“You have brains in your head. You have feet in your shoes. You can steer yourself in any direction you choose. You're on your own, and you know what you know. And you are the guy who'll decide where to go.”

“You’re off to Great Places! Today is your day! Your mountain is waiting, so…get on your way!”

“Today was good. Today was fun. Tomorrow is another one.”

“Life’s too short to wake up with regrets. So love the people who treat you right, forgive the ones who don't and believe that everything happens for a reason. If you get a chance, take it. If it changes your life, let it. Nobody said it'd be easy, they just promised it would be worth it.”

Song of the day: “The Wheels on the Bus.”
Before we get started...

- **Last time:**
  - Loose ends, numerical method for the solution of DAEs of multibody dynamics
    - Dealt with computation of some sensitivities that enter the Newton-Raphson/Modified Newton solution process
  - Started handling of frictional contact
    - Focused on the “penalty” approach

- **Today:**
  - Wrap up handling of frictional contact – focus on the complementarity approach
  - Wrapping up the “rigid body dynamics” component of the course

- **Reading:**
  - Online document, variational take on Coulomb’s Friction Law
  - Paper on Complementarity approach:

- **Homework-07:** uploaded later today
  - Assigned today, due on October 28
The “Penalty method” relies on a record (history) of tangential displacement $\delta_t$ to model static friction (see figure at right).
The “Penalty method” in Chrono: relies on a record (history) of tangential displacement to model static friction.

\[ F_n = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) (k_n \delta_n \mathbf{n} - \gamma_n m_{\text{eff}} \mathbf{v}_n) \]

\[ F_t = f \left( \frac{\delta_n}{D_{\text{eff}}} \right) (-k_t \delta_t - \gamma_t m_{\text{eff}} \mathbf{v}_t) \]

If \( |F_t| > \mu |F_n| \) then scale \( |\delta_t| \) so that \( |F_t| = \mu |F_n| \).

Visualize this \( \delta_t \) as creep.
Penalty Method – the Pros

- Backed by large body of literature and numerous validation studies

- No increase in the size of the problem
  - This is unlike the “complementarity” approach

- Can accommodate shock wave propagation
  - Can’t do w/ “complementarity” approach since it’s a pure “rigid body” solution

- Easy to implement
  - Entire numerical solution decoupled
    - Easy to scale up to large problems
    - Parallel-computing friendly – run in parallel on per contact basis
      - Memory communication intensive
Penalty Method – Cons

1. Numerical stability requires small integration time steps
   • Long simulation times

2. Choice of integration time step strongly influences results

3. Sensitive wrt information provided by the collision detection engine

4. There is some hand-waving when it comes to arbitrary shapes and the fact that the friction force is a multi-valued function
DEM, Further Reading


Gravity-driven Dense Granular Flows

D. Ertas, G. S. Grest, T. C. Halsey, D. Levine and L. E. Silbert
Paper Overview


- Analyzes dense granular flows on an incline with a rough bottom
- Inter-particle interactions between spheres modeled using linear damped spring or Hertzian force laws.
- 2D and 3D analysis

- Main obstacle in simulation: reaching and maintaining steady state
- Periodic and no slip boundary conditions (BCs) are imposed
  - Can’t deal with too many particles, from where the use of periodic BCs
- Side-wall effects are avoided
The Frictional Contact Model

- Consider two contacting bodies at $r_1$ and $r_2$, with velocities $v_1$ and $v_2$, and angular velocities $\omega_1$ and $\omega_2$

- Notation used:
  
  \[
  r_{12} = r_1 - r_2 \quad \hat{r}_{12} = \frac{r_{12}}{\|r_{12}\|} \quad v_{12} = v_1 - v_2
  \]

  $\delta$ – normal compression
  
  $v_n$ – normal velocity
  
  $v_t$ – relative surface velocity

- Quantities of interest
  
  $\delta = d - \|r_{12}\|$
  
  $v_n = (v_{12} \cdot \hat{r}_{12}) \hat{r}_{12}$
  
  $v_t = v_{12} - v_n - (\omega_1 + \omega_2) \times \frac{r_{12}}{2}$
Gravity-driven Dense Granular Flows

- $u_t$ – rate of change of elastic tangential displacement:

\[
\frac{du_t}{dt} = v_t - \frac{(u_t \cdot v_{12})r_{12}}{\|r_{12}\|^2}
\]

- Notation used: $k_n$ and $k_s$, and $\tau_n$ and $\tau_s$ are elastic constants and viscoelastic relaxation times for the normal and tangential components of the force. These are model parameters, you should somehow come up with them; not trivial, there is a lot of guessing involved (calibration)

- Normal Force $F_n$ computed as

\[
F_n = f\left(<\delta\frac{\delta}{d}\right)(\delta k_n \dot{r}_{12} - \tau_n v_n)
\]

