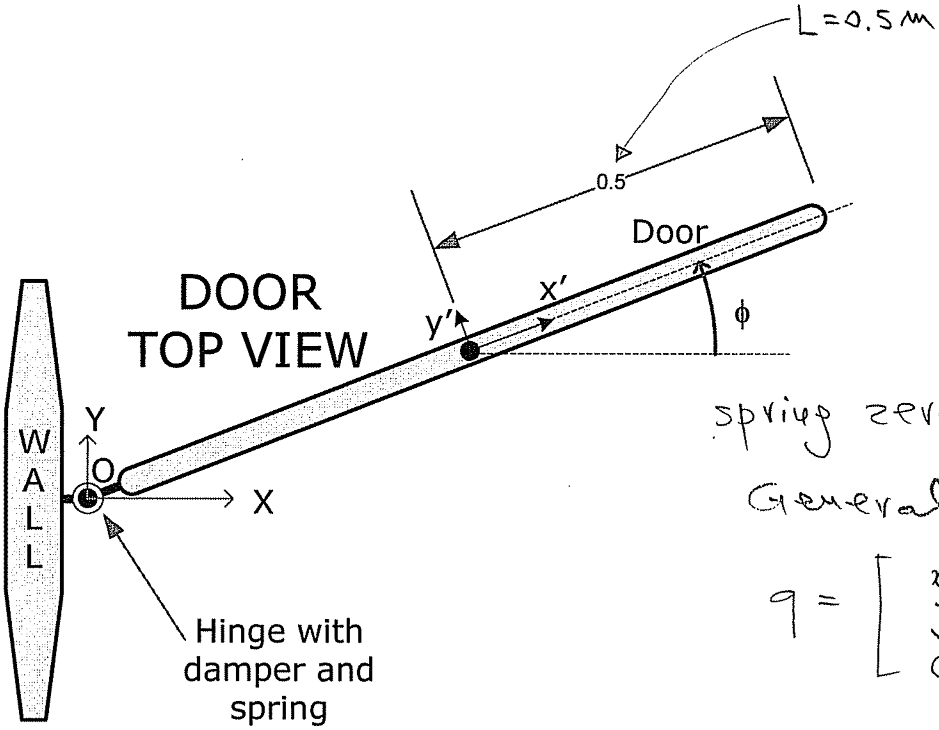


$m = 30\ kg$

$J = 2.5\ kg \cdot m^2$

$k = 8\ \frac{N \cdot m}{rad}$

$c = 1\ \frac{N \cdot m \cdot s}{rad}$

spring zero tension angle $\varphi_{free} = 0$.

Generalized coordinates used:

$$q = \begin{bmatrix} x \\ y \\ \varphi \end{bmatrix}$$

constraints:

A) kinematic : Absolute x & Absolute y constraint at point O.

$$r^O = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} -L \\ 0 \end{bmatrix} = \begin{bmatrix} x - Lc\varphi \\ y - Ls\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

B) Driving constraint on angle φ :

$$\Phi^D(q,t) = \varphi - \frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right) = 0.$$ → you want this motion for the first two seconds..

Therefore :

$$\Phi(q,t) = \begin{bmatrix} x - Lc\varphi \\ y - Ls\varphi \\ \varphi - \frac{\pi}{2}\sin\left(\frac{\pi}{4}t\right) \end{bmatrix} \Rightarrow \Phi_q = \begin{bmatrix} 1 & 0 & Ls\varphi \\ 0 & 1 & -Lc\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\Phi}(q,t) = \begin{bmatrix} \dot{x} + \dot{\varphi}Ls\varphi \\ \dot{y} - \dot{\varphi}Lc\varphi \\ \dot{\varphi} - \frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right) \end{bmatrix} \Rightarrow \nu = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right) \end{bmatrix}$$

$$\ddot{\Phi}(p,t) = \begin{bmatrix} \ddot{x} + \ddot{\varphi} L s\varphi + \dot{\varphi}^2 L c\varphi \\ \ddot{y} - \ddot{\varphi} L c\varphi + \dot{\varphi}^2 L s\varphi \\ \ddot{\varphi} + \frac{\pi^3}{32}\sin(\frac{\pi}{4}t) \end{bmatrix} \Rightarrow \gamma = \begin{bmatrix} -\dot{\varphi}^2 L c\varphi \\ -\dot{\varphi}^2 L s\varphi \\ -\frac{\pi^3}{32}\sin(\frac{\pi}{4}t) \end{bmatrix}$$

Generalized forces:

- Gravity is irrelevant in this case. The only nonzero term in the expression of the generalized force is coming out of the rotational spring-damper element present in the model:

$$n = -k(\varphi - \overset{\nearrow 0}{\varphi_{free}}) - c\dot{\varphi}$$

$$\Rightarrow \quad n = -(k\varphi + c\dot{\varphi})$$

Then, we conclude that $Q^A = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix}$

