ME751
Advanced Computational Multibody Dynamics

Section 9.2

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Quote of the day

“I only believe in statistics that I doctored myself.”

-- Sir Winston S. Churchill

Before we get started…

- Last Time:
  - Changing the RF for representing a Geometric Vector
  - The concept of orientation matrix **A**
  - Angular velocity ω of a rigid body
    - An attribute of a body, not of a reference frame attached to that body
    - Handy to express the time derivative of an orientation matrix: $\dot{\mathbf{A}} = \tilde{\omega}\mathbf{A}$
  - Degree of freedom count, relative to 3D rotation of a rigid body
    - Given three direction cosines, you can find the other six to assemble **A**

- Today:
  - Hopping from RF to RF
  - Describing the orientation of a body using Euler Angles
  - Connection between the angular velocity and time derivatives of Euler Angles

Next Assignment

- Due on September 23, 9:30 AM
- Assignment has two components
  - Component 1: read the proposal and provide feedback by Saturday evening
    - PDF of proposal to be emailed to you Friday by mid-day
    - Second version of the PDF will be emailed to you on Saturday at noon
      - Use either PDF version
      - Make comments and highlight things that don’t make sense (instructions in follow-up email)
  - Component 2: the usual pen and paper stuff, to be emailed to you

[Short Detour:]
Hopping from RF to RF

- The discussion framework:
  - Recall that when going from one L-RF_2 to a different L-RF_1, there is a transformation matrix that multiplies the representation of a geometric vector in L-RF_2 to get the representation of the geometric vector in L-RF_1 :

$$\bar{\mathbf{s}}_1 = \mathbf{A}_{12} \cdot \bar{\mathbf{s}}_2$$

- Question: What happens if you want to go from L-RF_3 to L-RF_2 and then eventually to the representation in L-RF_1 ?

- Why bother?
  - This comes into play when dealing with Euler Angles

[End Detour:]

Hopping from RF to RF

- Going from L-RF$_3$ to L-RF$_2$ to L-RF$_1$:

$$\bar{\mathbf{s}}_2 = \mathbf{A}_{23} \cdot \bar{\mathbf{s}}_3 \quad \oplus \quad \bar{\mathbf{s}}_1 = \mathbf{A}_{12} \cdot \bar{\mathbf{s}}_2 \quad \Rightarrow \quad \bar{\mathbf{s}}_1 = \mathbf{A}_{12}\mathbf{A}_{23} \cdot \bar{\mathbf{s}}_3$$

- The basic idea: you keep multiplying rotation matrices like above to hope from RF to RF until you arrive to your final destination

- Question [relevant in one week]: How do we compute $\mathbf{A}_{ij}$ if we have $\mathbf{A}_i$ and $\mathbf{A}_j$?
  - $\mathbf{A}_i$: hopping from L-RF$_i$ to G-RF
  - $\mathbf{A}_{ij}$: hopping from L-RF$_j$ to L-RF$_i$
  - Keep in mind the invariant here; i.e., the GV $\vec{s}$:

$$\left.\begin{array}{l} \mathbf{s} = \mathbf{A}_i \bar{\mathbf{s}}_i \\ \mathbf{s} = \mathbf{A}_j \bar{\mathbf{s}}_j \end{array}\right\} \quad \Rightarrow \quad \mathbf{A}_i \bar{\mathbf{s}}_i = \mathbf{A}_j \bar{\mathbf{s}}_j \quad \Rightarrow \quad \bar{\mathbf{s}}_i = \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{s}}_j \quad \Rightarrow \quad \boxed{\mathbf{A}_{ij} = \mathbf{A}_i^T \mathbf{A}_j}$$

