ME751
Advanced Computational Multibody Dynamics

October 10, 2016

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Quotes of the Day

[from Luning]

"They who know the truth are not equal to those who love it, and they who love it are not equal to those who delight in it." [Confucius] 知之者不如好之者，好之者不如乐之者

"To study and not think is a waste. To think and not study is dangerous." [Confucius] 学而不思则罔，思而不学则殆

"The virtuous man is driven by responsibility, the non-virtuous man is driven by profit." [Confucius] 君子喻於義，小人喻於利

"Nature does not hurry, yet everything is accomplished." [Lao Tzu] 无为而为之

"The highest good is like that of water. The goodness of water is that it benefits the ten thousand creatures; yet itself does not scramble, but is content with the places that all men disdain." [Lao Tzu] 上善若水，水善利万物而不争

Song of the day: "Twinkle, Twinkle, Little Star" and "Five Little Monkeys"

Before we get started…

- Last Time:
  - Loose ends, the $\mathbf{r} - \mathbf{p}$ formulation of the EOM
  - Super briefly talk about the EOM when using Euler Angles
  - Discussed TSDAs and RSDAs

- Today:
  - Simple example of deriving the EOM for a one body system
  - Inverse Dynamics Analysis
  - Equilibrium Analysis
  - Elements of the numerical solution of ordinary differential equations

- Reading:
  - Ed Haug's textbook: 11.5

[AO]
Example: EOM for a Dangling Cube

- See handout, available also online
- Units are all SI
- Cube of mass 6, length of edge is 2
- Hanging from a corner at point P
- A force applied at opposite corner, at point Q
- Moving under gravity $\mathbf{g}$

- What's the work order?
  - Formulate the EOM using the $\mathbf{r} - \boldsymbol{\omega}$ formulation
  - Get the linear system whose solution provides accelerations and Lagr. multipliers

End EOM
Beginning Inverse Dynamics Analysis

[New Topic]

Inverse Dynamics: The idea

- First of all, what does dynamics analysis mean?
  - You apply some forces/torques on a mechanical system and look at how the configuration of the mechanism changes in time
  - How it moves also depends on the ICs associated with that mechanical system

- In *inverse* dynamics, the situation is quite the opposite:
  - You specify a motion of the mechanical system and you are interested in finding out the set of forces/torques that were actually applied to the mechanical system to lead to this motion

- When is *inverse* dynamics useful?
  - Useful in controls.
    - Example – controlling the motion of a robot: you know how you want this robot to move; need to figure out what joint torques you should apply to make it move the way it should

Inverse Dynamics: The Math

- When can one talk about Inverse Dynamics?
  - Given a mechanical system, a prerequisite for Inverse Dynamics is that the number of degrees of freedom associated with the system is **zero**
    - You have as many generalized coordinates as constraints (THIS IS KEY)
  - This effectively makes the problem a Kinematics problem. Yet the analysis has a Dynamics component since you need to compute reaction forces

- The Process (3 step approach):
  - STEP 1: Solve for the accelerations using *exclusively* the set of constraints (the Kinematics part)
  - STEP 2: Computer next the Lagrange Multipliers using the Newton-Euler form of the EOM (the Dynamics part)
  - STEP 3: Once you have the Lagrange Multipliers, pick the ones associated with the very motions that you specified, and compute the reaction forces and/or torques you need to get the prescribed motion[s]

Inverse Dynamics: The Math

- Following the three steps outlined on the previous slide you can attack the Inverse Dynamics problem in the $\mathbf{r}$–$\bar{\omega}$ formulation, the $\mathbf{r}$–$\mathbf{p}$ formulation, or if you were in ADAMS, in the $\mathbf{r}$–$\bar{\epsilon}$ formulation
- We will demonstrate the Inverse Dynamics using the $\mathbf{r}$–$\bar{\omega}$ formulation (the other formulations are handled 100% identically)
  - STEP 1: Solve linear system for $\ddot{\mathbf{r}}$ and $\dot{\bar{\omega}}$:

$$\begin{bmatrix} \mathbf{\Phi_r} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix}_{6nb \times 6nb} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\bar{\omega}} \end{bmatrix}_{6nb \times 1} = \gamma_{6nb \times 1}$$