- Tangential Force $F_t$ computed as

\[
F_t = f\left(<\delta\frac{\delta}{d}\right)(-k_s u_t - \tau_s v_t)
\]

  - Function $f(x)$ selects between two possible models:

\[
f(x) = \begin{cases} 
1, & \text{for damped liner springs} \\
\sqrt{x}, & \text{for Hertzian contacts}
\end{cases}
\]
Gravity-driven Dense Granular Flows

- Coulomb yield criterion, $\mathbf{F}_t \leq \mu \mathbf{F}_n$, is satisfied by truncating $\mathbf{u}_t$

- Although there is a difference between $\mu_k$ (kinematic friction coefficient) and $\mu_s$ (static friction coefficient), the implementation works with only one value of the friction coefficient
  - Note that typically, $\mu_k \leq \mu_s$

- $k_{n,s}$, $\tau_{n,s}$ depend on the elastic moduli and diameter of particles

- In addition to the contact/friction force, you have to consider all other forces acting on the bodies (gravity, for instance)

- When dealing with large collection of granular materials, it makes sense to talk about the stress tensor in cell $k$ (with volume $V_k$)

$$\sigma_{\alpha\beta} = \frac{1}{V_k} \sum_{i \in V_k} \left[ \sum_{j \neq i} \frac{r_{ij}^\alpha F_{ij}^\beta}{2} + m_i (v_i^\alpha - \bar{v}^\alpha) (v_i^\beta - \bar{v}^\beta) \right]$$

- Notation:

  \[ F_{ij}^\beta = F_{n_{ij}}^\beta + F_{t_{ij}}^\beta \quad \Rightarrow \quad \text{frictional contact force acting between } i \text{ and } j \]

  \[ \bar{v}^\alpha \quad \Rightarrow \quad \text{direction } \alpha \text{ time averaged velocity of particles in volume } V \]
Radial and axial segregation of granular matter in a rotating cylinder

D. C. Rapaport
Overview


- Uses same DEM-P concept, yet the way the forces (normal and tangential) are calculated is different
  - Goes to say that there is no one way of computing them, tweaking usually involved in the process

- Handling mixture of granular particles of two different species
Normal Force Model

• For a pair of granular bodies $i$ and $j$ with diameters $d_i$ and $d_j$ the repulsive force between particles is

$$f_v = k_n (d_{ij} - |\mathbf{r}_{ij}|) \hat{\mathbf{r}}_{ij}, \quad |\mathbf{r}_{ij}| < d_{ij}$$

  - Notation:

  $$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \quad \& \quad \hat{\mathbf{r}}_{ij} = \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|}$$

  $$d_{ij} = \frac{(d_i + d_j)}{2}$$

  $$k_n = \text{normal stiffness coefficient}$$

• We also have a velocity dependent damping force:

$$f_d = -\gamma_n (\hat{\mathbf{r}}_{ij}.\mathbf{v}_{ij}) \hat{\mathbf{r}}_{ij}$$

  - Notation:

  $$\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$$

  $$\gamma_n = \text{damping coefficient}$$

• TOTAL normal force between particles:

$$f_n = f_v + f_d$$
Tangential Force Model

- **Frictional damping force** (acts transversely at the point of contact to oppose sliding while particles are within interaction range):
  \[
  f_s = -\min(\gamma_s^{C_iC_j} |v^s_{ij}|, \mu^{C_iC_j} |f_n|)\dot{v}^s_{ij}
  \]
  - Notation used:
    \[
    v^s_{ij} = v_{ij} - (\dot{r}_{ij} \cdot v_{ij})\dot{r}_{ij} - \left(\frac{d_i\omega_i + d_j\omega_j}{d_i + d_j}\right) \times r_{ij}
    \]
    \[
    \omega_i = \text{body angular velocity}
    \]
    \[
    \mu^{C_iC_j} = \text{static friction coefficient}
    \]
    \[
    \gamma_s^{C_iC_j} = \text{damping coefficient}
    \]

- **Tangential restoring force** (acts during collision and depends on cumulative relative displacement):
  \[
  f_g = -k_g u_{ij}
  \]
  - Notation used and remarks:
    \[
    u_{ij} = \int_{T_{coll}} v^s_{ij}(\tau) d\tau
    \]
    \[
    ||f_g|| \leq \mu^{C_iC_j} ||f_n|| \quad \rightarrow \quad \text{provides upper limit on value of } ||f_g||
    \]

- **Total transverse force**:
  \[
  f_t = f_s + f_g
  \]
The “Complementarity” Approach
aka
Differential Variational Inequality (DVI) Method
Two Shapes, and the Distance [Gap Function]

