D_{i,u}^T \dot{q}^{(l+1)}, D_{i,w}^T \dot{q}^{(l+1)}, (D_{i,u}^T \dot{q}^{(l+1)})^T \right] \), Eq. (16) can be written as

\[
\begin{align*}
g_i \in FC_i := \left\{ (x, y, z) \in \mathbb{R}^3 \mid x \geq \mu_i \sqrt{y^2 + z^2} \right\}
\end{align*}
\]  

(17)

We have \( g_i = D_i^T \dot{q}^{(l+1)} + b_i \), where \( b_i = \left[ \frac{\phi(q^{(l)})}{h}, 0, 0 \right]^T \). Substituting for \( \dot{q}^{(l+1)} \) from Eq. (5), we get

\[
g_i = N_i \gamma_i + r_i,
\]  

(18)

where \( N_i = D_i^T M^{-1} D_i \in \mathbb{R}^{3 \times 3} \) and \( r_i = D_i^T \left( \dot{q}^{(l)} + M^{-1} \left[ h (f^{(l)}_{ext} - f^{(l)}_{int}) \right] \right) + b_i \). Now, the reaction impulse of contact satisfies the maximum dissipation principle and thus is restricted to the Coulomb friction cone as
where $ FC_i $ is a convex cone. It should be noted that $ FC_i^\circ $ is a negative polar cone of $ FC_i $; that is $ g_i \in FC_i^\circ $ and $ \gamma_i \in FC_i $ implies that $ g_i^T \gamma_i \geq 0 $. Here $ FC_i^\circ $, the polar cone of a given cone $ FC_i $, is defined as $ FC_i^\circ := \{ x \in \mathbb{R}^3 \mid \langle x, y \rangle \leq 0, \forall y \in FC_i \} $.

From the KKT optimality conditions for the equilibrium constraint in Eq. (14) and using the complementarity condition in Eq. (16), it can be proved that $ g_i^T \gamma_i = 0 $ i.e. $ g_i \perp \gamma_i $; see [103, 124] for details. Using this relation, the complementarity constraints in Eq. (15) become a Cone Complementarity Problem (CCP) as

$$ \gamma_i \in FC_i \perp N_i \gamma_i + r_i \in -FC_i^\circ. $$

Eq. (20) is the CCP for one active contact. For more than one active contact in $ G_n $, the above relation can be expressed in compact form. To do this we define the vectors $ r \in \mathbb{R}^m $, $ \gamma \in \mathbb{R}^m $, $ g \in \mathbb{R}^m $, $ b \in \mathbb{R}^m $ as

$$ r = [r_1, r_2, \ldots, r_m]^T, $$

$$ \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_m]^T, $$

$$ g = [g_1, g_2, \ldots, g_m]^T, $$

$$ b = \left[ \frac{\phi_1}{h}, 0, 0, \frac{\phi_2}{h}, 0, 0, \ldots, \frac{\phi_m}{h}, 0, 0 \right]^T, $$

where $ m $ is the number of active contacts. The projection matrix can be assembled as $ D = [D_1, D_2, \ldots, D_m] $. Using these notations and also premultiplying by $ M^{-1} $ the first equation in Eq. (15), we get

$$ \bar{q}^{(i+1)} = M^{-1} D M^{-1} \bar{f} + f^{-1} \left[ h \left( \frac{\bar{f}_e - \bar{f}_i}{(i)} \right) \right]. $$
We have \( \mathbf{g} = \mathbf{D}^T \mathbf{q}^{(i+1)} + \mathbf{b} \). Substituting for \( \mathbf{q}^{(i+1)} \) using the expression above, we get

\[
\mathbf{g} = \mathbf{N} \mathbf{r}^+ ,
\]

where \( \mathbf{N} = \mathbf{D}^T \mathbf{M}^{-1} \mathbf{D} \in \mathbb{R}^{3m \times 3m} \) and \( \mathbf{r} = \mathbf{D}^T \left( \mathbf{q}^{(i)} + \mathbf{M}^{-1} \left[ h(\mathbf{f}_{\text{ext}}^{(i)} - \mathbf{f}_{\text{int}}^{(i)}) \right] \right) + \mathbf{b} \).

Using the properties of convex cone [103], we have \( \mathcal{Y} = \bigoplus_i \mathcal{F} \mathcal{C}_i \Rightarrow \mathcal{Y}^* = \bigoplus_i \mathcal{F} \mathcal{C}_i^* \) where \( \bigoplus \) denotes the Cartesian product. Thus, the cone complementarity problem (CCP) for the entire system can be expressed as

\[
\gamma \in \mathcal{Y} \perp \mathbf{N} \gamma + \mathbf{r} \in -\mathcal{Y}^* .
\]

### 6.2.7 Solution of the Cone Complementarity Problem

The CCP has been constructed from the solution of maximum dissipation principle. Note that it can be proved that Eq. (20) in fact represents the optimality condition of the following convex optimization problem with conic constraints [124]

\[
\min f(\gamma) = \frac{1}{2} \gamma^T \mathbf{N} \gamma + \mathbf{r}^T \gamma \quad \text{s.t.} \quad \gamma \in \mathcal{Y} .
\]

The matrix \( \mathbf{N} \) has been defined as \( \mathbf{N} = \mathbf{D}^T \mathbf{M}^{-1} \mathbf{D} \). Here the mass matrix \( \mathbf{M} \) is constant and symmetric. Generally in the rigid body contact problems, the projection matrix \( \mathbf{D} \) is rank deficient which makes \( \mathbf{N} \) a positive semidefinite matrix. In granular material simulation with possibly a millions of active contact, \( \mathbf{N} \) becomes very large. In such case, iterative solvers are preferred over direct solvers to solve this optimization problem. Among these iterative solvers the Jacobi and Gauss-Seidel methods as described in [103, 122, 125] are the most commonly used methods in the literature. The matrix free structure of these algorithms and their parallel implementation on GPU is discussed in [126]. More recently a set of methods are developed in
[124] which demonstrate significant improvements over existing methods in terms of accuracy and computation time. These methods include: the Gradient-Projected Minimum Residual Method (GPMinres) for frictionless problems, Accelerated Projected Gradient Descent Method (APGD), and the Primal-Dual Interior Point Method (PD) for frictional problems. The theoretical aspects and the implementation of these methods are thoroughly discussed in [124].

