Quote of the Day

“If you don't know where you are going, you might wind up someplace else.”
-- Yogi Berra
Before we get started…

- **Last Time:**
  - Discussed partial derivatives. There was a $p - \omega$ fork aspect we dealt with
  - Discussed computation of $\overline{\Pi}$
  - Quick remarks on Position Analysis + Newton Raphson
  - Wrapped up Kinematics Analysis

- **Today:**
  - Start discussing the Dynamics Analysis
  - Discuss about Virtual Displacements and Variation of a Function
    - We’re facing the same $p - \omega$ fork issue

- **New homework:** assigned today, due next Friday at 9:30 am
Purpose of Chapter 11

- At the end of this chapter you should understand what “dynamics” means and how you should go about carrying out a dynamics analysis.

- We’ll learn how to:
  - Formulate the equations that govern the time evolution of a system of bodies in 3D motion
    - These equations are differential equations and they are called the “equations of motion”
    - As many bodies as you wish, connected by any joints we’ve learned about…
  - Compute the reaction forces in any joint connecting any two bodies in the mechanism
  - Account for the effect of external forces in the equations of motion
The Idea, in a Nutshell…

- **Kinematics**
  - As *many* constraints as generalized coordinates
  - *No spare* degrees of freedom left
  - Position, velocity, acceleration found as the solution of algebraic problems
  - We do not care whatsoever about forces applied to the system
    - We are told what the motions are; this suffices for the purpose of kinematics

- **Dynamics**
  - You only have a *few* constraints imposed on the system
  - You have *extra* degrees of freedom
  - The system evolves in time as a result of external forces applied on it
  - We very much care about forces applied and inertia properties of the components of the mechanism (mass, mass moment of inertia)
A Relevant Question…

- Dynamics **key** question: How can I get the acceleration of each body of the mechanism?
  - Note: If you know the acceleration you can integrate it twice to get velocity and position information for each body
  - In other words, you want to get this quantity:
    \[
    \ddot{q}_i = \begin{bmatrix} \ddot{r}_i \\ \ddot{p}_i \end{bmatrix}
    \]

- Alternatively, you can get first
  \[
  \begin{bmatrix} \ddot{r}_i \\ \dot{\omega}_i \end{bmatrix}
  \]

- Then use the fact that there is a relationship of the type (see previous lecture)
  \[
  \dot{\mathbf{p}} \leftrightarrow \dot{\omega}
  \]
Looking Back; Looking Ahead

- Looking back: recall ME240 dynamics (for particle): \( F = m \cdot a \)
  - Right way to state this: \( m \cdot a = F \), which is the “equation of motion” (EOM)
  - Acceleration, which is what we care about, would then simply be \( a = F/m \)

- Looking ahead (next week):
  - Step 1: we’ll first show that the EOM for a rigid body is
    \[
    \mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} = 0_{3nb} \quad \text{Equations of Motion governing translation}
    \]
    \[
    \mathbf{J}\ddot{\mathbf{\omega}} - \mathbf{\dot{\mathbf{n}}} + \mathbf{\tilde{\omega}}\mathbf{J}\mathbf{\omega} = 0_{3nb} \quad \text{Equation of Motion governing rotation}
    \]
  - Step 2: formulate the equations of motion for a system of bodies interacting through contact, friction, and bilateral constraints
Virtual Displacements

- Rest of the lecture today:
  - Discuss the concept of “virtual displacements”
  - Discuss the concept of “consistent virtual displacements”

- Warm up, for deriving the EOM
Motivation

- Why do we have to talk about “virtual displacements” (VDs)?
  - Because they play a crucial role in evaluating the virtual work

- Why do we care about virtual work?
  - Because it is the crucial ingredient required to formulate the equations of motion (EOM)

- How are the EOM formulated actually?
  - Apply D'Alembert’s Principle; then fall back on the Principle of Virtual Work

- The Principle of Virtual Work:
  - Powerful tool used to get EOM in rigid and deformable body dynamics
  - “At equilibrium, the virtual work of forces acting on a system is zero”
Motivation
[Cntd.]