  - STEP 2: Solve for the Lagrange Multipliers:

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}}^T \\ \bar{\mathbf{\Pi}}^T(\mathbf{\Phi}) \end{bmatrix} \lambda = - \begin{bmatrix} \mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\ \bar{\mathbf{J}}\dot{\bar{\omega}} - \tau \end{bmatrix}$$

  - STEP 3: Recover the reaction forces and/or torques that should act on each body i so that the system experiences the motion you prescribed:

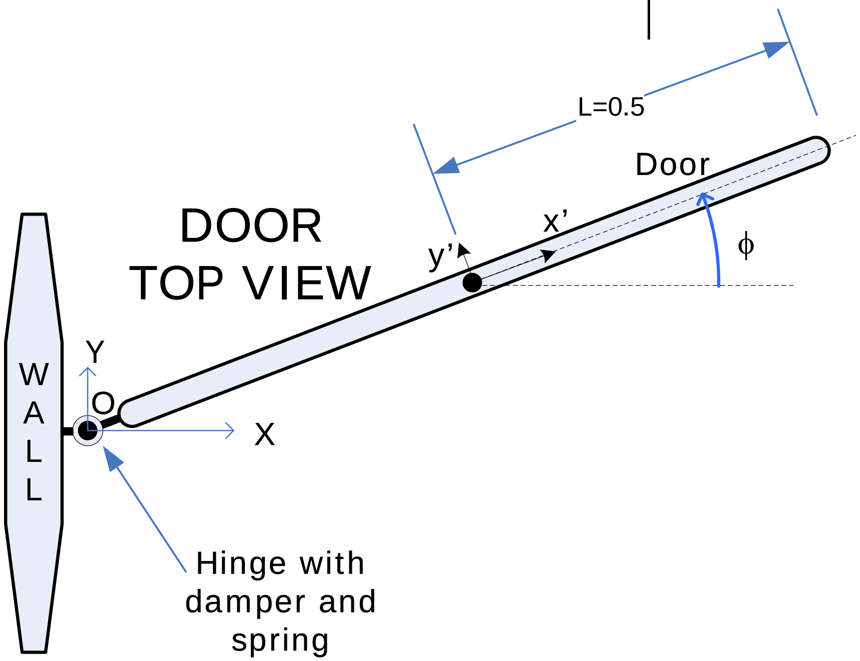
$$\mathbf{F}_i^r = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda \qquad \bar{\mathbf{n}}_i^r = -\bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda$$

[AO, cast as 2D problem]

Example: Inverse Dynamics

- Door Mass m = 30
- Mass Moment of Inertia J' = 2.5
- Spring/damping coefficients:

  K = 8 C = 1
- All units are SI.



- Zero Tension Angle of the spring:

$$\phi_{free} = 0$$

- Compute torque that electrical motor applies to *open* handicapped door
  - Apply motion for two seconds to open the door like

$$\Phi^D(t) = \phi - \frac{\pi}{2}\sin(\frac{\pi}{4}t)$$

End Inverse Dynamics
Beginning Equilibrium Analysis

[New Topic]

Equilibrium Analysis: The Idea

- A mechanical system is in equilibrium if the system is at rest, with zero acceleration

- What are you seeking here?
  - Find the equilibrium configuration $\mathbf{q}$
  - Reaction forces; that is, Lagrange Multipliers, in equilibrium configuration

- As before, it doesn't matter what formulation you use, in what follows we will demonstrate the approach using the $\mathbf{r} - \mathbf{p}$ formulation

- At equilibrium, we have that

$$\ddot{\mathbf{r}} = \dot{\mathbf{r}} = \mathbf{0}_{3\times 1} \qquad \text{and} \qquad \ddot{\mathbf{p}} = \dot{\mathbf{p}} = \mathbf{0}_{4\times 1}$$

Equilibrium Analysis: The Math

- Recall the matrix form **r**–**p** formulation of the EOM:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T \lambda &= \mathbf{F} \\ \mathbf{J}^{\mathbf{P}}\ddot{\mathbf{p}} + \mathbf{\Phi}_{\mathbf{p}}^T \lambda + \mathbf{P}^T \lambda^{\mathbf{P}} &= \hat{\tau} \end{aligned}$$