Here we use the projected Gauss-Jacobi algorithm discussed in [103, 124] to solve the cone constrained optimization problem stated in Eq. (23). A basic idea behind this algorithm is to solve a linear system \(N\gamma + r = 0\) iteratively and after each iteration the solution \(\gamma^k\) is projected onto feasible set \(Y\). The details of this algorithm are given below.

**Projected Gauss-Jacobi Algorithm**

The original Gauss Jacobi algorithm is one of the matrix-splitting methods to solve a linear system. In this algorithm a linear system is solved using the following iterative scheme

\[
N\gamma + r = (A + B)\gamma + r = 0 \\
\Rightarrow \gamma^{(k+1)} = -A^{-1}(B\gamma^{(k)} + r) .
\]

Here \(N = (A + B)\) and \(A\) is chosen to be diagonal so that its inverse can be computed easily. A weighted form of this iterative scheme is given as

\[
\gamma^{(k+1)} = (1 - \omega)\gamma^{(k)} - \omega A^{-1}(B\gamma^{(k)} + r) \\
= (1 - \omega)\gamma^{(k)} - \omega A^{-1}((A + B)\gamma^{(k)} - A\gamma^{(k)} + r) \\
= \gamma^{(k)} - \omega A^{-1}(N\gamma^{(k)} + r) ,
\]

where a parameter \(\omega > 0\) controls the convergence. This iterative scheme is extended to solve a CCP [103] using the following algorithm:

**Algorithm: Projected Gauss Jacobi** \((N, \eta, N_{max}, r, \gamma^{(0)})\)
for $k := 0$ to $N_{\text{max}}$
\[
\gamma^{(k+1)} = \Pi_Y (\gamma^{(k)} - \omega G(N\gamma^{(k)} + r))
\]
\[
\gamma^{(k+1)} = \lambda \gamma^{(k+1)} + (1 - \lambda) \gamma^{(k)}
\]
if $|\gamma^{(k+1)} - \gamma^{(k)}| < \tau$
\[
\text{break}
\]
endfor
return $\gamma^{k+1}$

In this algorithm the matrix $G$ is block diagonal. Each $3 \times 3$ block $G_i$ is given as
\[
G_i = \frac{1}{g_i} I, \text{ where for each active contact } i, \quad g_i = \frac{\text{tr}(D_i^T M_i^{-1} D_i)}{3}. \]
This choice was inspired by the Gauss-Jacobi idea of using the inverse of the diagonal of the system matrix, with the inclusion of averaging to improve performance for systems with large mass ratios [103]. Further note that maximum number of iterations $N_{\text{max}}$ and tolerance $\tau$ are used to check the convergence. The initial guess for vector of unknowns is taken as $\gamma^{(0)} = 0$. The iterative scheme in this algorithm was proved to converge under mild assumptions that can be met by a suitable choice of relaxation parameter $\omega$ [12]. The values $\omega = 1$ and $\lambda = 1$ are used in this work.

The projection $\Pi_Y$ projects every $\gamma_i$ into the Friction Cone $FC_i$ and is defined as
\[
\Pi_Y = [\Pi_{FC_i}, \Pi_{FC_g}, \ldots].
\]
The projection operator $\Pi_{FC_i} : \mathbb{R}^3 \to \mathbb{R}^3$ projects a vector onto the nearest point in the Friction Cone $FC_i$, see Figure 54. If a vector $\gamma_i$ is already inside the friction cone, it stays unchanged. If a vector is in the polar cone, it is projected to the origin of friction cone i.e. all attractive forces will be projected to zero.
If a vector \( \gamma_i \) is neither in the friction cone nor in the polar cone, the projection operator will map the part of \( \gamma_i \) that is orthogonal to the friction cone to zero i.e. it is projected onto the surface of the friction cone as shown in Figure 54. The closed form expressions for the projection derived in \([103, 124]\) are summarized here:

\[
\Pi_{FC_i}(\gamma_i) = \begin{cases} 
\gamma_i & \text{if } \gamma_{i,d} \leq \mu_i \gamma_{i,n} \\
0 & \text{if } \gamma_{i,d} \leq -\frac{1}{\mu_i} \gamma_{i,n} \\
\Pi_n, \mu_i \frac{\gamma_{i,u}}{\gamma_{i,d}} \Pi_n, \mu_i \frac{\gamma_{i,w}}{\gamma_{i,d}} \Pi_n & \text{if } \gamma_{i,d} > \mu_i \gamma_{i,n}, \gamma_{i,d} > -\frac{1}{\mu_i} \gamma_{i,n}
\end{cases}
\]  

(26)

where \( \Pi_n = \frac{\gamma_{i,d} \mu_i + \gamma_{i,n}}{\mu_i^2 + 1} \) and \( \gamma_{i,d} = \sqrt{\gamma_{i,u}^2 + \gamma_{i,w}^2} \) is the magnitude of tangent reaction impulse.
6.3 DVI Implementation

We have seen how the complementarity condition of the Coulomb friction model leads to the CCP. Using this, the DVI time stepping scheme given in Eq. (15) becomes

\[
M(q^{(i+1)} - \dot{q}^{(i)}) = h\dot{f}^{(i)}_{\text{ext}} - h\dot{f}^{(i)}_{\text{int}} + \sum_{i \in G_A} D_i \gamma_i
\]

\[
i \in G_A : \gamma_i \leq FC_i \perp N_i \gamma_i + r_i \leq FC_i^o
\]

\[
q^{(i+1)} = q^{(i)} + h\dot{q}^{(i+1)}.
\]

This is a key equation as it recapitulates the DVI formulation discussed in this chapter. The solution steps of the DVI problem are summarized in the following pseudo-code.