- Imagine that a force $\vec{F}^P$ acts on the rigid body at point $P$. The work done by this force is

$$\delta W^{F_P} = \delta \vec{r}^P \cdot \vec{F}^P$$

- Here $\delta \vec{r}^P$ represents a very small displacement of the point $P$. Causes for this displacement:

  - The principle of virtual work requires that I should be in a position to consider any small displacement of point $P$
  - This generic displacement is called *virtual displacement*
  - The size of the virtual displacement of point $P$ is infinitesimally small
  - The fact that a body $i$ is connected to other bodies through joints intuitively suggests that a virtual displacement of body $i$ is related to a virtual displacement of body $j$ if bodies $i$ and $j$ are connected through some type of constraint (joint)
Virtual Translation + Virtual Rotation

- Important observation: since the body is rigid, the small displacement of point $P$ is fully described in terms of a small translation $\delta \mathbf{r}$ and a small change of orientation $\delta \mathbf{A}$ of the L-RF.

- Specifically, assume that the change in the L-RF position and orientation are as follows:

  \[ \mathbf{r} \rightarrow \mathbf{r} + \delta \mathbf{r} \]

  \[ \mathbf{A} \rightarrow \mathbf{A} + \delta \mathbf{A} \]

- Read the construct “$\delta$ of blah” as “a small variation in blah.”
Virtual Displacement of Point P

- Original position of $P$:
  \[ r^P = r + A\bar{s}^P \]

- Position of $P$ after the small change in the position and orientation of the rigid body:
  \[ r^P + \delta r^P = (r + \delta r) + (A + \delta A)\bar{s}^P \]

- Net change in position of point $P$:
  \[ \delta r^P = (r^P + \delta r^P) - r^P = \delta r + \delta A \bar{s}^P \]

- Quick remarks:
  - Dimensions: $\delta r$ is $3 \times 1$, and $\delta A$ is $3 \times 3$
  - The change in orientation, $\delta A$, is not quite random. This is because the new matrix $A + \delta A$, which corresponds to the new orientation after the rigid body is nuded, should represent an actual orientation, that is, it must satisfy the orthonormality condition
  \[ (A + \delta A)^T (A + \delta A) = I_3 \]
Comments on Change in Orientation, $\delta A$

- First, keep in mind that the changes in position and orientation are small.

- Translation, in mathematical lingo: products of two changes in position and/or orientation are ignored
  \[ \delta r^T \delta r \approx 0 \quad \quad (\delta A)^T \delta A \approx 0_{3 \times 3} \quad \quad \delta A (\delta A)^T \approx 0_{3 \times 3} \]

- Key Result: There is a vector that is the generator of the matrix $A \delta A$. This vector is called **virtual rotation**: $\delta \tilde{\pi}$

- Proof:
  \[ (A + \delta A)^T (A + \delta A) = I_3 \Rightarrow A^T A + A^T \delta A + (\delta A)^T A + (\delta A)^T \delta A = I_3 \]
  \[ \Rightarrow \quad A^T \delta A + (\delta A)^T A = 0_{3 \times 3} \quad \Rightarrow \quad A^T \delta A = -(\delta A)^T A = -[A^T \delta A]^T \]

- Thus, the matrix $A^T \delta A$ is skew symmetric. As such, there should be a vector, denote it $\delta \tilde{\pi}$, so that
  \[ \hat{\delta \tilde{\pi}} = A^T \delta A \]

- The vector $\delta \tilde{\pi}$ is called the **virtual rotation vector**, and therefore the change in the orientation matrix $\delta A$ can be expressed in terms of the virtual rotation vector as
  \[ \delta A = A \delta \tilde{\pi} \]
The Invariance Property of $\delta \pi$

- Note that when representing the virtual rotation vector in the G-RF, one gets

$$\tilde{\delta \pi} = A \tilde{\delta \pi} A^T \quad \Rightarrow \quad \tilde{\delta \pi} = (\delta A) A^T \quad \Rightarrow \quad \delta A = \tilde{\delta \pi} A$$

- The virtual rotation vector $\delta \pi$ was implicitly defined by the identity $\tilde{\delta \pi} = (\delta A) A^T$. This somewhat suggests that $\delta \pi$ is related to the matrix $A$. What follows proves that this is not the case, instead, $\delta \pi$ is an attribute of the rigid body the L-RF is attached to.