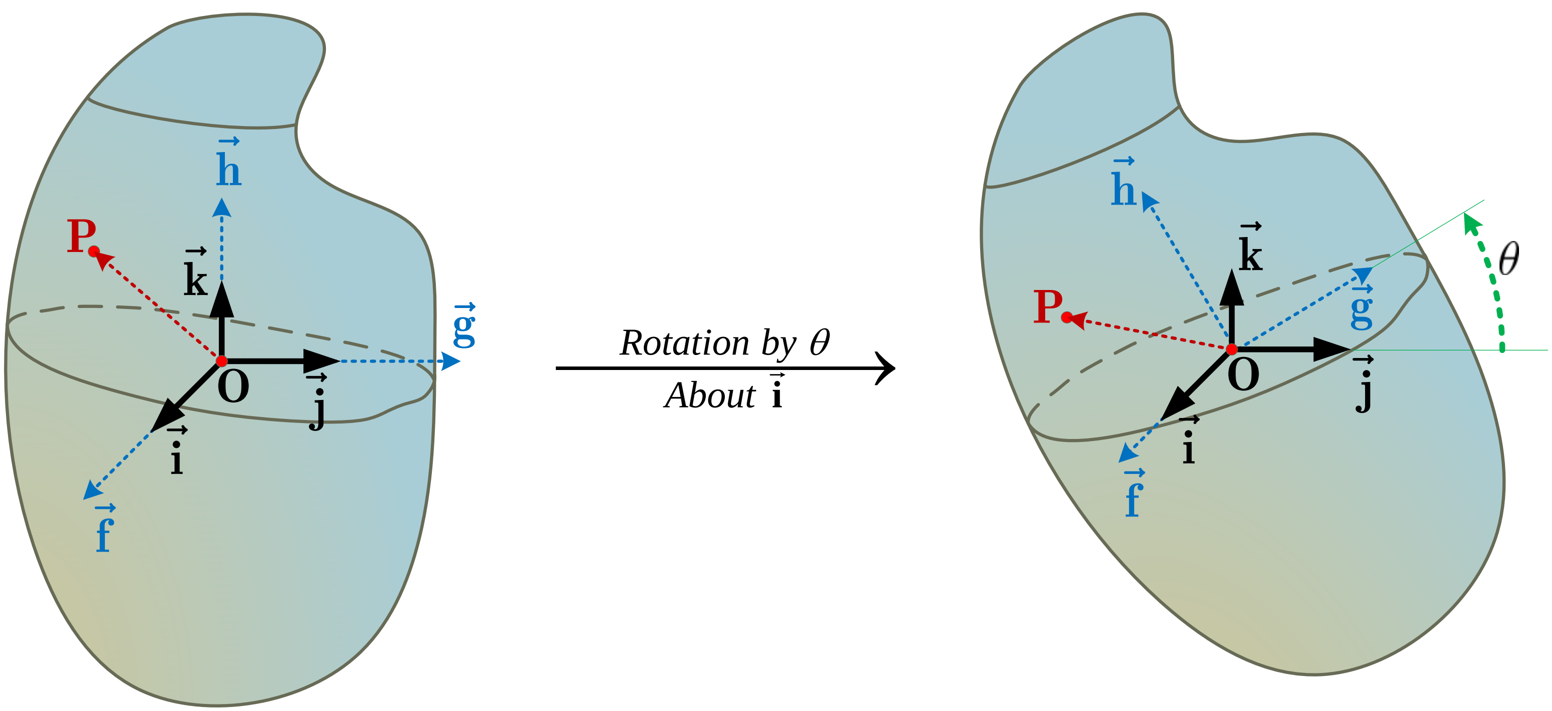
A Matter of Notation

- A rotation matrix is tied to two RFs: a source, and a destination
- It makes sense to expect to see two subscripts attached to $\mathbf{A}$: one for the destination RF, one for the source RF
- On the previous slide we used the notation $\mathbf{A}_i$ for the matrix that gives the transformation from L-RF$_i$ to the G-RF
- A more consistent notation would have been $\mathbf{A}_{0i}$, to indicate that the destination is "0"; i.e., the G-RF, and the source is L-RF$_i$
- This notation convention; i.e., having two subscripts, would help put things in perspective when we discussed the matrices $\mathbf{A}_{ij} = \mathbf{A}_i^T \mathbf{A}_j$ on the previous slide: L-RF$_i$ is the destination, L-RF$_j$ is the source
- NOTE: Because of convenience, we'll continue to use $\mathbf{A}_i$ instead of $\mathbf{A}_{0i}$

[AO]

Exercise:

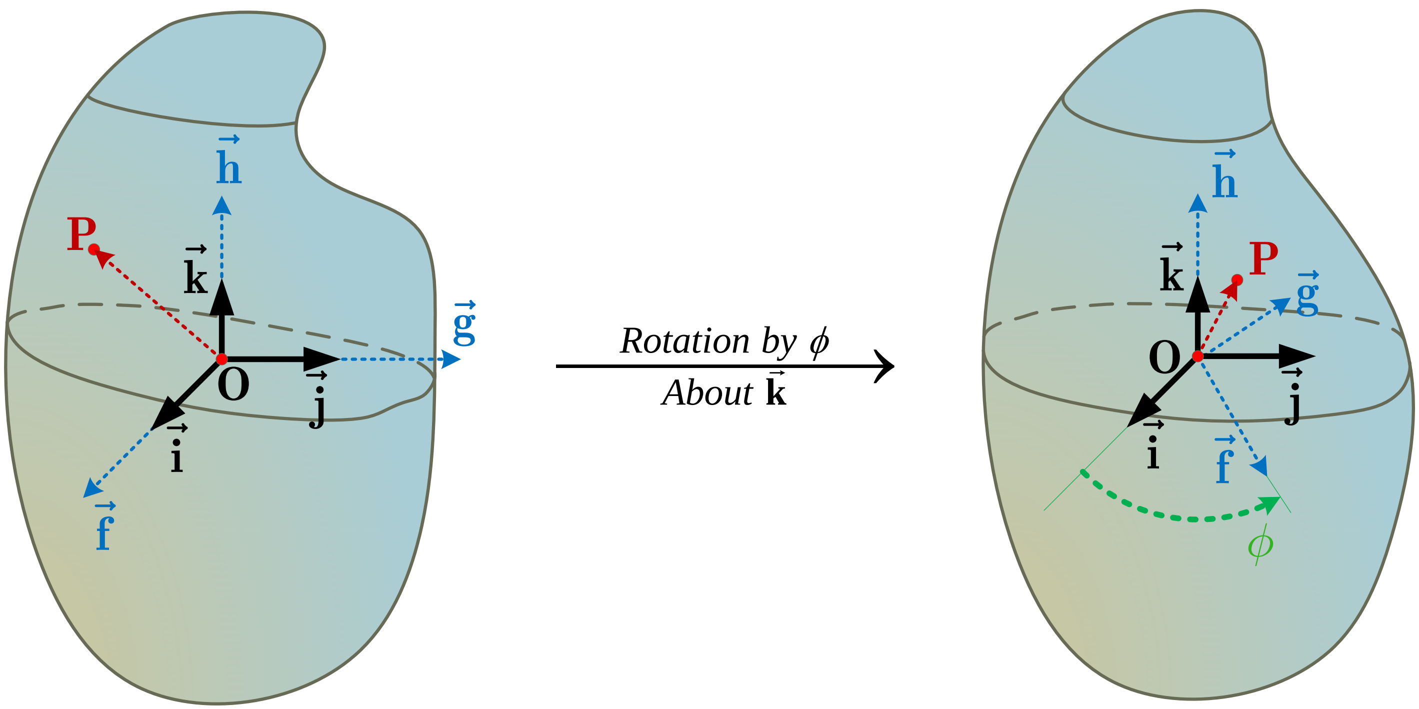
- A body is rotated about the Ox axis by an angle „ (see below). What is the expression of the orientation matrix associated with a L-RF attached to the body?
- If the expression of „(t) as a function of time is prescribed, what is the angular velocity of the body in its rotation?



[Homework]

Exercise:

- A body is rotated about the Oz axis by an angle ¨ .
 What is the expression of the orientation matrix associated with a L-RF attached to the body?
- If the expression of ¨ (t) as a function of time is prescribed, what is the angular velocity of the body in its rotation?