- Notation: $\partial A$ represents set of points making up the boundary of body $A$
- Shape body $A$: collection of points $S$ with $r_A^S = r_A + A_A s_A^S$, $s_A^S \in \partial A$
- Shape body $B$: collection of points $S$ with $r_B^S = r_B + A_B s_B^S$, $s_B^S \in \partial B$

Distance function in a given configuration $q_A$ and $q_B$

$$\Phi(q_A(t), q_B(t)) \equiv \min_{s_A^S \in \partial A, s_B^S \in \partial B} \|r_A^S - r_B^S\|_2$$

Contact when distance function is zero

$$\Phi(q_A(t^*), q_B(t^*)) = 0$$
General Comments, DVI

- Differential Variational Inequality (DVI): a set of differential equations that hold in conjunction with a collection of constraints

  - Recall the constrained equations of motion we dealt with: we had the Newton-Euler equations of motion

  - Their solution also satisfied a set of kinematic constraints coming from joints
    - These constraints are called bilateral constraints

  - When dealing with contacts, the non-penetration condition will be captured as a unilateral constraint. That is,
    - At point of contact, relative to body 1, body 2 can move outwards, but not inwards

  - The variational attribute stems from the optimization problem approach embraced to pose the Coulomb friction model
Nomenclature: in the past we worked with kinematic constraints; we’ll be more specific now and call these constraints bilateral constraints. In DVI we also have non-penetration constraints, which are unilateral constraints and assume the form of inequalities.

Notation: We’ll call $\mathcal{A}$ the set of all active unilateral constraints present in the system. Think of these as active contacts. They’ll be denoted by

$$\Phi_i(q) \quad i \in \mathcal{A}$$

Note that the nonpenetration condition is expressed as (the distance between two bodies should also be positive)

$$\Phi_i(q) \geq 0, \quad i \in \mathcal{A}$$

Notation: We’ll call $\mathcal{B}$ the set of all bilateral constraints present in the system. These expression of these constraints will be denoted by $\Psi(q, t)$. Just like before we have that

$$\Psi_i(q, t) = 0, \quad i \in \mathcal{B}$$

Remark: While the bilateral constraints typically don’t change in time (a spherical joint stays a spherical joint throughout the simulation), the unilateral constraints appear and disappear; i.e., contacts are made and then broken. In other words, $\mathcal{A}$ depends on the state $q$ of the system.
DVI-Based Methods: Notation Used

- The DVI approach presented will draw on the $\mathbf{r} - \bar{\omega}$ formulation.

- Notation:

\[
\mathbf{q} = \begin{bmatrix}
\mathbf{r}_1 \\
\mathbf{p}_1 \\
\vdots \\
\mathbf{r}_{nb} \\
\mathbf{p}_{nb}
\end{bmatrix}
\quad \dot{\mathbf{q}} = \begin{bmatrix}
\dot{\mathbf{r}}_1 \\
\dot{\mathbf{p}}_1 \\
\vdots \\
\dot{\mathbf{r}}_{nb} \\
\dot{\mathbf{p}}_{nb}
\end{bmatrix}
\quad \mathbf{v} = \begin{bmatrix}
\dot{\mathbf{r}}_1 \\
\bar{\omega}_1 \\
\vdots \\
\dot{\mathbf{r}}_{nb} \\
\bar{\omega}_{nb}
\end{bmatrix}
\]

- Recall that $\dot{\mathbf{p}}_i = \frac{1}{2} \mathbf{G}^T(p_i)\bar{\omega}_i$, where

\[
\mathbf{G}(p_i) = \begin{bmatrix}
-e_{1,i} & +e_{0,i} & +e_{3,i} & -e_{2,i} \\
-e_{2,i} & -e_{3,i} & +e_{0,i} & +e_{1,i} \\
-e_{3,i} & +e_{2,i} & -e_{1,i} & +e_{0,i}
\end{bmatrix}.
\]
Body A – Body B Contact Scenario
Defining the Normal and Tangential Forces

- What happens when a contact occurs?
- A point of contact is identified and a local reference frame is defined (see picture on previous slide). Axes of the local reference frame:
  - \( \mathbf{u}_i \) and \( \mathbf{w}_i \) are two mutually perpendicular unit vectors in the tangent plan at the contact point
  - \( \mathbf{n}_i \) - unit vector, defines the normal direction in the local reference frame. Convention: it points towards the interior of the body

- A normal force appears along the direction normal to the plane of contact
  - Magnitude of the force is \( \hat{\gamma}_{i,n} \). Specifically,
    \[
    \mathbf{F}_{i,N} = \hat{\gamma}_{i,n} \mathbf{n}_i
    \]

- A friction force appears in the tangent plane.
  - This force in the tangent plane has two components along the axes \( \mathbf{u}_i \) and \( \mathbf{w}_i \): \( \hat{\gamma}_{i,u} \) and \( \hat{\gamma}_{i,w} \), respectively. Specifically,
    \[
    \mathbf{F}_{i,T} = \hat{\gamma}_{i,u} \mathbf{u}_i + \hat{\gamma}_{i,w} \mathbf{w}_i
    \]