- First, assume that there are two different virtual rotation vectors: $\delta \pi_1$, which goes along with L-RF$_1$, and $\delta \pi_2$, which goes along with L-RF$_2$, where the two L-RFs are rigidly attached to the same body.

- Then, since $A_2 = A_1 C$, we have $\delta A_2 = \delta A_1 C$, which implies that

$$\tilde{\delta \pi}_2 A_2 = \tilde{\delta \pi}_1 A_1 C$$

- Since $A_2 = A_1 C$, we get that

$$\tilde{\delta \pi}_2 = \tilde{\delta \pi}_1 \quad \Rightarrow \quad \delta \pi_2 = \delta \pi_1$$

- In other words, the virtual rotation is an attribute of the body, not of the L-RF rigidly attached to it.
Putting Things in Perspective

[Nomenclature issues]

- Virtual Translation: an infinitesimal translation $\delta \mathbf{r}$ of the L-RF. Performed with the time held fixed.

- Virtual Change in Orientation: an infinitesimal change in the orientation of the body captured in a change $\delta \mathbf{A}$ of the orientation matrix $\mathbf{A}$ associated with the L-RF. The virtual change $\delta \mathbf{A}$ in orientation is performed with the time held fixed.

- Virtual Rotation: a vector quantity $\delta \mathbf{\pi}$ that is the generator of $\mathbf{A}^T \delta \mathbf{A}$. In other words, $\delta \mathbf{\pi} = \mathbf{A}^T \delta \mathbf{A}$, from where $\delta \mathbf{A} = \mathbf{A} \delta \mathbf{\pi}$.

- Virtual Displacement: the combination of a virtual translation and a virtual rotation.

- Virtual Variation of a function (expression): change in the value of a function that depends on the location and orientation of a body in the system as a result of a virtual displacement applied to that body.
Variational Calculus

- We have a function (or expression) that depends on the location and orientation of the bodies in a mechanical system.

- Examples of such expressions:

\[
\Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, f(t)) = \bar{a}_i^T A_i^T A_j \bar{a}_j - f(t)
\]

\[
d_{ij} = r_j + A_j \bar{s}_j^Q - r_i - A_i \bar{s}_i^P
\]

\[
\Phi^{CD}(c, i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = c^T d_{ij} - f(t)
\]

- The fundamental question that we want to answer today: what is the variation in the value of the function when the location and orientation of the bodies in the system slightly change as a result of applying a virtual displacement?

- The answer to this question is the subject of the calculus of variations.
Formulas, Calculus of Variations

Rule 1 Variation of a constant quantity \( c \) (applies to scalars \( c \) or matrices \( C \) as well):

\[ \delta(c) = 0 \]

- Example use: calculate the variation of \( c^T \mathbf{d}_{ij} \)

Rule 2 Variation of a sum of two vectors:

\[ \delta(u + v) = \delta u + \delta v \]

- Example use: calculate the variation of \( \mathbf{r}_i + \mathbf{A}_i \beta^T \mathbf{r}_i \)

Rule 3 Variation of the product of two matrices:

\[ \delta(UV) = (\delta U)V + U(\delta V) \]

- Example use: calculate the variation of \( \mathbf{A}(\mathbf{p}) = \mathbf{E}\mathbf{G}^{T} \)

Rule 4 Variation of the product of matrix times a vector:

\[ \delta(Uv) = (\delta U)v + U(\delta v) \]

- Example use: calculate the variation of \( \mathbf{G}\mathbf{p} \)

Rule 5 Variation of the product of two vectors:

\[ \delta(u^Tv) = v^T(\delta u) + u^T(\delta v) \]

- Example use: calculate the variation of \( \mathbf{a}_i^T\mathbf{a}_j \)

Not difficult to prove. We’ll skip though.
Virtual Variation, Basic GCons: $\Phi^{DP1}$

- Recall that
  $$\Phi^{DP1}(i, \ddot{a}_i, j, \ddot{a}_j, f(t)) = \ddot{a}_i^T \dot{A}_i^T A_j \ddot{a}_j - f(t) = a_i^T a_j - f(t) = 0$$

- Assume that body $i$ experiences a virtual displacement characterized by $\left[ \begin{array}{c} \delta r_i \\ \delta \pi_i \end{array} \right]$, and the body $j$ experiences a virtual displacement characterized by $\left[ \begin{array}{c} \delta r_j \\ \delta \pi_j \end{array} \right]$. Therefore, $A_i \rightarrow A_i + \delta A_i$, and $A_j \rightarrow A_j + \delta A_j$.