- At equilibrium, since the velocity and acceleration are zero, the above linear system assumes the form

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}}^T & \mathbf{0}_{3nb \times nb} \\ \mathbf{\Phi}_{\mathbf{p}}^T & \mathbf{P}^T \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda^{\mathbf{P}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\tau} \end{bmatrix}_{7nb \times 1}$$

- Additionally, recall that we have to be in a consistent configuration; i.e., the set of generalized coordinates must satisfy

$$\mathbf{\Phi}^F(\mathbf{r}, \mathbf{p}, t) = \mathbf{0}_{(nc+nb)\times 1}$$

- Counting the equations and unknowns:
  - The number of equations in the red linear system and red nonlinear system: $8nb + nc$
  - The number of unknowns: $8nb + nc \quad \rightarrow \quad 3nb$ for $\mathbf{r}$, $4nb$ for $\mathbf{p}$, nc for λ, nb for $\lambda^{\mathbf{P}}$
  - He have the same number of unknowns as equations
  - Solving the set of equations in blue together with the equations in red amounts to solving a nonlinear system with $8nb + nc$ unknowns. The system is nonlinear since the set of constraints $\mathbf{\Phi}^F(\mathbf{r}, \mathbf{p}, t)$ are nonlinear in $\mathbf{p}$

Equilibrium Analysis:
Closing Remarks

- Consider the velocity kinematic constraint equation and account for the fact that $\dot{\mathbf{q}} = \mathbf{0}_{7nb\times 1}$:

$$\mathbf{\Phi}_{\mathbf{q}} \cdot \dot{\mathbf{q}} = -\mathbf{\Phi}_t \qquad \Rightarrow \qquad \mathbf{0} = -\mathbf{\Phi}_t$$

- In other words, the set of constraints cannot depend *explicitly* on time. Concretely, you cannot have driving constraints in the set of constraints $\mathbf{\Phi}$, which intuitively makes sense, since at equilibrium everything should be at rest and time should not show up in this context

- How do people usually go about solving the equilibrium problem

  - Approach 1 (finicky): solve the nonlinear system on the previous slide. Finicky since providing a good initial guess in Newton-Raphson for Equilibrium Analysis of complex systems is daunting
  - Approach 2 (dumb, but powerful): in order to see where the system stops, add a lot of damping in the system so that you dissipate its energy. Somewhere it'll stop. That's an equilibrium configuration
  - Approach 3 (not that common): pose the equilibrium problem as an optimization problem $\longrightarrow$ find the system configuration that has minimal potential energy. Drawback: works only for conservative systems (not that many...)
  - Approach 4 (mix of 1 and 2): start with 2 and once you get close to the equilibrium point switch to 1. Never tried this.

- Keep things in perspective: the outcome of the equilibrium analysis is the configuration $\mathbf{q}$, and the Lagrange Multipliers. The latter are used to find the reaction forces associated with the geometric constraints in the equilibrium configuration

[AO, cast as 2D problem]

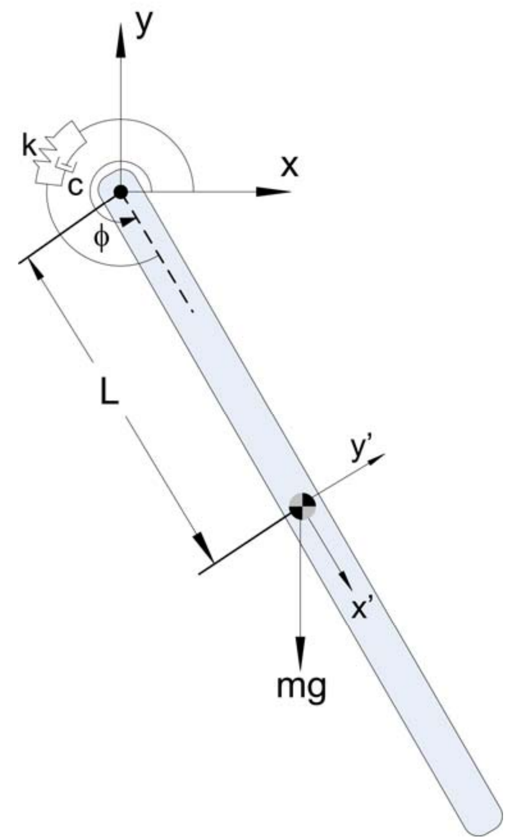
Example: Equilibrium Analysis

- Find the equilibrium configuration of the pendulum below
  - Pendulum connected to ground through a revolute joint and rotational spring-damper element

- Free angle of the spring:

$$\phi_{free} = 0$$

- Spring constant: k=25
- Mass m = 10
- Length L=1
- All units are SI.