- NOTE: The point of contact, \( \mathbf{n}_i \), \( \mathbf{u}_i \), and \( \mathbf{w}_i \) are obtained at the end of the collision detection task, which is executed at the beginning of each time step
DVI-Based Methods
The Contact Model

- The contact model is as follows: a contact is modeled by one inequality constraint, which states that either the distance between two bodies is greater than zero \( \Phi_i(q) > 0 \), in which case the normal force is zero \( \hat{\gamma}_{i,n} = 0 \), or vice-versa; i.e., if the distance is zero, the contact force is nonzero.

  - Condition above captured in the following complementarity condition:

    \[
    \hat{\gamma}_{i,n} \geq 0, \quad \Phi_i(q) \geq 0, \quad \Phi_i(q)\hat{\gamma}_{i,n} = 0,
    \]

  - Another way to state the complementarity condition:

    \[
    0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(q) \geq 0
    \]
DVI-Based Methods: The Friction Model

- The friction model considered is Coulomb’s:

\[
\mu_i \hat{\gamma}_{i,n} \geq \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2}
\]

\[
F_{i,T}^T \cdot v_{i,T} = -\|F_{i,T}\| \|v_{i,T}\|
\]

\[
\|v_{i,T}\| \left( \mu_i \hat{\gamma}_{i,n} - \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \right) = 0
\]

- First condition states that the friction force is within the friction cone
- Second condition states that the friction force and the velocity between two bodies are collinear and of opposite direction
- The third condition captures the stick-slip condition. If the velocity is greater than zero, it means that the friction force saturated; i.e., \( \mu_i \hat{\gamma}_{i,n} = \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \); this is the sliding scenario. Conversely, if the bodies stick to each other, then the relative tangential velocity is zero, \( v_{i,T} = 0_3 \), and the friction force is not saturated \( \mu_i \hat{\gamma}_{i,n} > \sqrt{\hat{\gamma}_{i,u}^2 + \hat{\gamma}_{i,w}^2} \).
Coulomb’s Model Posed as the Solution of an Optimization Problem

- The First Key Twist (out of three)

- Assume that $\hat{\gamma}_{i,n}$ and $v_{i,T}$ are given and you pose the following optimization problem in variables $x$ and $y$:

  - Minimize the function $v_{i,T}^T (xu_i + yw_i)$ subject to the constraint $\sqrt{x^2 + y^2} \leq \mu_i \hat{\gamma}_{i,n}$

- If you pose the first order Karush-Kuhn-Tucker optimality conditions for this optimization problem you end up precisely with the set of three conditions that define the Coulomb friction model

- It follows that $\hat{\gamma}_{i,u}$ and $\hat{\gamma}_{i,w}$, that is, two out of the three multipliers unknown are computed as the solution of an optimization problem that can be posed as soon as $\hat{\gamma}_{i,n}$ and $v_{i,T}$ are available

- Using math lingo, we have that

  $$(\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \text{argmin}_{\sqrt{x^2+y^2}\leq\mu_i\hat{\gamma}_{i,n}} v_{i,T}^T (xu_i + yw_i).$$
The DVI Problem:  
The EOM, in Fine Granularity Form

- The time evolution of the dynamical system is governed by the following differential variational inequality (DVI)

\[
B = 1, \ldots, nb : \quad m_B \ddot{\mathbf{r}}_B = \sum_{i \in \mathcal{B}(B)} \left[ \Psi_r^{(i)} \right]^T \dot{\mathbf{r}}^i, + f_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in \mathcal{A}(B)} \left( \dot{\gamma}_{i,n} \mathbf{n}_i + \dot{\gamma}_{i,u} \mathbf{u}_i + \dot{\gamma}_{i,w} \mathbf{w}_i \right)
\]

\[
B = 1, \ldots, nb : \quad \ddot{\mathbf{J}}_B \dot{\mathbf{\omega}}_B = \sum_{i \in \mathcal{B}(B)} \dot{\mathbf{\Pi}}_B^T(\Psi^{(i)}) \mathbf{\dot{r}}^i, + \tau_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in \mathcal{A}(B)} \dot{s}_{i,B} A_B^T(\dot{\gamma}_{i,n} \mathbf{n}_i + \dot{\gamma}_{i,u} \mathbf{u}_i + \dot{\gamma}_{i,w} \mathbf{w}_i)
\]

\[
B = 1, \ldots, nb : \quad \dot{\mathbf{p}}_B = \frac{1}{2} G^T(\mathbf{p}_B) \ddot{\mathbf{\omega}}_B
\]

\[
i \in \mathcal{B} : \quad \Psi_i(\mathbf{q}, t) = 0
\]

\[
i \in \mathcal{A} : \quad 0 \leq \dot{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \geq 0,
\]

\[
i \in \mathcal{A} : \quad (\dot{\gamma}_{i,u}, \dot{\gamma}_{i,w}) = \arg\min_{\mu_i \dot{\gamma}_{i,n} \geq \sqrt{x^2 + y^2}} \mathbf{v}^T (x \mathbf{D}_{i,u} + y \mathbf{D}_{i,w})
\]
Notation Conventions