Euler Angles

Putting Things in Perspective, What Comes Next

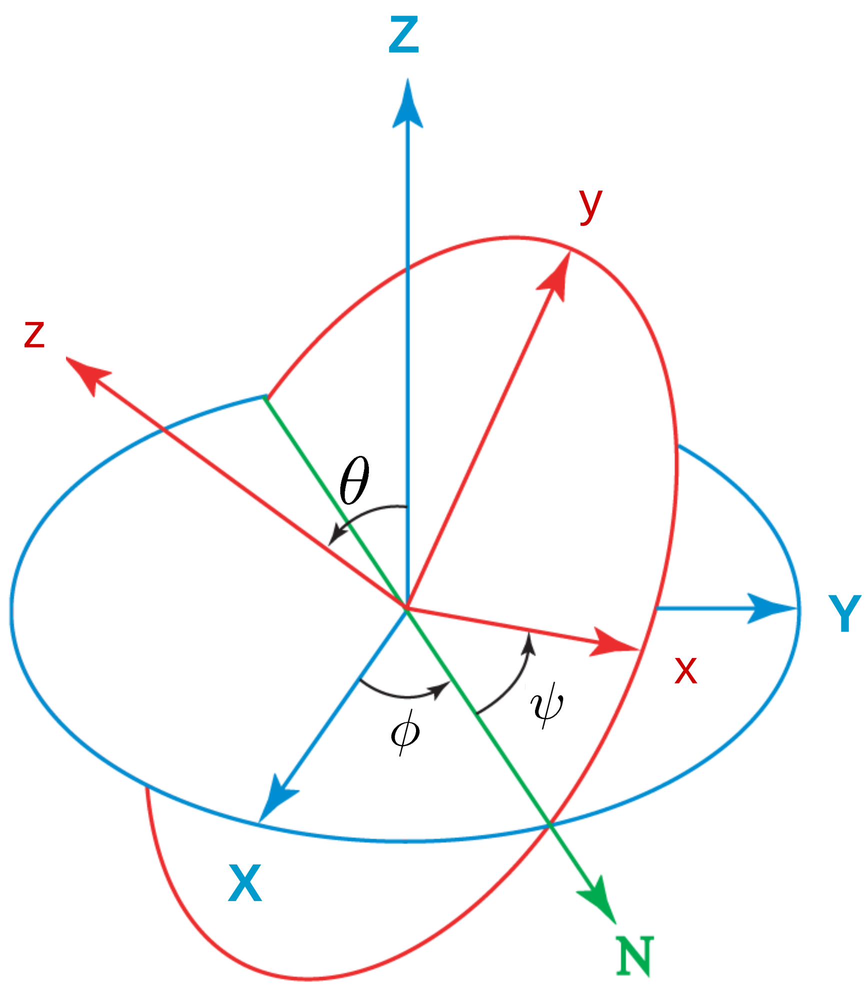
- We need three generalized coordinates (GCs) to uniquely define the orientation of a body

- We are about to choose the three GCs to be the set of Euler Angles

- When using this set of GCs, we are interested in:
  - Producing the orientation matrix **A** for a given Euler Angles configuration
  - Expressing the connection between the angular velocity of a body and the time derivative of the GCs considered

Euler Angles

- Euler Angles, definition:
  - A set of three angles used to describe the orientation of a reference frame in 3D space
- Draws on the following observation:
  - You can **<u>align (superimpose) the global reference frame to any arbitrary reference frame</u>** through a sequence of THREE rotation operations
    - Wait till slide 18 to see why the case



- In picture: start from blue RF, end in the red RF after three rotations
- The sequence of three rotations that we'll consider is
  - About axes Z then X then Z again (called the 3-1-3 sequence)
  - We'll denote the three Euler Angles by ˘, „, and ¨ , respectively

[http://prt.fernuni-hagen.de/lehre/KURSE/PRT001/course_main_sh3d/node10.html]→

Euler Angles, Quick Remarks

- Some notation conventions:
  - The G-RF is denoted by OXYZ
  - The L-RF that we land on after three ˘, „, ¨ rotations denoted by O’x’y’z’
- Details regarding the three rotations:
  - O’x’y’z’ is obtained by rotating O’’x’’y’’z’’ around the O’’z’’ axis by angle ¨
  - O’’x’’y’’z’’ is obtained by rotating O’’’x’’’y’’’z’’’ around the O’’’x’’’ axis by angle „
  - O’’’x’’’y’’’z’’’ is obtained by rotating OXYZ around OZ axis by angle ˘

Euler Angles,
Quick Remarks [Cntd.]