We have all the quantitie needed to solve the inverse dynamics problem:

$$\Phi_q \cdot \ddot{q} = \gamma \Rightarrow \begin{bmatrix} 1 & 0 & L s\varphi \\ 0 & 1 & -L c\varphi \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} -\dot{\varphi}^2 L c\varphi \\ -\dot{\varphi}^2 L s\varphi \\ -\frac{\pi^3}{32}\sin(\frac{\pi}{4}t) \end{bmatrix} \qquad (*)$$

As far as the time evolution of φ, based on the expression of φ and the last equation in $\Phi_q \cdot \dot{q} = \nu$ and $\Phi_q \cdot \ddot{q} = \gamma$, we get

$$\boxed{\begin{aligned} \varphi &= \frac{\pi}{2}\sin(\frac{\pi}{4}t) \\ \dot{\varphi} &= \frac{\pi^2}{8}\cos(\frac{\pi}{4}t) \\ \ddot{\varphi} &= -\frac{\pi^3}{32}\sin(\frac{\pi}{4}t). \end{aligned}} \qquad (**)$$

Next, based on Eq.(*) (see previous page),

$$\ddot{x} + L\ddot{\varphi}s\varphi = -\dot{\varphi}^2 Lc\varphi \quad \Rightarrow \quad \ddot{x} = -L\ddot{\varphi}s\varphi - \dot{\varphi}^2 Lc\varphi$$

$$\ddot{y} - L\ddot{\varphi}c\varphi = -\dot{\varphi}^2 Ls\varphi \quad \Rightarrow \quad \ddot{y} = L\ddot{\varphi}c\varphi - \dot{\varphi}^2 Ls\varphi.$$

Therefore, based on (**), we obtain the expression of $\ddot{x}$ and $\ddot{y}$ as functions of time. We therefore know how $\ddot{q}$ changes in time and are ready to compute the Lagrange multipliers. Before we do that, first note that we also know q and $\dot{q}$:

$$q = \begin{bmatrix} x \\ y \\ \varphi \end{bmatrix} = \begin{bmatrix} Lc\varphi \\ Ls\varphi \\ \frac{\pi}{2}\cdot\sin(\frac{\pi}{4}t) \end{bmatrix} \qquad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -\dot{\varphi}Ls\varphi \\ \dot{\varphi}Lc\varphi \\ \frac{\pi^2}{8}\cos(\frac{\pi}{4}t) \end{bmatrix}$$

Next, write the EOM:

$$M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J' \end{bmatrix} \qquad Q^A = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix}$$

$$M\ddot{q} + \Phi_q^T\lambda = Q^A$$

Then, $$\Phi_q^T\lambda = Q^A - M\ddot{q} = \begin{bmatrix} 0 \\ 0 \\ -k\varphi - c\dot{\varphi} \end{bmatrix} - \begin{bmatrix} m\ddot{x} \\ m\ddot{y} \\ J'\ddot{\varphi} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ Ls\varphi & -Lc\varphi & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} -m\ddot{x} \\ -m\ddot{y} \\ -k\varphi - c\dot{\varphi} - J'\ddot{\varphi} \end{bmatrix}$$

Then,

$$\lambda_1 = -m\ddot{x} = m(L\ddot{\varphi}\,s\varphi + \dot{\varphi}^2 L\,c\varphi)$$

$$\lambda_2 = -m\ddot{y} = m(-L\ddot{\varphi}\,c\varphi + \dot{\varphi}^2 L\,s\varphi)$$

$$\lambda_3 = -k\varphi - c\dot{\varphi} - J'\ddot{\varphi} - L\,s\varphi\cdot\lambda_1 + L\,c\varphi\,\lambda_2$$

$$= -k\varphi - c\cdot\dot{\varphi} - J'\ddot{\varphi} - L\,s\varphi\, m(L\ddot{\varphi}\,s\varphi + \dot{\varphi}^2 L\,c\varphi) + L\,c\varphi\cdot m(-L\ddot{\varphi}\,c\varphi + \dot{\varphi}^2 L\,s\varphi)$$

$$= -k\varphi - c\dot{\varphi} - J'\ddot{\varphi} - mL^2\ddot{\varphi} = -\left[(mL^2 + J')\ddot{\varphi} + c\dot{\varphi} + k\varphi\right]$$

For our problem $c = 1$ & $k = 8$

We are interested in the reaction force/torque associated with the third constraint (the "driving" constraint).

$$\Phi^D(q,t) = \varphi - \frac{\pi}{2}\cdot\sin\left(\frac{\pi}{4}t\right)$$

$$\Rightarrow \quad \Phi^D_r = [0 \quad 0] \quad \& \quad \Phi^D_\varphi = [1].$$