End Inverse Dynamics Analysis Start, Numerical Integration Methods

Dynamics Analysis

- Dynamics Analysis, Framework:
  - The state of a mechanical system (position, velocity) changes in time under the influence of internal and external **forces and/or prescribed motions**
  - The goal is to determine how the state of the system changes in time
  - Almost always you will only be able to determine the state of the mechanical system at a collection of grid points in time
    - That is, not everywhere, yet can have as many grid points as you wish (and afford)
  - Time evolution is obtained as the solution of the EOM (Newton-Euler equations derived before)

Dynamics vs. Kinematics

- Kinematics Analysis
  - Prescribed motions exclusively determine how the system changes in time
    - The concept of force/torque does not factor in anywhere
  - For a Kinematics Analysis to be possible, the NDOF should be zero
  - Its solution provided at each time step by a sequence of 3 **algebraic** problems:
    - Nonlinear system of equations provides the position at each time step
    - Linear system of equations provides the velocity configuration at each time step
    - Linear system of equations provides the acceleration configuration at each time step

- Dynamics Analysis
  - External forces/torques dictate how the system evolves in time
  - It is more general than Kinematics:
    - A Kinematics problem can be solved using the methods of Dynamics, but not the other way around
  - Its solution obtained at each time step by numerical integration (solving a differential equation)

30,000 Feet Perspective

- When carrying out Dynamics Analysis, what you can compute is the acceleration of each part in the model
- Acceleration represents the second time derivative of your coordinates
- Oversimplifying the problem, in ME751 you get second time derivative

$$\ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$$

- Problem is reduced to a set of first order differential equations by introducing a helper variable **v** (the velocity):

$$\dot{\mathbf{q}} = \mathbf{v}$$

- With this, the original second order differential problem becomes:

$$\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y}, t) \quad \text{where} \quad \mathbf{y} = \begin{bmatrix} \mathbf{q} \\ \mathbf{v} \end{bmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{y}, t) \equiv \begin{bmatrix} \mathbf{v} \\ \mathbf{f}(\mathbf{v}, \mathbf{q}, t) \end{bmatrix}$$

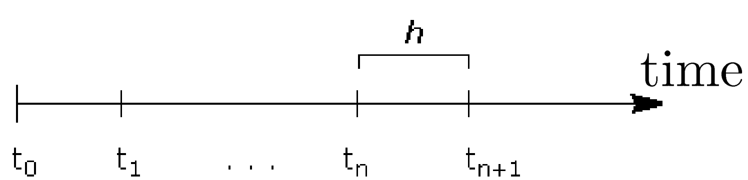
Numerical Integration
~The Problem ~

- Initial Value Problem: (IVP)

$$\begin{cases} \dot{y}(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

- IVP: Stating the Problem
  - You are looking for a function $y(t)$ that depends on time (changes in time), whose time derivative is equal to a function $f(t, y)$ that is given to you (see equation above)
  - You are given the derivative of a function. Can you tell what the function is?

- In ME751, the best you can hope for is to find an approximation of the unknown function $y(t)$ at a sequence of discrete points (as many of them as you wish)
  - The numerical algorithm produces an approximation of the value of the unknown function $y(t)$ at the each grid point. That is, the numerical algorithm produces an approximation for $y(t_1)$, $y(t_2)$, $y(t_3)$, etc.; i.e., y_1, y_2, y_3, etc.



Road Map

- Basic Concepts in Numerical Integration

- Basic Methods for Numerical Integration
  - Runge-Kutta
  - AB & AM Methods
  - BDF Methods

- Text used:
  - Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, by U. Ascher and L. Petzold, SIAM, 1998