[1/2]

- To stick with the presentation in the paper of Anitescu and Tasora, we’ll use the following notation

  - Point 1: Instead of sticking with transpose of Jacobians, we’ll use gradients, which are defined precisely as the transpose of the Jacobians. Specifically,

    \[
    \nabla_q \Psi_i = \Psi_{i,q}^T = \left[ \frac{\partial \Psi_i}{\partial \mathbf{q}} \right]^T \\
    \nabla_q \Phi_i = \Phi_{i,q}^T = \left[ \frac{\partial \Phi_i}{\partial \mathbf{q}} \right]^T
    \nabla_q \Psi_i = \Psi_{i,q}^T = \left[ \frac{\partial \Psi_i}{\partial \mathbf{q}} \right]^T
    \]

  - Point 2: We’ll use the transformation matrix \( \mathbf{L}(\mathbf{q}) \) to link the time derivative of the level zero unknowns in the \( \mathbf{r} - \mathbf{p} \) formulation to the level one unknowns in the \( \mathbf{r} - \mathbf{\omega} \) formulation:

    \[
    \dot{\mathbf{q}} = \mathbf{L}(\mathbf{q}) \mathbf{v}
    \]

  - Point 3: To keep the notation simpler (and probably confuse you), we’ll group the translational and rotational equations of motion in one big matrix-vector equation (nothing changed, except the notation) in order to have less symbols and equations to deal with

  - Point 4: We’ll use the following notation (\( h \) is the integration step-size)

    \[
    \gamma_{i,n} = h \gamma_{i,n} \\
    \gamma_{i,u} = h \gamma_{i,u} \\
    \gamma_{i,w} = h \gamma_{i,w} \\
    \gamma_{i,\mathbf{w}} = h \gamma_{i,\mathbf{w}} \\
    \gamma_{i,b} = h \gamma_{i,b}
    \]

    * Recall that time \( \times \) force (like in \( \gamma_{i,n} = h \gamma_{i,n} \)) is impulse, and it’s impulse that changes the momentum of a body
• Define the transformation matrix $A_{i \rightarrow G}$ that given the representation of a geometric vector in the contact reference frame associated with contact $i$ is used to generate its representation in the GRF:

\[
A_{i \rightarrow G} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix}
\]

- Note that the frictional contact force at contact $i$ as felt by body $A$ is simply

\[
F_{f_c}^{t,A} = n_i \hat{\gamma}_{i,n} + u_i \hat{\gamma}_{i,u} + w_i \hat{\gamma}_{i,w} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix} \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix} = A_{i \rightarrow G} \cdot \hat{\gamma}_i \quad \text{where} \quad \hat{\gamma}_i = \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix}
\]

• A projection matrix $D_i$ is defined for each contact $i \in \mathcal{A}$ to project the contact forces onto the equations of motion, both for translation and rotation. If we assume that contact $i$ acts between body $A$ and body $B$,

\[
D_i \equiv \begin{bmatrix}
0 \\
\vdots \\
A_{i \rightarrow G} \\
\tilde{s}_{i,A} A_{A,i \rightarrow G} \\
\vdots \\
0 \\
-A_{i \rightarrow G} \\
-\tilde{s}_{i,B} A_{B,i \rightarrow G} \\
0
\end{bmatrix}_{6nb \times 3}
\]

• Notation used in expression of $D_i$: the vectors $\tilde{s}_{i,A}$ and $\tilde{s}_{i,B}$ represent the location of the contact point in the local reference frame of body $A$ and $B$, respectively

• The columns of $D_i$ are denoted by $D_{i,n}$, $D_{i,u}$, $D_{i,w}$ and are each vectors of dimension $6nb$:

\[
D_i = \begin{bmatrix} D_{i,n} & D_{i,u} & D_{i,w} \end{bmatrix}_{6nb \times 3}
\]
Frictional Contact: The Problem Setup