**Algorithm: DVI**

1. Set \( t = 0 \), step counter \( l = 0 \), provide initial values for \( q^{(i)} \) and \( \dot{q}^{(i)} \).

2. For each beam, compute \( f_{\text{int}}^{(i)} \) and \( f_{\text{ext}}^{(i)} \) (i.e. gravity force in this model, which is constant; computed just once at the beginning).

3. Perform collision detection between beams, obtaining a set of all active contacts \( G_A \). For each contact \( i \), compute \( D_i = [D_{i,v} \quad D_{i,w}] \), and residual \( \phi_i(q^{(i)}) \), which also provides \( b_i = \left[ \frac{\phi_i(q^{(i)})}{h}, 0, 0 \right]^T \).

4. Use ‘Projected Gauss Jacobi Algorithm’ to solve the cone complementarity problem and obtain unknown impulse \( \gamma_i \).

5. Compute velocities using \( q^{(i+1)} = q^{(i)} + M^{-1} \left[ h(f_{\text{ext}}^{(i)} - f_{\text{int}}^{(i)}) + \sum_{i \in G_A} D_i \gamma_i \right] \).
6. Update positions using 
\[ q^{(l+1)} = q^{(l)} + h \dot{q}^{(l+1)}. \]

7. Increment \( t := t + h \), \( l := l + 1 \), and repeat from step 2 until \( t > t_{\text{end}} \).

6.4 Numerical Experiments and Results

Several numerical experiments were carried out in order to validate the DVI approach for contact/impact modeling with the ANCF beams. In these numerical experiments the beam-to-rigid body (Model-1) and the beam-to-beam (Model-2) contact/impact problems are considered. The model details and simulation parameters are given in Table 4. The schematic of Model-1 is shown in Figure 55. In this model, a rigid horizontal cylinder with radius of 0.01 m is fixed at (0.5, -0.1). For all the numerical experiments the flexible pendulums/beams are assumed to be in horizontal configuration with no initial velocity.

Table 4: Model and simulation parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model-1</th>
<th>Model-2</th>
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</thead>
<tbody>
<tr>
<td>Contact type</td>
<td>Flex-rigid</td>
<td>Flex-flex</td>
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<td>Length (m)</td>
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<td>1</td>
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<tr>
<td>Cross-section Area(m²)</td>
<td>( \pi \times 0.01^2 )</td>
<td>( \pi \times 0.01^2 )</td>
</tr>
<tr>
<td>Material Density (kg/m³)</td>
<td>7200</td>
<td>7200</td>
</tr>
<tr>
<td>Modulus of elasticity (Pa)</td>
<td>2.0E7</td>
<td>2.0E9</td>
</tr>
<tr>
<td>Second moment of area (m⁴)</td>
<td>7.85E-9</td>
<td>7.85E-9</td>
</tr>
<tr>
<td>Tip Mass (kg)</td>
<td>1</td>
<td>--</td>
</tr>
<tr>
<td>External Force</td>
<td>Gravity + Contact</td>
<td>Gravity + Contact</td>
</tr>
<tr>
<td>Step-Size</td>
<td>1.0E-5</td>
<td>1.0E-5</td>
</tr>
</tbody>
</table>
6.4.1 Energy Analysis

The energy analysis is the first step in the validation of DVI approach. In the first set of numerical experiments the effect of coefficient of restitution \( (c_r) \) on energy preservation characteristics is studied. It should be noted that the complementarity condition for the contact case assumes \( c_r = 0 \), whereas in the impact case, one can choose any value of \( c_r \) between 0 to 1 (i.e. \( 0 \leq c_r \leq 1 \)). The energy analysis of both models for the frictionless contact case \( (c_r = 0) \) is shown in Figure 56 and Figure 57. The total energy of a model is comprised of the potential, kinetic, and strain energy of the ANCF beams. It can be seen from these figures that there is an energy loss at each contact occurrence due to its inelastic nature. Thus the total energy of the system decreases for the \( c_r = 0 \) case. In case of Model-1, the beam remains in contact with the rigid cylinder due to inertia of attached tip-mass. The energy analysis results of this model for longer simulation period is shown in Figure 58. The continuous loss in the kinetic energy is observed in this case.
Figure 56: Energy analysis - frictionless perfectly inelastic contact (Model-1)

Figure 57: Energy analysis - frictionless perfectly inelastic contact (Model-2)
Figure 58: The energy loss due to perfectly inelastic contact (Model-1)

The energy analysis of both models for the frictionless impact case with $c_r = 1$ is shown in Figure 59 and Figure 60. This is a case of perfectly elastic impact with no friction, so there should not be any energy loss. It can be seen from these figures that the total energy is conserved in this case, as expected. There is no energy loss observed even for the longer simulation time, see Figure 60. These results also validate the fact that the integration scheme used in these simulations is symplectic (energy preserving) in nature.
Figure 59: Energy analysis - frictionless elastic impact, beam-to-rigid cylinder

Figure 60: Energy analysis - frictionless elastic impact, beam-to-beam
6.4.2 Validation of DVI approach