- This variation in the attitude of bodies $i$ and $j$ will lead to a variation in the value of $\Phi^{DP1}$. Specifically, $\ddot{a}_i^T \dot{A}_i^T \dot{A}_j \ddot{a}_j \rightarrow \ddot{a}_i^T (A_i + \delta A_i)^T (A_j + \delta A_j) \ddot{a}_j$.

- Therefore,
  $$\delta \Phi^{DP1}(i, \ddot{a}_i, j, \ddot{a}_j, f(t)) = \ddot{a}_i^T (A_i + \delta A_i)^T (A_j + \delta A_j) \ddot{a}_j - \ddot{a}_i^T \dot{A}_i^T \dot{A}_j \ddot{a}_j$$
  $$= \ddot{a}_i^T \dot{A}_i^T \delta A_j \ddot{a}_j + \ddot{a}_i^T (\delta A_i)^T A_j \ddot{a}_j$$
  $$= \ddot{a}_i^T \dot{A}_i^T A_j \delta \pi_j \ddot{a}_j + \ddot{a}_j^T \dot{A}_j^T A_i \delta \pi_i \ddot{a}_i$$
  $$= -\ddot{a}_i^T \dot{A}_i^T A_j \ddot{a}_j \delta \pi_j - \ddot{a}_j^T \dot{A}_j^T A_i \ddot{a}_i \delta \pi_i$$

- Compare to $\dot{\Phi}^{DP1}(i, \ddot{a}_i, j, \ddot{a}_j, f(t))$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{DP1}$ and $\delta \Phi^{DP1}$.
[Short Detour]:

On the Variation of $d_{ij}$, that is, $\delta d_{ij}$

- Recall that

$$d_{ij} = r_j + A_j s_j^Q - r_i - A_i s_i^P = r_j + s_j^Q - r_i - s_i^P$$

- Assume that body $i$ experiences a virtual displacement characterized by $[\delta r_i \delta \pi_i]$, and the body $j$ experiences a virtual displacement characterized by $[\delta r_j \delta \pi_j]$. Therefore, $r_i \rightarrow r_i + \delta r_i$ and $A_i \rightarrow A_i + \delta A_i$. Likewise, $r_j \rightarrow r_j + \delta r_j$ and $A_j \rightarrow A_j + \delta A_j$.

- This variation in the attitude of bodies $i$ and $j$ will lead to a variation in the value of $d_{ij}$. Specifically, $d_{ij} \rightarrow d_{ij} + \delta d_{ij}$. In other words,

$$r_j + A_j s_j^Q - r_i - A_i s_i^P \rightarrow r_j + \delta r_j + (A_j + \delta A_j)s_j^Q - [r_i + \delta r_i + (A_i + \delta A_i)s_i^P]$$

- Therefore,

$$\delta d_{ij} = (d_{ij} + \delta d_{ij}) - d_{ij} = \delta r_j + \delta A_j s_j^Q - \delta r_i - \delta A_i s_i^P$$

$$= \delta r_j + A_j \delta \pi_j s_j^Q - \delta r_i - A_i \delta \pi_i s_i^P$$

$$= \delta r_j - A_j \delta \pi_j s_j^Q - \delta r_i + A_i \delta \pi_i s_i^P \delta \pi_i$$

- Compare to $\dot{d}_{ij}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{d}_{ij}$ and $\delta d_{ij}$.
Virtual Variation, Basic GCons: $\Phi^{DP2}$

- Recall that
  \[
  \Phi^{DP2}(i, \bar{a}_i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = \bar{a}_i^T A_i^T d_{ij} - f(t) = a_i^T d_{ij} - f(t) = 0
  \]

- Assume that body $i$ experiences a virtual displacement characterized by $\begin{bmatrix} \delta r_i \\ \delta \pi_i \end{bmatrix}$, and the body $j$ experiences a virtual displacement characterized by $\begin{bmatrix} \delta r_j \\ \delta \pi_j \end{bmatrix}$. Therefore, $r_i \rightarrow r_i + \delta r_i$ and $A_i \rightarrow A_i + \delta A_i$. Likewise, $r_j \rightarrow r_j + \delta r_j$ and $A_j \rightarrow A_j + \delta A_j$.