- The resulting problem that we have to deal with now looks like this

\[
\begin{align*}
\dot{q} &= \mathbf{L}(q)v \\
M\dot{\mathbf{v}} &= \mathbf{f}(t, q, \mathbf{v}) + \sum_{i \in \mathcal{B}} \tilde{\gamma}_{i,b} \nabla \Psi_i + \sum_{i \in \mathcal{A}} (\tilde{\gamma}_{i,n} \mathbf{D}_{i,n} + \tilde{\gamma}_{i,u} \mathbf{D}_{i,u} + \tilde{\gamma}_{i,w} \mathbf{D}_{i,w})
\end{align*}
\]

\[i \in \mathcal{B} : \quad \Psi_i(q, t) = 0\]

\[i \in \mathcal{A} : \quad 0 \leq \tilde{\gamma}_{i,n} \perp \Phi_i(q) \geq 0,\]

\[\left(\tilde{\gamma}_{i,u}, \tilde{\gamma}_{i,w}\right) = \arg\min_{\mu_i \tilde{\gamma}_{i,n} \geq \sqrt{x^2 + y^2}} v^T (x \mathbf{D}_{i,u} + y \mathbf{D}_{i,w})\]
The Discretization Process

- Recall that in our Ford F-150 direction solution, we used the Newton-Euler form of the equations of motion in conjunction with the level zero constraints (the position constraint equations).

- In the proposed approach to find the time evolution of the multi-body system we’ll use instead the level one constraints (velocity level constraints).

- Implications:

  - Since the level zero constraints are not enforced, there will be drift in the solution.
  - Stabilization terms, that penalize the violation of the level zero constraints, are added to the level one bilateral and unilateral constraints.
  - The bilateral and unilateral constraints are massaged into the following (note that a superscript \((l)\) denotes the time step; used to be a subscript, typically \(n\), yet \(n\) stands for ’normal’ now):

\[
\begin{align*}
    i \in \mathcal{B} & : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0 \\
    i \in \mathcal{A} & : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D^T_{i,n} v^{(l+1)} \geq 0 .
\end{align*}
\]

  * The above reformulation becomes the Second Key Twist (out of three).
  * There is much to be said here, modifying the expression of the constraints and working with level one constraints are two issues that demand more attention than what we pay here.
The Discretization Process

- The discretized equations look like this:

\[ M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla\Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w}) \]

\[ i \in B \quad : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla\Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0 \]

\[ i \in A \quad : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0 \]

\[ (\gamma_{i,u}, \gamma_{i,w}) = \arg\min_{\mu_i \gamma_{i,n} \geq \sqrt{x^2 + y^2}} v^T (x D_{i,u} + y D_{i,w}) \]

\[ q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}. \]

- First four equations above combine to form an optimization problem with equilibrium constraints

- Why an optimization problem?
  - Because of the way the Coulomb friction model is posed

- What type of optimization problem?
  - This represents a nonlinear optimization problem
  - Can be linearized if the friction cone is discretized and represented as a multifaceted pyramid (bad idea: problem size increases, anisotropy creeps in)

- What are the 'equilibrium constraints'?
  - Your typical optimization problem might display algebraic equality or inequality constraints
  - Above, we are solving an optimization problem for which the constraints represent the discretization of a set of differential equations
The NCP → CCP Metamorphosis

The Third Key Twist (out of three)

• Dealing with some generic nonlinear optimization problem like the one above is daunting

• Can we do some trick and recast it as a simpler optimization problem for which (i) we are guaranteed that a solution exists (ideally, it would be unique, in some sense), and (ii) there are tailored algorithms that we can use to efficiently find the solution

• Coming from the left field, it turns out that if you introduce a relaxation over the complementarity constraints the problem that you have to solve can be posed as a cone complementarity problem (CCP). To this end, rather than working with

\[ i \in \mathcal{A} : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D^T_{i,n} v^{(l+1)} \geq 0 \]

Work with this:

\[ i \in \mathcal{A} : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D^T_{i,n} v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0 \]
The Cone Complementarity Problem

- The resulting problem that we have to deal with now looks like this

\[
M(\mathbf{v}^{(l+1)} - \mathbf{v}^{(l)}) = hf(t^{(l)}, \mathbf{q}^{(l)}, \mathbf{v}^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla \Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B \quad : \quad \frac{1}{h} \Psi_i(\mathbf{q}^{(l)}, t) + \nabla \Psi_i^T \mathbf{v}^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
\]

\[
i \in A \quad : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(\mathbf{q}^{(l)}) + D_{i,n}^T \mathbf{v}^{(l+1)} - \mu_i \sqrt{(\mathbf{v}^T D_{i,u})^2 + (\mathbf{v}^T D_{i,w})^2} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \text{argmin}_{\mu_i \gamma_{i,n} \geq \sqrt{x^2+y^2}} \mathbf{v}^T (x D_{i,u} + y D_{i,w})
\]

\[
\mathbf{q}^{(l+1)} = \mathbf{q}^{(l)} + hL(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)}.
\]
Cone Complementarity Problem (CCP)

- The new discretization that draws on the modified complementarity condition converges when $h \to 0$ to the solution of the original discretization.