- There are other Euler Angles choices
  - 1-2-3, 1-3-1, etc.
  - The approach we follow for our 3-1-3 choice to find **A** given the three angles of rotation applies exactly to any other choice of rotation sequence
  - NOTE: ADAMS, a widely used MBD package, uses the 3-1-3 sequence

- Remember that the last two rotations („ and ¨) are measured with respect to rotated reference frames
  - θ about the X axis of O'''x'''y'''z''' to get O''x''y''z''
  - ψ about the Z axis of O''x''y''z'' to get O'x'y'z'

Expressing A using Euler Angles (Part 1 of 3)

- Recall that I have a 3-1-3 rotation sequence.
- The last sequence, is a rotation of angle ψ about z'' to get O'x'y'z'.
- Therefore, the rotation matrix that relates O'x'y'z' to O''x''y''z'' is

$$\mathbf{A}_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- In other words, if I have a vector represented as **a'** in O'x'y'z', it will be represented in O''x''y''z'' as

$$\mathbf{a}'' = \mathbf{A}_3\mathbf{a}'$$

Expressing A using Euler Angles (Part 2 of 3)

- Just dealt with the last “3” in the 3-1-3 rotation sequence (angle ¨)
- Focus next on the “1” rotation in the 3-1-3 rotation sequence
  - This is the rotation of angle θ
- The rotation of angle θ about the axis O’’’x’’’, takes O’’’x’’’y’’’z’’’ into O’’x’’y’’z’’

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- In other words, if I have a vector represented in O’’x’’y’’z’’ as **a’’**, the same vector will be represented in O’’’x’’’y’’’z’’’ as

$$\mathbf{a}''' = \mathbf{A}_2\mathbf{a}''$$

Expressing A using Euler Angles (Part 3 of 3)

- Just dealt with the “1” in the 3-1-3 rotation sequence (angle „)
- Focus next on the first “3” rotation in the 3-1-3 rotation sequence
  - This is the rotation of angle ˘
- The rotation of angle ˘ about the axis OX, takes OXYZ into O’’’x’’’y’’’z’’’

$$\mathbf{A}_1 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- In other words, if I have a vector represented in O’’’x’’’y’’’z’’’ as **a’’’**, the same vector will be represented in OXYZ as

$$\mathbf{a} = \mathbf{A}_1 \mathbf{a}'''$$

Expressing A using Euler Angles
~ Putting it All Together ~

- A Z-X-Z (or 3-1-3) rotation sequence takes OXYZ to O`x`y`z`
- Recall that
  - Given **a**', you get **a**'' as

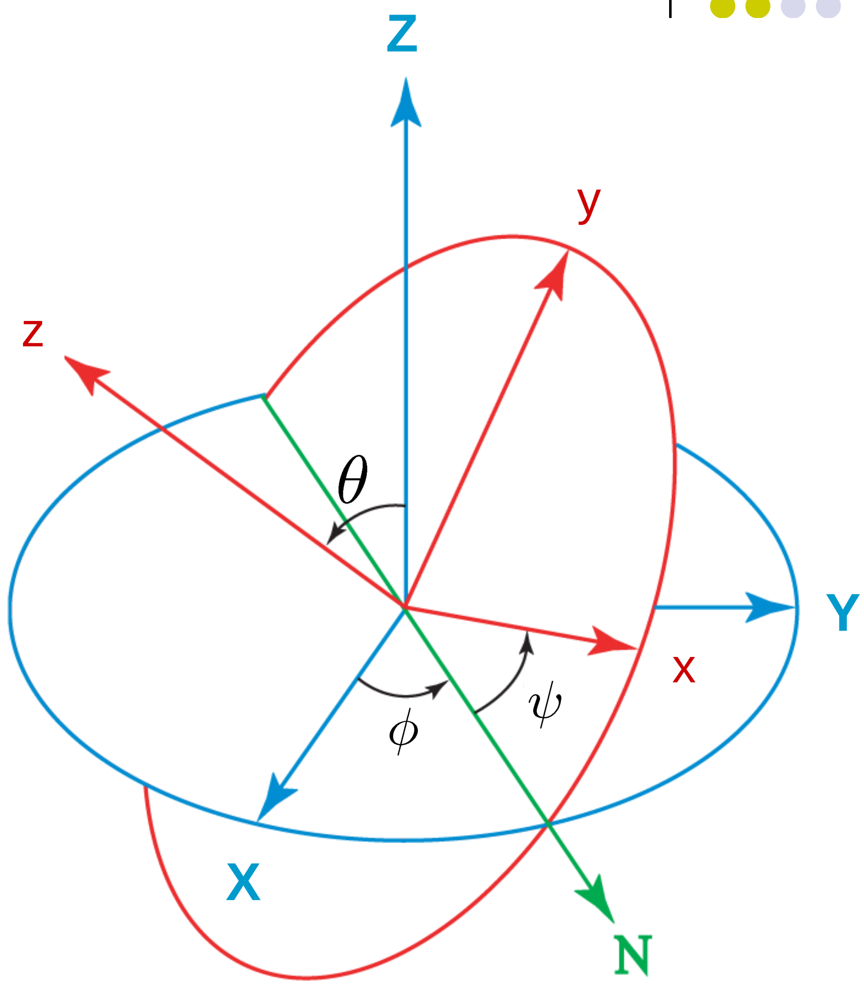
$$\mathbf{a}'' = \mathbf{A}_3\mathbf{a}'$$