- This variation in the attitude of bodies $i$ and $j$ will lead to a variation in the value of $\Phi^{DP2}$. Specifically, $\Phi^{DP2} \rightarrow \Phi^{DP2} + \delta \Phi^{DP2}$.

- We have that (see Rule 5, Rule 2)
  \[
  \delta \Phi^{DP2} = a_i^T \delta d_{ij} + d_{ij}^T \delta a_i \\
  = a_i^T \left[ \delta r_j - A_j \bar{s}_j^Q \delta \pi_j - \delta r_i + A_i \bar{s}_i^P \delta \pi_i \right] - d_{ij}^T A_i \bar{s}_i^P \delta \pi_i \\
  = a_i^T \delta r_j - a_i^T A_j \bar{s}_j^Q \delta \pi_j - a_i^T \delta r_i + \left[ (a_i^T A_i - d_{ij}^T A_i) \bar{s}_i^P \right] \delta \pi_i
  \]

- Compare to $\dot{\Phi}^{DP2}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{DP2}$ and $\delta \Phi^{DP2}$.
Virtual Variation, Basic GCons: $\Phi^D$

- Recall that the GCon-CD assumes the expression

$$\Phi^D(i, s^P_i, j, s^Q_j, f(t)) = d^T_{ij}d_{ij} - f(t) = 0$$

- Assume that body $i$ experiences a virtual displacement characterized by $\begin{bmatrix} \delta r_i \\ \delta \pi_i \end{bmatrix}$, and the body $j$ experiences a virtual displacement characterized by $\begin{bmatrix} \delta r_j \\ \delta \pi_j \end{bmatrix}$. Therefore, $r_i \rightarrow r_i + \delta r_i$ and $A_i \rightarrow A_i + \delta A_i$. Likewise, $r_j \rightarrow r_j + \delta r_j$ and $A_j \rightarrow A_j + \delta A_j$.

- This variation in the attitude of bodies $i$ and $j$ will lead to a variation in the value of $\Phi^D$. Specifically, $\Phi^D \rightarrow \Phi^D + \delta \Phi^D$.

- We have that (see Rule 2, Rule 5)

$$\delta \Phi^D = d^T_{ij}(\delta d_{ij}) + (\delta d^T_{ij})d_{ij}$$

$$= 2d^T_{ij}\delta d_{ij}$$

$$= 2d^T_{ij}\left[\delta r_j - A_j s^Q_j \delta \pi_j - \delta r_i + A_i s^P_i \delta \pi_i\right]$$

$$= 2d^T_{ij}\delta r_j - 2d^T_{ij}A_j s^Q_j \delta \pi_j - 2d^T_{ij}\delta r_i + 2d^T_{ij}A_i s^P_i \delta \pi_i$$

- Compare to $\dot{\Phi}^D$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^D$ and $\delta \Phi^D$. 


Virtual Variation, Basic GCons: $\Phi^{CD}$

- Recall that the GCon-CD assumes the expression
  \[
  \Phi^{CD}(c, i, \tilde{s}_i^P, j, \tilde{s}_j^Q, f(t)) = c^T d_{ij} - f(t) = 0
  \]

- Assume that body $i$ experiences a virtual displacement characterized by $[\delta r_i \quad \delta \pi_i]$, and the body $j$ experiences a virtual displacement characterized by $[\delta r_j \quad \delta \pi_j]$. Therefore, $r_i \rightarrow r_i + \delta r_i$ and $A_i \rightarrow A_i + \delta A_i$. Likewise, $r_j \rightarrow r_j + \delta r_j$ and $A_j \rightarrow A_j + \delta A_j$.

- This variation in the attitude of bodies $i$ and $j$ will lead to a variation in the value of $\Phi^{CD}$. Specifically, $\Phi^{CD} \rightarrow \Phi^{CD} + \delta \Phi^{CD}$.

