The Coulomb friction model states that:

1) \( \mu N \geq \sqrt{u^2 + \nu^2} \), \( \mu N = \mu F_N \).

2) \( \mu N \cdot (\mu F_N - \sqrt{u^2 + \nu^2}) = \mu N \cdot (\mu F_N - \mu F_N) = 0 \)

3) \( \langle F, U \rangle = - F_N \cdot \nu \cdot F_N \) (\( \nu \) are colinear and opposite).

The idea is to prove that a, b, c above are equivalent to the first order key conditions of a suitably chosen minimization problem:

- minimize power dissipated by friction force given that the friction force should not lie outside the friction force.
Minimize $\alpha^T (y u + y w t w) $

Subject to the constraint $\mu u \geq \sqrt{y_u^2 + y_w^2}$

Proof that first order KKT is equivalent to $a, b, c$ above.

$Z(u, y, \lambda) = \alpha^T (y u + y w t w) - \lambda (\mu u - \sqrt{y_u^2 + y_w^2})$

$$\frac{\partial Z}{\partial u} = \alpha^T y u = 0 \Rightarrow \alpha^T \mu u = \|F\| = -\lambda y u . \quad (1)$$

$$\frac{\partial Z}{\partial w} = 0 \Rightarrow \alpha^T y w = -\lambda y w . \quad (2)$$

$$\lambda \geq 0 \quad \Rightarrow \quad \lambda (\mu u - \|F\|) = 0$$

$$\mu u - \|F\| \geq 0$$

Case 1: $\lambda = 0 \Rightarrow \alpha^T \mu u = 0 \Rightarrow \|F\| = 0 \Rightarrow \mu u = 0 . \quad (3)$$

Then we conclude that $\|F\| = 0 . \quad (3)$

Case 2: $\lambda > 0$

Then $\mu u = \|F\|$. (Based on 3). $\Rightarrow$ both a and b hold.

Proof c): Using (1) & (2) we get

$$(\alpha^T \mu u + \lambda^T y w), \|F\| = -\lambda (y_u^2 + y_w^2) = -\lambda \|F\|^2 .$$
Then, \( \mathbf{n}^T \mathbf{F} - \lambda \| \mathbf{F} \|^2 = 0 \),
\[\mathbf{n}^T \mathbf{F} = -\lambda \| \mathbf{F} \|^2. \quad (x)\]

Also based on \((1)\) & \((2)\),

\[\left[ (\mathbf{n}^T \mathbf{u})^2 + (\mathbf{n}^T \mathbf{w})^2 \right] \| \mathbf{F} \|^2 = \lambda^2 (\mathbf{y}_u^2 + \mathbf{y}_w^2) = \lambda^2 \| \mathbf{F} \|^2 \]
\[\Rightarrow \quad \| \mathbf{F} \|^2 = \lambda^2 \cdot \| \mathbf{F} \|^2 \]
\[\Rightarrow \quad \mathbf{n} \| \mathbf{F} \|^2 - \lambda^2 \cdot \| \mathbf{F} \|^2 = \lambda^2 \cdot \| \mathbf{F} \|^2 \]
\[\Rightarrow \quad \| \mathbf{n} \|^2 - \lambda = 0 \]
\[\Rightarrow \quad \| \mathbf{n} \|^2 = \lambda \] since \( \lambda > 0 \)

Therefore, based on \((x)\),

\[\mathbf{n}^T \mathbf{F} = -\| \mathbf{F} \| \| \mathbf{F} \|^2, \] which is \( 0 \).

To conclude, the conditions that the friction force, that is, the component \( \mathbf{y}_u \) & \( \mathbf{y}_w \), should satisfy in the Coulomb's model can be posed as the first order optimality conditions of an optimization problem.