The simulation results of the DVI approach are validated against the results of FEAP, ABAQUS and the penalty method. In the first set of numerical experiments, a frictional contact with $\mu = 0.3$ is considered in Model-1. The comparison of simulation results of the DVI approach and the penalty method is shown in Figure 61 and Figure 62. It should be noted that the Hertzian contact parameters as discussed in Chapter 5 are used in the penalty method. These figures show that the simulation results obtained using both the approaches match well. The average CPU time for 2 sec long simulation in Matlab are reported in Table 5. In the penalty method the step size is limited to smaller values due to the instabilities induced by high stiffness of contact spring, whereas the DVI approach does not suffer from these stability issues.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Step Size (sec)</th>
<th>CPU Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVI Method</td>
<td>1.E-3</td>
<td>8.52</td>
</tr>
<tr>
<td></td>
<td>1.E-4</td>
<td>74.60</td>
</tr>
<tr>
<td></td>
<td>1.E-5</td>
<td>781.89</td>
</tr>
<tr>
<td>Penalty Method</td>
<td>1.E-3</td>
<td>Unstable</td>
</tr>
<tr>
<td></td>
<td>1.E-4</td>
<td>72.88</td>
</tr>
<tr>
<td></td>
<td>1.E-5</td>
<td>765.30</td>
</tr>
</tbody>
</table>
Figure 61: X coordinate of beam tip (DVI-Penalty method comparison)

Figure 62: Y coordinate of beam tip (DVI-Penalty method comparison)
These results indicate that for a general mechanical model with contact/impact, the DVI approach is more stable than the penalty method. However, note that unlike the rigid body case, the DVI step-size is limited here due to high frequency content of flexible body. For the same step-size, DVI simulation is slower than the penalty method as expected. In general, the DVI approach can be more efficient, if it allows a step-size one order larger than that used in the penalty method.

In the second set of numerical experiments, the ANCF and DVI implementation is validated against FEAP and ABAQUS. In this experiment a frictionless impact case with $c_r = 1$ is considered in Model-2. In ABAQUS the explicit general hard contact algorithm with the default stiffness value is used to compute normal contact force. In FEAP mortar contact elements are used to model contact. In DVI approach each beam is modeled using 8 ANCF beam elements and 100 spheres along beam centerline. In ABAQUS each beam is modeled using 100 BEAM31 (2-node 3D) elements. In FEAP 100 2-node 3D beam elements (shear on) are used to model each beam. The comparison of the simulation results is shown in Figure 63 and Figure 64. The results from ANCF and DVI approach are quite close to the ABAQUS and FEAP results. Since the results are within an acceptable range of accuracy, the DVI approach can be considered as a robust and accurate alternative to the various contact algorithms used in the commercial softwares.
Figure 63: X displacement of beam tip (DVI, FEAP, and ABAQUS results)

Figure 64: Y displacement of beam tip (DVI, FEAP, and ABAQUS results)
6.5 **Concluding Remarks**

A methodology to combine ANCF with a DVI approach to model frictional contact/impact between flexible beams is outlined in this chapter. A detailed derivation of DVI approach for both frictionless and frictional contact/impact is provided along with implementation details. The complementarity condition in the traditional DVI approach is only valid for purely inelastic impact. A new complementarity condition is introduced which includes the coefficient of restitution to model a generalized impact problem.

The numerical experiments confirm that total energy of the system is conserved for the frictionless impact case due to introduction of new complementarily condition. The results of ANCF+DVI for the flex-flex impact case are validated against FEAP and Abaqus. The results for the flex-rigid contact case also match well with those obtained using a penalty method. Moreover, the DVI approach provides more stable solution compared to the penalty method.

Though the penalty method is straightforward to implement and expeditious, choosing the correct values of penalty parameters is a challenging task. Note that in the penalty method, too low spring stiffness may produce large, physically inadmissible penetrations, and too high spring stiffness may lead to numerical instability during the solution process. The DVI method does not suffer from these drawbacks as a non-penetration criteria is satisfied exactly using the appropriate complementarily condition.

Moreover, in the DVI approach the stick-slip phenomenon of the Coulomb friction model is modeled exactly using the principles of non-smooth dynamics. The penalty method cannot model the exact stick-slip phenomenon as it uses the regularization techniques.
7 Demonstration of Technology

The purpose of this chapter is to further demonstrate the capabilities of ANCF in the flexible multibody dynamics framework. The models considered for demonstration purposes are representative of real-world engineering problems. These models can be easily adapted in the system design process with knowledge of realistic parameters.

Author has recently implemented the ANCF technology in MSC ADAMS, a commercial MBS software [79]. This implementation enables us to solve the specific engineering problems using large multibody dynamics models. In this chapter, an overview of the ANCF implementation in Adams solver is presented first. Then, the details of ANCF models in ADAMS are discussed and their simulation results are shown.

7.1 ANCF Implementation in Adams

The ADAMS C++ solver is a powerful numerical analysis program that automatically solves the equations governing kinematic, static, quasi-static, and dynamic system simulations. The ADAMS solver is the solution kernel for all ADAMS products which facilitates building, testing, and refining virtual prototypes of mechanical systems[79]. The current Solver has a structured modular design which uses object oriented features of C++.

The ADAMS solver is mainly comprised of two layers: ‘As’ and ‘API’ as shown in Figure 65. The As layer is the main layer where all the solver functionality exists. Each class in the API layer wraps a private As object. The wrapped object is public but cannot be referenced
by the driver code. The AMD/Asol layer serves as interface or communicator between ADAMS solver and other User-Interface (UI) or CAD softwares.
The hierarchy of main modeling objects in the As layer is shown in Figure 66. A typical ADAMS model consists of bodies (Inertial object) connected to each other by forces (Force object) and joints (Constraint object), which are collectively referred to as connectors. Connectors attach to bodies at marker (Marker object) locations. The characteristics of a force may be described using function expressions and user subroutines that contain measures measuring marker kinematics such as position, orientation, velocity, etc. The user defined variables (Variable object) and the differential equations (Differential object) can be added to the equation set (System object) expressed in DAE format.

The ANCF beam/cable object has been added to these modeling objects to model highly deformable flexible bodies. The user can easily model the nonlinear beam by defining material properties, geometric properties and the node locations. Each node contains position, velocity and acceleration states, which are used in the DAE that governs motion of the system. These DAEs are solved using BDF or HHT integration methods similar to the ones discussed in this thesis.