- We have that (see Rule 1, Rule 5)
  \[
  \delta \Phi^{CD} = c^T \delta d_{ij}
  \]
  \[
  = c^T [\delta r_j - A_j \tilde{s}_j^Q \delta \pi_j - \delta r_i + A_i \tilde{s}_i^P \delta \pi_i]
  \]
  \[
  = c^T \delta r_j - c^T A_j \tilde{s}_j^Q \delta \pi_j - c^T \delta r_i + c^T A_i \tilde{s}_i^P \delta \pi_i
  \]

- Compare to $\dot{\Phi}^{CD}$ to see the parallel between the 'dot' and 'delta' operators; i.e., between $\dot{\Phi}^{CD}$ and $\delta \Phi^{CD}$.
Virtual Variation, Basic GCons: Putting It All Together

- Gather now all the virtual translations and rotations in two big vectors:

\[
\delta \mathbf{r} = \begin{bmatrix}
\delta r_1 \\
\vdots \\
\delta r_{nb}
\end{bmatrix}_3 \quad \text{and} \quad \delta \mathbf{\pi} = \begin{bmatrix}
\delta \pi_1 \\
\vdots \\
\delta \pi_{nb}
\end{bmatrix}_3
\]

- We want to express the variation of a basic constraint \( \Phi^\alpha \), where \( \alpha \in \{DP1, DP2, D, CD\} \), in terms of \( \delta \mathbf{r} \) and \( \delta \mathbf{\pi} \).

- The key observation is that \( \delta \Phi^\alpha \) assumes the form

\[
\delta \Phi^\alpha = \begin{bmatrix}
0_{1 \times 3} \cdots 0_{1 \times 3} & \Phi_{r_i}^\alpha & 0_{1 \times 3} \cdots 0_{1 \times 3} & \Phi_{r_j}^\alpha & 0_{1 \times 3} \cdots 0_{1 \times 3} & \Pi_i & 0_{1 \times 3} \cdots 0_{1 \times 3} & \Pi_j & 0_{1 \times 3} \cdots
\end{bmatrix}
\]

Related to variations in position \( \delta \mathbf{r} \)

\[\uparrow \quad \text{Body 1, Transl.} \quad \uparrow \quad \text{Body i, Transl.} \quad \uparrow \quad \text{Body j, Transl.} \quad \uparrow \quad \text{Body i, Rotation} \quad \uparrow \quad \text{Body j, Rotation} \quad \]
Virtual Variation, Basic GCons: Putting It All Together

- Using the notation:

\[
\delta \mathbf{r} = \begin{bmatrix}
\delta r_1 \\
\vdots \\
\delta r_{\text{nb}}
\end{bmatrix}_{3\ \text{nb}} \quad \text{and} \quad \delta \mathbf{\bar{\pi}} = \begin{bmatrix}
\delta \bar{\pi}_1 \\
\vdots \\
\delta \bar{\pi}_{\text{nb}}
\end{bmatrix}_{3\ \text{nb}}
\]

- We express the variation of a basic constraint \( \Phi^\alpha \), where \( \alpha \in \{DP1, DP2, D, CD\} \), in terms of \( \delta \mathbf{r} \) and \( \delta \mathbf{\bar{\pi}} \) as

\[
\delta \Phi^\alpha = [ \Phi_r \quad \Pi(\Phi^\alpha) ] \cdot \begin{bmatrix}
\delta \mathbf{r} \\
\delta \mathbf{\bar{\pi}}
\end{bmatrix} = \mathbf{R} \begin{bmatrix}
\delta \mathbf{r} \\
\delta \mathbf{\bar{\pi}}
\end{bmatrix}
\]

- Equivalently,

\[
\delta \Phi^\alpha = [ \Phi_r \quad \Pi(\Phi^\alpha) ] \cdot \begin{bmatrix}
\delta \mathbf{r} \\
\delta \mathbf{\bar{\pi}}
\end{bmatrix} = \mathbf{R} \begin{bmatrix}
\delta \mathbf{r} \\
\delta \mathbf{\bar{\pi}}
\end{bmatrix}
\]

- Recall that by definition (see previous lecture), \( \Pi(\Phi^\alpha) \) is the coefficient matrix that multiplies \( \mathbf{\bar{\omega}} \) in the time derivative \( \dot{\Phi}^\alpha \).
End, Variations in a Function due to Virtual Displacements $\delta \mathbf{r}$ and $\delta \pi$

Begin, Variations in a Function due to Virtual Displacements $\delta \mathbf{r}$ and $\delta \mathbf{p}$
The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta p$ Virtual Rotation

- Framework: assume you have a vector quantity that depends on $p$. Assume that the value of $p$ changes to $p + \delta p$. What is the variation in the quantity that depends on $p$ due to the said change?

