on the Coulomb friction model.

Input:

- friction coefficient μ.
- Normal force γ_n
- two unit vectors in the contact tangent plane, t_u & t_w
- relative sliding velocity v.

Notation:

- the components of the friction force along the unit vectors t_u & t_w : γ_u & γ_w.
- the friction force

$$F = \gamma_u t_u + \gamma_w t_w$$

The Coulomb friction model states that:

a) $\mu \gamma_n \geq \sqrt{\gamma_u^2 + \gamma_w^2} = \|F\|$.

b) $\|v\| \cdot (\mu \gamma_n - \sqrt{\gamma_u^2 + \gamma_w^2}) = \|v\| \cdot (\mu \gamma_n - \|F\|) = 0$

c) $\langle F, v \rangle = - \|F\| \cdot \|v\|$ (F and v are colinear and opposite).

The idea is to prove that a, b, c above are equivalent to the first order KKT conditions of a suitably chosen minimization problem:

- minimize power dissipated by friction force given that the friction force should not lie outside the friction cone

Minimize $v^T(\gamma_u t_u + \gamma_w t_w)$

subject to the constraint $\mu \gamma_n \geq \sqrt{\gamma_u^2 + \gamma_w^2}$

Proof that first order KKT is equivalent to a, b, c above.

$$\mathcal{L}(\gamma_u, \gamma_w, \lambda) = v^T(\gamma_u t_u + \gamma_w t_w) - \lambda\left(\mu\gamma_n - \sqrt{\gamma_u^2 + \gamma_w^2}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_u} = v^T \cdot t_u + \lambda \cdot \frac{\gamma_u}{\sqrt{\gamma_u^2 + \gamma_w^2}} = 0 \Rightarrow v^T \cdot t_u \cdot \|F\| = -\lambda \gamma_u. \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_w} = 0 \Rightarrow v^T \cdot t_w \cdot \|F\| = -\lambda \gamma_w. \qquad (2)$$

$$\left.\begin{array}{l} \lambda \geq 0 \\ \mu\gamma_n - \|F\| \geq 0 \end{array}\right\} \quad \lambda(\mu\gamma_n - \|F\|) = 0 \qquad (3)$$

Case 1 : $\lambda = 0 \Rightarrow \left.\begin{array}{l} v^T \cdot t_u \cdot \|F\| = 0 \\ v^T \cdot t_w \cdot \|F\| = 0 \end{array}\right\} \Rightarrow \|v\| = 0 \Rightarrow v = 0.$

Then we conclude that $\mu\gamma_n \geq \|F\|$

$\|v\| \cdot (\mu\gamma_n - \|F\|) = 0$ (since $v = 0$)

$\langle F, v \rangle = -\|F\| \cdot \|v\| = 0$

Case 2 : $\lambda > 0$.

Then $\mu\gamma_n = \|F\|$. (based on 3). $\Rightarrow$ both a & b hold.

Prove c) : using (1) & (2) one gets

$$(v^T \cdot t_u \gamma_u + v^T t_w \gamma_w) \cdot \|F\| = -\lambda(\gamma_u^2 + \gamma_w^2) = -\lambda \|F\|^2.$$

Then, $v^T \cdot F \cdot \|F\| = -\lambda \|F\|^2 \Rightarrow v^T \cdot F = -\lambda \|F\|$. (*)

Also based on (1) & (2)

$$\left[(v^T t_u)^2 + (v^T t_w)^2\right] \cdot \|F\|^2 = \lambda^2 \cdot (\gamma_u^2 + \gamma_w^2) = \lambda^2 \cdot \|F\|^2$$

$$\Rightarrow \quad \|v\|^2 = \lambda^2.$$

Since $\lambda > 0 \Rightarrow \|v\| = \lambda$.

Therefore, based on (*),

$v^T \cdot F = -\|v\| \cdot \|F\|$, which is c.

To conclude, the conditions that the friction force, that is the components γ_u & γ_w, should satisfy in the Coulomb model can be posed as the first order optimality conditions of an optimization problem.

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