The modeling of various mechanical joints with ANCF beams is a challenging task. For this purpose, a marker is attached to each node of a beam and the kinematic expressions of this marker are used in the bilateral constraints of a mechanical joint. Additional features of a beam such as variable cross-section, initially-curved, distributed loads, structural damping, etc. are also supported in this implementation.

A detailed discussion about ANCF implementation in the ADAMS solver is beyond the scope of this thesis.
7.2 ANCF Model Applications

7.2.1 A Satellite with Self Deployable Solar Panel

The first ANCF application considered here is a satellite model with self deployable solar panels, see Figure 67. The model consists of a satellite body, 2 solar panels and 4 flexible beams, i.e., inner and outer arms on either side, that are modeled using ANCF beam elements. The inner beams are connected to the satellite body through the revolute joints and the outer beams are connected to the solar panels through the fixed joints. Also, the inner and outer beams are connected to each other through revolute joints.

The satellite body has a mass $M = 2267.96$ kg, and moment of inertia $J_{xx} = J_{yy} = J_{zz} = 1404.67$ kg.m$^2$. Each solar panel is modeled as a rigid body with mass $M = 11$ kg, and moment of inertia $J_{xx} = J_{yy} = J_{zz} = 5.85E-5$ kg.m$^2$. The beams have circular cross-section with $A = 2.1871E-4$ m$^2$, area moment of inertia $I_{yy} = I_{zz} = 7.7253E-8$ m$^4$, Young’s modulus $E = 2.0E11$ N/m$^2$, and density $\rho = 7800$ kg/m$^3$. The lengths of inner and outer beams are $L_1 = 2.3$ m and $L_2 = 2.5$ m, respectively.

The motion drivers [79] are added to each revolute joint which produce a desired constrained motion of the beams as shown in Figure 68. A 30 sec long simulation of this model is carried out in ADAMS using HHT-I3 integrator. The snapshots from this simulation are shown in Figure 69.
Figure 67: A satellite model with ANCF beams

Figure 68: Constrained angular motion at revolute joints
Figure 69: Snapshots from satellite model simulation
7.2.2 An Elevator Cable Model

Another ANCF application considered here is the elevator traveling cable model. This model is used to simulate the motion of a long cable attached to an elevator car (cage). The industry-typical cable properties and elevator car velocity profiles are used in this model.

This model has an 80m long cable, modeled using 100 ANCF beam elements. The cable has a circular cross-section with $A = 3.1457E-4$ m$^2$, area moment of inertia $I_{yy} = I_{zz} = 7.86425E-9$ m$^4$, Young’s modulus $E = 2.0E7$ N/m$^2$, and density $\rho = 50.9$ kg/m$^3$. An elevator car is modeled as a rigid body with mass $M = 1.87E5$ kg, and moment of inertia $J_{xx} = J_{yy} = J_{zz} = 3.9E5$ kg.m$^2$.

The cable is in the horizontal configuration initially as shown in Figure 70. Then, the left wall is moved towards the right wall and an elevator car is moved to its highest point. This dynamic analysis is followed by a static analysis to get the proper initial position of the cable before the elevator descent starts, see Figure 70.

Figure 70: Elevator traveling cable model (initial position)
A 128 sec long simulation of this model is carried out in ADAMS using the HHT-I3 integrator. The time evolution of Y coordinate of the car CM is shown in Figure 71. Note that the first 90 sec of the simulation are used for the dynamic settling of the cable. The elevator car motion is generated using industry–typical velocity profiles.

For a good cable model, it is expected that the cable does not contact the wall while the elevator car descends. The X displacements of different points on cable shown in Figure 72 confirm that the cable does not contact the wall. Snapshots from the simulation of the model are shown in Figure 73. Note that if too stiff cable and too jerky velocity profile is used then the cable would contact the wall. This model can be effectively used in designing the elevator traveling cable systems.

Figure 71: An elevator car motion
Figure 72: X displacements of cable (at quarter span, midspan, and 3 quarters span) in elevator car descent motion

Figure 73: Snapshots from elevator cable model simulation
8 Conclusions

This thesis contributes to three active research thrusts in the area of MBS: (i) numerical integration methods to solve DAEs, (ii) modeling of geometrically nonlinear elastic bodies using ANCF, and (iii) modeling of the frictional contact/impact between flexible bodies. Specifically, six different implicit low order numerical integration methods to solve index-3 and stabilized index-2 DAEs are discussed. The formulation of gradient deficient ANCF beam element for straight as well as initially curved beam is investigated. Moreover, two widely used approaches for handling frictional contact in the MBS, namely the penalty method and the DVI method, are extended to solve the frictional contact/impact problem of ANCF flexible bodies.

The specific accomplishments of this work are summarized as follows:

• Numerical Integration Methods
  - The implementation of new integration methods like HHT-ADD and HHT-SI2 in Matlab, and their comparison with existing numerical integration methods such as BDF, Newmark, and HHT-I3, in terms of order of convergence, energy preservation, constraint satisfaction and efficiency.

• ANCF
  - The Matlab implementation of original ANCF, gradient deficient ANCF and initially curved ANCF beam within the index-3 DAE framework.
  - The locking problems of the original ANCF, the convergence analysis of gradient deficient ANCF, and the validation of ANCF against ABAQUS and FEAP are shown through various numerical experiments.

• ANCF + Contact/Impact using penalty method
- The development and implementation of penalty-based Hertzian contact model to solve the frictional contact/impact problem of ANCF flexible bodies.

- Use of spherical decomposition for the contact discretization of the ANCF flexible body.

- Validation of results against ABAQUS and FEAP for beam-to-beam and beam-to-rigid frictional contact/impact cases

**ANCF + Contact/Impact using DVI**

- Development and implementation of a DVI formulation to solve the frictional contact/impact problem of ANCF flexible bodies.