- After some more massaging, with the exception of the very last one, the equations on the previous slide combine to lead to the following CCP (for the intermediates steps that you need to get to the CCP see paper of Anitescu and Tasora or come and talk to me):

  - Introduce the convex hypercone...

    \[
    \mathcal{Y} = \left( \bigoplus_{i \in \mathcal{A}(q^{(l)})} \mathcal{F}C^i \right) \bigoplus \left( \bigoplus_{i \in \mathcal{B}(q^{(l)})} \mathcal{B}C^i \right)
    \]

    where \( \mathcal{F}C^i \) is the \( i \)-th friction cone \( \mathcal{B}C^i \) is \( \mathbb{R} \)

  - ... and its polar hypercone

    \[
    \mathcal{Y}^o = \left( \bigoplus_{i \in \mathcal{A}(q^{(l)})} \mathcal{F}C^{i_o} \right) \bigoplus \left( \bigoplus_{i \in \mathcal{B}(q^{(l)})} \mathcal{B}C^{i_o} \right)
    \]

- The CCP that needs to be solved at each time step is as follows (note that matrix \( \mathbf{N} \) and vector \( \mathbf{d} \) are computed based on state information at time-step \( t^{(l)} \)):
  
  * Find the Lagrange hyper-multiplier \( \gamma \) that satisfies:

    \[
    \gamma \in \mathcal{Y} \perp -(\mathbf{N}\gamma + \mathbf{d}) \in \mathcal{Y}^o
    \]
The Optimization Angle

- CCP represents first order optimality condition of a quadratic problem with conic constraints

\[
\begin{align*}
N &= D^T M^{-1} D \\
r &= b + D^T M^{-1} k \\
\gamma &= [\gamma_1^T, \gamma_2^T, \ldots, \gamma_{N_c}^T]^T \in \mathbb{R}^{3N_c} \\
\gamma^* &= \arg\min_{\gamma \in \mathcal{T}_i, 1 \leq i \leq N_c} \left( \frac{1}{2} \gamma^T N\gamma + r^T \gamma \right)
\end{align*}
\]

- \(N \in \mathbb{R}^{3N_c \times 3N_c}\) is symmetric and positive semi-definite
- \(N\) and \(r \in \mathbb{R}^{3N_c}\) do not depend on \(\gamma\). They are computed once at beginning of each time step
- Problem has a global solution \(\gamma^*\)
- Problem doesn’t have a unique solution
Wrapping it Up, Complementarity Approach

- Life becomes simple once the frictional contact forces at the interface between shapes are available
  - Velocity at new time step $l + 1$ computed as
    \[
    v^{(l+1)} = M^{-1} (k + D\gamma)
    \]
  - Once velocity available, the new set of generalized coordinates computed as
    \[
    q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}
    \]
Complementarity Approach: Putting Things in Perspective

- Perform collision detection
- Formulate equations of motion; i.e., pose DVI problem
- DVI discretized to lead to nonlinear complementarity problem (NCP)
- Relax NCP to get CCP
- Equivalently, solve QP with conic constraints to compute $\gamma$
- Once friction and contact forces available, velocity available
- Once velocity available, positions are available (numerical integration)
3 Second Dynamics – 1 million spheres dropping in a bucket [Commercial Software Simulation]
1 Million Bodies, Parallel Simulation on GPU card

- 20 second long simulation
- Two hours to finish simulation
- GPU Card: GTX680
- Optimization problem for $\gamma$
  - Approx. 4 million variables
  - Solved at each time step
  - Problem looks like

$$
\gamma^* = \arg\min_{\gamma_i \in \gamma_i, 1 \leq i \leq N_c} \left( \frac{1}{2} \gamma^T N \gamma + r^T \gamma \right)
$$

The Make-or-Break Ingredient: Constrained QP Solver

\[ \gamma^* = \arg\min_{\gamma_i \in T_i, 1 \leq i \leq N_c} \left( \frac{1}{2} \gamma^T N \gamma + r^T \gamma \right) \]
Complementarity Approach: First Order Solution Methods

- Existing methods
  - Jacobi
  - Gauss-Seidel

- Methods investigated in the lab
  - Gradient Projected Minimum Residual [FRICTIONLESS ONLY]
  - Modified Proportioning with Reduced Gradient Projection [FRICTIONLESS ONLY]
  - Preconditioned Spectral Projected Gradients with Fallback (Barzilai-Borwein)
  - Kucera-Dostal
  - Nesterov Method (Accelerated Projected Gradient Descent – APGD)

New Solution Algorithm: APGD – Built around Nesterov

- APGD: Accelerated Projected Gradient Descent
  - Idea: instead of descending along the gradient, use a linear combination of previous descent directions

- Proved to be, up to a factor $c$, the best first order optimization method
  - Convergences like $O(1/k^2)$ as opposed to $O(1/k)$