- Specifically, assume the vector quantity of interest is $u$, and $u$ depends on $p$ and possibly time $t$:
  \[ u = u(p, t) \]

- I am interested at a fixed time $t$ in the $\delta u$ below given $p$, $\delta p$, and the expression of $u(p)$:
  \[
  p \longrightarrow u(p, t) \quad p + \delta p \longrightarrow u(p + \delta p, t) = u(p, t) + \delta u \\
  \delta u = ?
  \]
The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta p$ Virtual Rotation

[Cntd.]:

- The answer to question of interest, $\delta u(p) = $?, is obtained using a Taylor series expansion:

  $$u(p + \delta p, t) = u(p, t) + u_p \delta p + \ldots$$

  $$\approx u(p, t) + u_p \delta p$$

- Then

  $$\delta u(p) = u(p + \delta p, t) - u(p, t) = u_p \delta p$$

- In the argument above, we rely on the fact that the virtual rotations, that is, the perturbations $\delta p$, are small and therefore higher order terms that contain entries of $\delta p$, that is, $\delta e_0$, $\delta e_1$, $\delta e_2$, or $\delta e_3$, can be safely approximated to be zero.

- Important observation: note that the time does not play a role in figuring out what the variation in $u$ is. In other words, looking into the variation of $u$ is an exercise that is carried out at a certain time $t$, and time is held fixed.

- Note that the same argument applies if $u$ is a scalar function that depends on $p$. In that case,

  $$\delta u(p, t) = u_p \delta p$$
Exercise

- Calculate the variation of the function \( u(p) = A(p)\bar{s} \) due to a variation \( \delta p \) in the Euler Parameters. The vector \( \bar{s} \) does not depend on \( p \).
Exercise

- Calculate the variation of the function $u(p) = p^T p - 1$ due to a variation $\delta p$ in the Euler Parameters
Quick Question

- Note that when interested in variations as induced by virtual rotations of the $\delta p$ flavor (as opposed to the $\delta \pi$ flavor), it is very straightforward to produce the quantity of interest:

$$\delta u(p) = u_p \delta p$$

- Why did not we take the same approach for the $\delta \pi$?
  - We couldn’t do this direct approach for the the same reason we couldn’t find a set of three variables whose time derivative is the angular velocity $\vec{\omega}$
  - Specifically, there is no concept of partial derivative $u_\pi$ to work with, and therefore we have to resort to the process that in the end expresses the variation $\delta u$ or the time derivative $\dot{u}$ using $\bar{\Pi}(u)$ and $\delta \pi$, or $\bar{\Pi}(u)$ and $\bar{\omega}$, respectively
Virtual Variation, Basic GCons: $\Phi^{DP1}$
[The $\delta p$ Flavor]

- Recall that

$$\Phi^{DP1}(i, \tilde{a}_i, j, \tilde{a}_j, f(t)) = \tilde{a}_i^T A_i^T A_j \tilde{a}_j - f(t) = a_i^T a_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^{DP1}}{\partial r_i} = 0_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial p_i} = a_j^T B(p_i, \tilde{a}_i)$$

$$\frac{\partial \Phi^{DP1}}{\partial r_j} = 0_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial p_j} = a_i^T B(p_j, \tilde{a}_j)$$