- Improvement in the original DVI formulation to handle the impact problems with nonzero coefficient of restitution.

- Discussion on the cone complementarity problem (CCP) for frictional contact between ANCF flexible bodies and Matlab implementation of projected Gauss-Jacobi algorithm to solve the CCP.

- Validation of results through comparison with ABAQUS, FEAP, and penalty method results for beam-to-beam and beam-to-rigid frictional contact/impact cases.

**Demonstration of Technology**

- Implementation of gradient deficient ANCF beam in ADAMS

- ADAMS simulation of a satellite model and an elevator cable model to demonstrate the real-life applications of ANCF.
8.1 Future Work

The work carried out here suggests several interesting directions for future work. The penalty method and DVI approach used herein can be easily extended to solve the frictional contact/impact problem of the ANCF plate/shell. The theoretical formulation of gradient deficient ANCF plate element and the corresponding DVI formulation are included in Appendix. These contact formulations can also be used along with generalized ANCF elements, and other classical beam, plate and shell elements that are widely employed in the finite element analysis field.

Note that the contact formulations discussed here are not limited to simple contact detection schemes such as spherical decomposition. These formulations can be used along with more complex contact detection algorithms used in the field of finite element analysis or MBS.

The DVI formulation discussed here uses a first order semi-implicit Euler method. This formulation can be generalized to support other numerical integration methods used in MBS. The DVI formulation for the impact problem uses a coefficient of restitution. The correlation between the coefficient of restitution and the damping parameters used in penalty-based contact models can be established through a series of numerical simulations.

Also, there is a great scope for the parallel implementation of ANCF and the proposed contact formulations. The scaling of these formulations in large multibody systems with thousands of flexible bodies interacting through frictional contact can be studied to gauge the extent to which parallel computing can speed up the simulation of complex mechanical systems.
APPENDIX

A1. Use of Quasi-Newton Approach

In the implicit numerical integration method, a system of nonlinear equations needs to be solved at each time step. Generally, a Newton-like iterative method is used to solve the system of nonlinear equations, which calls for the solution a linear system at each iteration. The computation of the Jacobian matrix at each iteration can be costly in Newton’s method and will eventually lead to overall inefficiency of the algorithm.

A Quasi-Newton approach instead of Newton’s method is used to increase the efficiency of the numerical method. In this approach, the Jacobian matrix is approximated by neglecting the computationally intensive terms that enter its expression. This approach can be effectively used in conjunction with Newmark, HHT-I3 and NSTIFF integration methods. The details of implementation of this approach in HHT-I3 algorithm are given in this section.

In the HHT-I3 method, the iterative algorithm of Newton’s method requires at each iteration ($k$), the solution of the linear system

\[
\begin{bmatrix}
\dot{\mathbf{M}} & \mathbf{F}_q^T \\
\mathbf{F}_q & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{u}}_1 \\
\dot{\mathbf{u}}_2 \\
\mathbf{e}_1 \\
\mathbf{e}_2
\end{bmatrix}
= \begin{bmatrix}
\dot{\mathbf{u}}_1^0 \\
\dot{\mathbf{u}}_2^0 \\
\mathbf{e}_1^0 \\
\mathbf{e}_2^0
\end{bmatrix},
\]

where $\mathbf{e}_i$ are the residuals in satisfying the set of the discretized equation of motion and constraint equations which are scaled such that

\[
\begin{align*}
\mathbf{e}_1 &= \frac{1}{1 + a} (\mathbf{M} \ddot{\mathbf{q}}_{n+1} + (\mathbf{F}_q^T \dot{\mathbf{q}} - \mathbf{Q})_{n+1} - \frac{a}{1 + a} (\mathbf{F}_q^T \dot{\mathbf{q}} - \mathbf{Q})_n \\
\mathbf{e}_2 &= \frac{1}{bh^2} \mathbf{F}(\mathbf{q}, t).
\end{align*}
\]
This scaling is done in order to improve the conditioning of the Jacobian matrix for Newton’s method. The matrix $\hat{M}$ in Eq. (1) is defined as

$$
\hat{M} = \frac{1}{1 + a} M + \frac{bh}{\|q\|^2} \frac{1}{1 + a} (M \dot{q} \dot{q}^T + (F_q \lambda)_q) - \frac{\|Q\|}{\|q\|^2} h g \frac{\|Q\|}{\|q\|^2}.
$$

(3)

For a non-stiff system and a sufficiently small $h$, the terms involving $h$ can be neglected in $\hat{M}$. This approach saves the computational overheads of calculating $(M \dot{q} \dot{q}^T, (F_q \lambda)_q, \|Q\|/\|q\|$, and $\|Q\|/\|q\|$, at each iteration without any significant loss in accuracy. The new Jacobian matrix for a Quasi-Newton method is

$$
J_{new} = \begin{bmatrix}
\hat{M} & F_q^T \dot{u}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}^T
\dot{u}
\end{bmatrix},
$$

(4)

where $\hat{M} = \frac{1}{1 + a} M$ is a diagonal matrix for rigid bodies. With this new Jacobian matrix, the solution of the linear system in Eq. (1) is obtained by first computing $D\lambda$ as the solution of the linear system

$$
(F_q \hat{M}^{-1} F_q^T) D\lambda = e_2 - F_q \hat{M}^{-1} e_1,
$$

(5)

and then $D\dot{q}$ can be easily computed as the solution of the linear system

$$
\hat{M} \Delta \dot{q} = -(e_1 + F_q^T \lambda). \Delta q)
$$

(6)

The coefficient matrix $(F_q \hat{M}^{-1} F_q^T)$ in Eq. (5), denoted by $E \hat{M}^{-1} m^m m$ is called the acceleration reduced matrix [14]. Since the mass matrix $M$ and hence $\hat{M}^{-1}$ is positive definite, the reduced matrix $E$ is also positive definite as long as the constraints in $F$ are linearly
independent. A direct solver like Cholesky method or an iterative solver like the Conjugate Gradient method can be effectively used to solve the linear system in Eq. (5).