- Projected version implemented owing to presence of conic constraints

Performance Results: Jacobi vs. GS vs. APGD

- Benchmark Problem: 4000 rigid spheres
- Heavy block/slab rests on packed spheres
- Results obtained at one step
- Mass of block varied
  - $10^3$ kg to $10^6$ kg
- The three-way race: how far can you get in 1000 iterations when solving the QP?

$$\gamma^* = \arg\min_{\gamma_i \in \gamma_i, 1 \leq i \leq N_c} \left( \frac{1}{2} \gamma^T N \gamma + r^T \gamma \right)$$
Performance Results: Jacobi vs. GS vs. APGD

\[ \gamma^* = \underset{\gamma_i \in \mathcal{T}_i}{\text{argmin}} \left( \frac{1}{2} \gamma^T \mathbf{N} \gamma + \mathbf{r}^T \gamma \right) \]

<table>
<thead>
<tr>
<th>Mass [kg]</th>
<th>Jacobi</th>
<th>Gauss Seidel</th>
<th>APGD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 \times 10^3</td>
<td>-28.29</td>
<td>-117.70</td>
<td>-220.14</td>
</tr>
<tr>
<td>1 \times 10^4</td>
<td>-35.63</td>
<td>-162.99</td>
<td>-883.54</td>
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<tr>
<td>1 \times 10^5</td>
<td>-37.02</td>
<td>-176.94</td>
<td>-3199.27</td>
</tr>
<tr>
<td>1 \times 10^6</td>
<td>-37.15</td>
<td>-210.23</td>
<td>-4696.48</td>
</tr>
</tbody>
</table>

Performance Results: Jacobi vs. GS vs. APGD

- Mass of block is 1000 kg
- The three-way race:
  - How much effort does it take to converge the solution within a $7 \times 10^{-6}$ tolerance

$$\gamma^* = \arg\min_{\gamma \in \mathbb{R}_i, 1 \leq i \leq N_c} \left( \frac{1}{2} \gamma^T N \gamma + r^T \gamma \right)$$

<table>
<thead>
<tr>
<th>Solver</th>
<th>Residual</th>
<th>Iterations</th>
<th>Time[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jacobi</td>
<td>$7.54 \times 10^{-6}$ (UtC)</td>
<td>500 000</td>
<td>24 300</td>
</tr>
<tr>
<td>Gauss Seidel</td>
<td>$6.99 \times 10^{-6}$</td>
<td>11 485</td>
<td>494.8</td>
</tr>
<tr>
<td>APGD</td>
<td>$6.97 \times 10^{-6}$</td>
<td>202</td>
<td>10.6</td>
</tr>
</tbody>
</table>
General Comments: Penalty and DVI

- There is hand waving when it comes to handling friction and contact
  - Both in Penalty and DVI

- Handling frictional contact is equally art and science
  - To get something to run robustly requires tweaking
  - Takes some time to understand strong/weak points of each approach

- Doing justice to this topic would require several more lectures

- Continues to be an active area of research


~ Rigid Body Dynamics ~
Putting Things in Perspective
ME751 – Topics Covered so Far [1/2]

- 3D vectors and locating points attached to moving rigid bodies
- Describe the orientation of a body in 3D space
- Express geometric constraints associated with the relative motion of two bodies (four building blocks – DP1, DP2, D, CD)
- Kinematics Analysis, carried out for zero DOF systems in which one or more motions are prescribed
  - Position Analysis (requires solution of nonlinear system)
  - Velocity Analysis (requires solution of linear system)
  - Acceleration Analysis (requires solution of linear system)
ME751 – Topics Covered so Far [2/2]

- Formulate EOM for a system of interconnected and mutually interacting rigid bodies

- Discussed numerical solution of the Dynamics problem
  - Dealing w/ differential-algebraic equations (DAEs)
  - The workhorse was the BDF family of implicit integration formulas
  - We used the F-150 direct approach but more sophisticated approaches exist

- Briefly discussed the friction and contact for rigid bodies
  - Penalty approach – simple, integration step-sizes are very short
  - Complementarity approach – involved, integration step-sizes are long though
ME751: Looking Ahead

- Multi-body systems with deformable (compliant) bodies
  - Floating frame of reference formulation
  - Absolute nodal coordinate formulation (ANCF)

- Covered in nine lectures by Antonio
  - Starting Monday, wrapping up on November 11
ME751: Things We’ll Not Cover

- We didn’t do justice to the frictional contact problem
  - Would require a couple of lectures

- Real-time simulation
  - Useful in controls, gaming/virtual reality applications

- Fluid-Solid Interaction Problems
  - Smoothed Particle Hydrodynamics (SPH)
  - Broad spectrum of applications

- Advanced computing aspects
  - GPU computing for granular dynamics and SPH
  - Parallel computing at large