- Putting it all together, $\delta \Phi^{DP1} = \Phi^{DP1}_q \delta q$, where,

$$\Phi^{DP1}_q = \left[ \begin{array}{cccc}
0_{1 \times 3} & \ldots & 0_{1 \times 3} & \ldots & 0_{1 \times 3} & \ldots & 0_{1 \times 4} & \frac{\partial \Phi^{DP1}}{\partial p_i} & 0_{1 \times 4} & \ldots & 0_{1 \times 4} & \frac{\partial \Phi^{DP1}}{\partial p_j} & 0_{1 \times 4} & \ldots & 0_{1 \times 4}
\end{array} \right]$$

Partials with respect to $r$  
Partials with respect to $p$
[Short Detour]:

Computing $\delta d_{ij}$

- Recall that

$$d_{ij} = r_j + A_j s_j^Q - r_i - A_i s_i^P = r_j + s_j^Q - r_i - s_i^P$$

- Recall also that

$$[d_{ij}]_{q_i,q_j} = [-I_3 \quad -(s_i^P)_{p_i} \quad I_3 \quad (s_j^Q)_{p_j}]$$

$$= [-I_3 \quad -B(p_i,s_i^P) \quad I_3 \quad B(p_j,s_j^Q)]$$

- It follows that

$$\delta d_{ij} = [-I_3 \quad -B(p_i,s_i^P) \quad I_3 \quad B(p_j,s_j^Q)] \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = [d_{ij}]_q \cdot \delta q$$
Virtual Variation, Basic GCons: $\Phi^{DP2}$

[The $\delta p$ Flavor]

- Recall that

$$\Phi^{DP2}(i, \bar{a}_i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = \bar{a}_i^T A_i^T \mathbf{d}_{ij} - f(t) = a_i^T \mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{DP2}_{qi,qj}(a_i, d_{ij}) = [ -a_i^T \quad d_{ij}^T B(p_i, s_i^P) - a_i^T B(p_i, \bar{s}_i^P) \quad a_i^T \quad a_i^T B(p_j, \bar{s}_j^Q) ]$$

- It follows that

$$\delta \Phi^{DP2} = [ -a_i^T \quad d_{ij}^T B(p_i, \bar{s}_i^P) - a_i^T B(p_i, \bar{s}_i^P) \quad a_i^T \quad a_i^T B(p_j, \bar{s}_j^Q) ] \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = \Phi^{DP2}_{q} \cdot \delta \mathbf{q}$$
Virtual Variation, Basic GCons: $\Phi^D$

[The $\delta \mathbf{p}$ Flavor]

- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = d_{ij}^T d_{ij} - f(t) = 0$$

- It also that

$$\Phi_{a_i, a_j}^D = \begin{bmatrix} -2d_{ij}^T & -2d_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{s}_i^P) & 2d_{ij}^T & 2d_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{s}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta \Phi^D = \begin{bmatrix} -2d_{ij}^T & -2d_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{s}_i^P) & 2d_{ij}^T & 2d_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{s}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r}_i \\ \delta \mathbf{p}_i \\ \delta \mathbf{r}_j \\ \delta \mathbf{p}_j \end{bmatrix} = \Phi_{a}^D \cdot \delta \mathbf{q}$$
Virtual Variation, Basic GCons: $\Phi^{CD}$

[The $\delta p$ Flavor]

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(c, i, s^p_i, j, s^Q_j, f(t)) = c^T d_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{CD}_{q_i, q_j} = \begin{bmatrix} -c^T & -c^T B(p_i, s^p_i) & c^T & c^T B(p_j, s^Q_j) \end{bmatrix}$$

- It follows that

$$\delta \Phi^{CD} = \begin{bmatrix} -c^T & -c^T B(p_i, s^p_i) & c^T & c^T B(p_j, s^Q_j) \end{bmatrix} \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = \Phi^{CD}_{q} \cdot \delta q$$
Virtual Variation, Euler Parameter Normalization Constraint: $\Phi^p$

- Recall that the Euler Parameter normalization constraint assumes the expression

$$\Phi^p_i = p^T_i p_i - 1 = 0$$

- Recall also that

$$(\Phi^p_i)_{qi} = \begin{bmatrix} 0_{1\times3} & 2p^T_i \end{bmatrix}$$

- It follows that

$$\delta\Phi^p_i = \begin{bmatrix} 0_{1\times3} & 2p^T_i \end{bmatrix} \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \end{bmatrix} = (\Phi^p_i)_{qi} \cdot \delta q$$