A Quasi-Newton approach in HHT-I3 method is used for 2 sec long simulations of all the models described earlier. The average CPU time is measured and compared with that of HHT-I3 using regular Newton’s method. It should be noted that the Cholesky method is used to solve the linear system given in Eq. (5). The results of the numerical experiments are summarized in Table 6. It is observed that the use of the Quasi-Newton approach in HHT-I3 makes it more efficient except in case of a flexible-body system. This is because the mass matrix using floating frame of reference formulation is not constant and computing its inverse at each iteration is computationally expensive. However, this approach is well suited for flexible models using ANCF formulation where the mass matrix is constant.

Table 6: Average CPU time comparison

<table>
<thead>
<tr>
<th>Model</th>
<th>Time-step (sec)</th>
<th>Avg. CPU time (sec)</th>
<th>Newton’s method</th>
<th>Quasi-Newton</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double Pendulum</td>
<td>$2^{-14}$</td>
<td>42.2</td>
<td>37.9</td>
<td></td>
</tr>
<tr>
<td>Slider Crank</td>
<td>$2^{-12}$</td>
<td>10.5</td>
<td>8.3</td>
<td></td>
</tr>
<tr>
<td>Flexible Slider Crank</td>
<td>$2^{-14}$</td>
<td>42.8</td>
<td>45.7</td>
<td></td>
</tr>
<tr>
<td>Seven Body Mechanism</td>
<td>$2^{-14}$</td>
<td>42.1</td>
<td>32.8</td>
<td></td>
</tr>
</tbody>
</table>

A novel approach of using Quasi-Newton method is proposed here which can be used in conjunction with HHT-I3 and NSTIFF algorithms to increase their efficiency. This approach reduced the computation time significantly for rigid multibody simulation and can be efficiently
used for flexible multibody simulation with ANCF formulation. The use of sparse matrix techniques and the parallel implementation of the Cholesky method or the Conjugate Gradient method can achieve further speed up in the simulation.
A2. Gradient Deficient ANCF Plate Element

The gradient deficient (GD) ANCF plate element [8], also called a thin-plate element, is based on the Kirchhoff’s plate theory. In this element, there are four nodes, one at each corner. Each of these nodes has nine generalized coordinates: three to describe the position, \( \mathbf{r}_i \), of the node in Cartesian coordinates, three to describe the slope, \( \frac{\partial \mathbf{r}_i}{\partial x} \), of the plate in the local x-direction, and three to describe the slope, \( \frac{\partial \mathbf{r}_i}{\partial y} \), of the plate in the local y-direction. The four nodes result in a total of 36 generalized coordinates. Mathematically, the vector of generalized coordinates is written as

\[
\mathbf{e} = \left[ \mathbf{e}_1^T \mathbf{e}_2^T \mathbf{e}_3^T \mathbf{e}_4^T \right]^T \in \mathbb{R}^{36} \quad \text{with} \quad \mathbf{e}_i = [\mathbf{r}_i^T \frac{\partial \mathbf{r}_i}{\partial x} \frac{\partial \mathbf{r}_i}{\partial y}]^T.
\]

For thin plates, the deformation along the thickness of the plate can be neglected. The normal vector of the mid-surface of an element can be defined by the cross product of the gradients in the x and y-direction at the location of the desired normal. The global position vector \( \mathbf{r} \) of any material point on the plate mid-surface can be defined by using the element shape functions and the nodal coordinate vector as follows

\[
\mathbf{r} = \mathbf{S}(x, y) \mathbf{e}, \quad (7)
\]

where \( \mathbf{S} \) is the element shape function matrix expressed in terms of the element spatial coordinates, \( \xi \) and \( \eta \). Specifically,

\[
\mathbf{S} = \begin{bmatrix}
S_1 \mathbf{I} & S_2 \mathbf{I} & S_3 \mathbf{I} & S_4 \mathbf{I} & S_5 \mathbf{I} & S_6 \mathbf{I} & S_7 \mathbf{I} & S_8 \mathbf{I} & S_9 \mathbf{I} & S_{10} \mathbf{I} & S_{11} \mathbf{I} & S_{12} \mathbf{I}
\end{bmatrix}, \quad (8)
\]
\[ S_1 = -\left(\xi - 1\right)\left(\eta - 1\right)\left(2\eta^2 - \eta + 2\xi^2 - \xi - 1\right) \quad S_2 = -l_\xi \left(\xi - 1\right)^2 \left(\eta - 1\right) \]
\[ S_3 = -w_\eta \left(\eta - 1\right)^2 \left(\xi - 1\right) \quad S_4 = \xi \left(2\eta^2 - \eta + 3\xi + 2\xi^2 \right)\left(\eta - 1\right) \]
\[ S_5 = -l_\xi^2 \left(\xi - 1\right)\left(\eta - 1\right) \quad S_6 = w_\xi \eta \left(\eta - 1\right)^2 \]
\[ S_7 = -\xi \eta \left(1 - 3\xi - 3\eta + 2\eta^2 + 2\xi^2 \right) \quad S_8 = l_\xi^2 \eta \left(\xi - 1\right) \]
\[ S_9 = w_\xi \eta^2 \left(\eta - 1\right) \quad S_{10} = \eta \left(\xi - 1\right)\left(2\xi^2 - \xi - 3\eta + 2\eta^2 \right) \]
\[ S_{11} = l_\xi \eta \left(\xi - 1\right) \quad S_{12} = -w_\eta^2 \left(\xi - 1\right)\left(\eta - 1\right), \]

where \( I \) is the 3x3 identity matrix and \( \xi = x/l, \eta = y/w \), with \( l \) and \( w \) being length and width of the element, respectively.