Then $F^D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & $T^D = -(\Phi^D_\varphi)^T\lambda_3 = -\lambda_3$

$$\Rightarrow T^D = (mL^2 + J')\ddot{\varphi} + c\dot{\varphi} + k\varphi$$

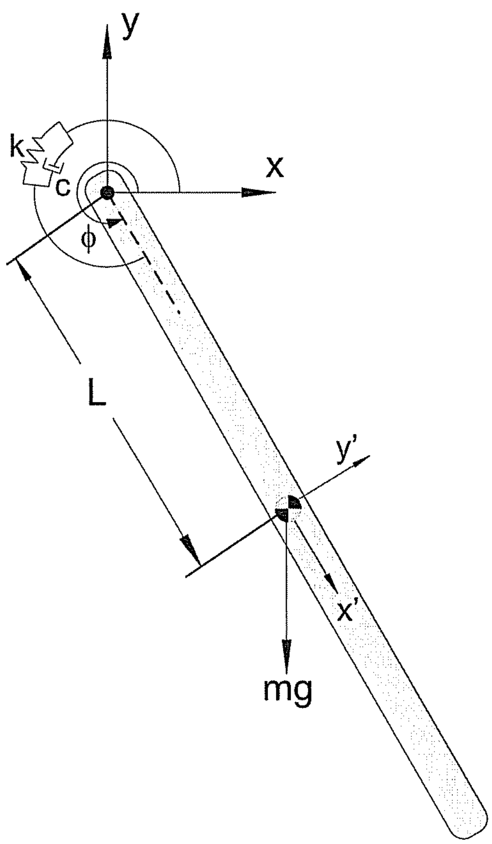
$$= [30\cdot(0.25) + 2.5]\cdot\left(-\frac{\pi^3}{32}\sin\left(\frac{\pi}{4}t\right)\right) + \frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right) + 4\pi\cdot\sin\left(\frac{\pi}{4}t\right)$$

$$= \left(4\pi - \frac{10\pi^3}{32}\right)\sin\left(\frac{\pi}{4}t\right) + \frac{\pi^2}{8}\cos\left(\frac{\pi}{4}t\right).$$

$$\approx 2.877\sin\left(\frac{\pi}{4}t\right) + 1.234\cos\left(\frac{\pi}{4}t\right) \quad [N\cdot m].$$

Apply this torque for the first two seconds to get the specified time evolution $\varphi = \frac{\pi}{2}\cdot\sin\left(\frac{\pi}{4}t\right)$.

Equilibrium Example.



$m = 10$

$g = 9.81$

$k = 25$

$L = 1$

$\phi_{free} = 0$

Generalized coordinates:

$$q = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$$

Constraints:
- revolute joint at origin

$$\Phi(q) = \begin{bmatrix} x - L c\phi \\ y - L s\phi \end{bmatrix} = 0_{2\times1}$$

Generalized forces acting on the pendulum:

$$Q^A = \begin{bmatrix} 0 \\ -mg \\ -k(\phi - \phi_{free}) - c\dot{\phi} \end{bmatrix}.$$

At equilibrium,

$$\begin{cases} \Phi_q^T \lambda = Q^A \\ \Phi(q) = 0 \end{cases}$$

We have 5 unknowns: $x, y, \phi, \lambda_1, \lambda_2$

We have 5 equations (nonlinear).

$$\Phi_q = \begin{bmatrix} 1 & 0 & L s\phi \\ 0 & 1 & -L c\phi \end{bmatrix} \Rightarrow \Phi_q^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ L s\phi & -L c\phi \end{bmatrix}$$

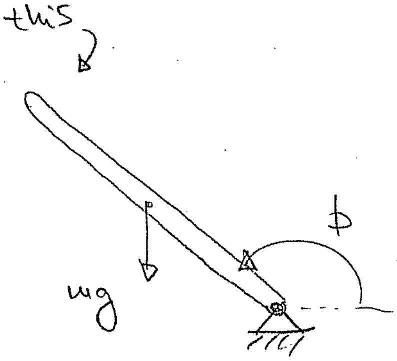
$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ L s\phi & -L c\phi \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ -k(\phi - \phi_{free}) - \cancelto{0}{c\dot{\phi}} \end{bmatrix} \\ x - L c\phi = 0 \\ y - L s\phi = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -mg \\ \cancel{L\lambda_1 S\phi} - L\lambda_2 C\phi = -k(\phi - \phi_{free}) \\ x - L\,C\phi = 0 \\ y - L\,S\phi = 0 \end{cases}$$

Then, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix}$

$$-L(-mg)\,C\phi = -k\phi \quad \Rightarrow \quad \boxed{mgL\,C\phi = -k\phi}$$

The equilibrium configuration will therefore look like this

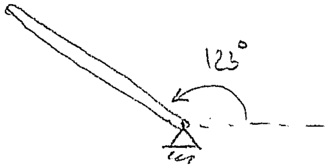


For the given values, the equation that needs to be solved is

$$98.1 \cdot C\phi + 25\,\phi = 0$$

Using's MATLAB's "fsolve", one gets two solutions:

$\phi^{(1)} = 2.1509$ [rad] $\Downarrow$ $\phi^{(1)} = 123.24°$ $\Downarrow$



$\phi^{(2)} = 3.5698$ [rad] $\Downarrow$ $\phi^{(2)} = 204.53°$ $\Downarrow$