Using the principle of virtual work of the continuum, the element equation of motion is determined to be

\[ \mathbf{M}\ddot{\mathbf{e}} + \mathbf{Q}_e = \mathbf{Q}_e^e, \quad (9) \]

where \( \mathbf{Q}_e \) is the vector of generalized element elastic forces, \( \mathbf{Q}_e^e \) is the vector of generalized element external forces, and \( \mathbf{M} \) is the symmetric consistent element mass matrix defined as

\[ \mathbf{M} = \left[ \int_\Omega \rho \mathbf{S}^T \mathbf{S} \mathbf{d}V \right], \quad (10) \]

which remains constant. The external force vector due to a concentrated force \( \mathbf{F} \) is calculated using the following equation

\[ \mathbf{Q}_e = \mathbf{S}^T \mathbf{F}, \quad (11) \]

and the external force vector due to a force that acts over the entire volume of the element is given by

\[ \mathbf{Q}_e = \int_\Omega \mathbf{S}^T \mathbf{F} \mathbf{d}V. \quad (12) \]
Following Kirchhoff theory, the strain energy of a thin plate can be written as the sum of two terms: one term is due to membrane and shear deformations at the plate mid-surface, whereas the other term is due to the plate bending and twist. The strain energy of a gradient deficient plate element is

\[
U = \frac{1}{2} \int_{V_o} \varepsilon^T E_{\varepsilon} \varepsilon dV_o + \frac{1}{2} \int_{V_o} \kappa^T E_{\kappa} \kappa dV_o ,
\]

(13)

where \( \varepsilon \in \mathbb{R}^3 \) is the strain vector, \( \kappa \in \mathbb{R}^3 \) is the curvature vector, and \( E_{\varepsilon}, E_{\kappa} \) are the corresponding constitutive matrices. The detailed expressions of these terms are given in [8]. The vector of generalized element elastic forces is subsequently obtained by differentiating the strain energy with respect to the generalized coordinates:

\[
Q_s = \left( \frac{\partial U}{\partial \varepsilon} \right)^T .
\]

(14)
A3. DVI Formulation with GD ANCF plate elements

To model the frictional contact between GD ANCF plates, a spherical decomposition approach is used for the contact discretization of the colliding plates. Specifically, each flexible plate can be considered as a grid of spheres that overlap and which are distributed equally along the x and y axes of the gradient deficient ANCF plate elements. At each time step the collision detection between the spheres in all the plates is performed and a signed distance, $\phi$ and a contact normal, $n$ is determined for each active contact. The contacts are considered as active, if $\phi < \epsilon$ for a given threshold $\epsilon$. The set of all active contacts will be referred to by $G_A$.

**Generalized Contact Coordinates**

Consider an active contact, $i \in G_A$, between two spheres which belong to two different plate elements $b_k$ and $b_i$. Let the velocities of these two spheres be $v_1$ and $v_2$. Let $n_i \in \mathbb{R}^3$ be the normal at the contact pointing toward the exterior of the first sphere. Let $u_i, w_i \in \mathbb{R}^3$ be two vectors along the tangential directions in contact plane which are computed using Gram-Schmidt orthonormalization. Here $n_i, u_i, w_i$ are mutually orthogonal vectors of unit length which form a contact coordinate system. Let $\hat{\gamma}_i = [\hat{\gamma}_{i,n}, \hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}]^T$ be the reaction force in contact coordinates. A normal relative velocity of two spheres in contact is given by

$$v_{rel,n} = n_i^T (v_1 - v_2) = n_i^T S(x_1, y_1) \dot{e}_{b_k} - n_i^T S(x_2, y_2) \dot{e}_{b_i} ,$$

where $(x_1, y_1)$ and $(x_2, y_2)$ are the spatial coordinates of colliding spheres and $\dot{e}_{b_k}, \dot{e}_{b_i} \in \mathbb{R}^{36}$ are the vectors of nodal velocities of the elements $b_k$ and $b_i$, respectively. Let the vector of generalized coordinates be $q = [q_1^T \ldots q_n^T]^T \in \mathbb{R}^p$. Here $n$ is the total number of bodies (plates) and $p$ is the total number of degrees of freedom present in the system.
\[ \mathbf{q} = [\ldots \dot{\mathbf{e}}_h \ldots \dot{\mathbf{e}}_h \ldots]^T \in \mathbb{R}^n \] be the vector of generalized velocities. Equation (15) can be written as

\[ \mathbf{v}_{rel,n} = \mathbf{D}_{i,n}^T \mathbf{q}, \quad (16) \]

where \( \mathbf{D}_{i,n} = \begin{bmatrix} 0 \ldots 0, \mathbf{n}^T_i \mathbf{S}(x_1, y_1), 0 \ldots 0, \mathbf{n}^T_i \mathbf{S}(x_2, y_2), 0 \ldots 0 \end{bmatrix}^T \) is the generalized normal vector.

Similarly, the expressions for the generalized tangent vectors can be obtained as

\[ \mathbf{D}_{i,u} = \begin{bmatrix} 0 \ldots 0, \mathbf{u}^T_i \mathbf{S}(x_1, y_1), 0 \ldots 0, -\mathbf{u}^T_i \mathbf{S}(x_2, y_2), 0 \ldots 0 \end{bmatrix}^T, \]

\[ \mathbf{D}_{i,w} = \begin{bmatrix} 0 \ldots 0, \mathbf{w}^T_i \mathbf{S}(x_1, y_1), 0 \ldots 0, -\mathbf{w}^T_i \mathbf{S}(x_2, y_2), 0 \ldots 0 \end{bmatrix}^T. \]

The matrix \( \mathbf{D}_i = [\mathbf{D}_{i,n} \mathbf{D}_{i,u} \mathbf{D}_{i,w}] \) is a projection matrix which transforms contact coordinate system to the generalized coordinate system.

Once this projection matrix is defined, the rest of DVI formulation for ANCF plates follows the same steps given in Chapter 6.
LIST OF REFERENCES