  - Given **a**'' you get **a**''' as

$$\mathbf{a}''' = \mathbf{A}_2\mathbf{a}''$$

  - Given **a**''' you get a as

$$\mathbf{a} = \mathbf{A}_1\mathbf{a}'''$$



  - Therefore,

$$\mathbf{a} = \mathbf{A}_1\mathbf{a}''' = \mathbf{A}_1(\mathbf{A}_2\mathbf{a}'') = \mathbf{A}_1(\mathbf{A}_2(\mathbf{A}_3\mathbf{a}')) = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{a}' \Rightarrow \mathbf{A} = \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$$

Expressing A using Euler Angles
~ Putting it All Together ~

- Using the expression of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, one gets for the expression of the orientation matrix **A**

$$\mathbf{A} = \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Carrying out the above multiplications we get

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

- [Homework]: Provide two proofs that $\mathbf{A}^T \mathbf{A} = \mathbf{A}\,\mathbf{A}^T = \mathbf{I}$

Expressing A using Euler Angles
~ Two Questions ~

- Some questions that ought to be asked

1. Should the order (sequence) ˘ $ „#$ ˇ be observed?

2. You can always find one set of ˘, „, ˇ angles to bring G-RF into L-RF
   - Is this set unique?
   - If not, do you run into any issue when you have more 3-1-3 rotations that relate the G-RF and the L-RF?

Expressing A using Euler Angles
~ Two Answers ~

- Answers to previous two questions:

1. The order of the rotation should be observed since it’s important. If you change the order this will land you in a completely different rotational configuration

2. The only time when you run into a problem is when „=k`  (the “1” rotation in the 3-1-3 sequence is any integer multiple of `).
   - In this case it’s not clear how to split the rotations about the Z axes (the first “3” and the last “3” in the 3-1-3 sequence)
   - In this case you cannot rely on this set of generalized coordinates since there is a loss of uniqueness associated with them

[AO]

Exercise

- The purpose of this exercise is to demonstrate that there is no **global** one-to-one mapping between the nine direction cosines of **A** and the three generalized coordinates ˘, „, ¨ used to describe the orientation of a L-RF:
  - Take ϕ=20, θ=0, and ψ=60. Compute **A**
  - Take ϕ=40, θ=0, and ψ=40. Compute **A**
  - Both cases lead to the <u>same</u> **A**
    - This is called the singularity associated with the Euler angles, and it is unavoidable
    - A solution is to start using a different L-RF_{new} when the initial L-RF is associated with a value of „ ˜ k`, where k is any integer
      - This is possible since a L-RF is just an accessory for you to characterize the attitude (orientation) of a rigid body
  - NOTE: People have been attempting to use different generalized coordinates to determine in a more robust fashion the orientation of a body with respect to a global reference frame

Quick Remark

- The fact that a set of three GCs cannot be found to express ***globally*** the orientation of a body with respect to the G-RF proved in 1964:
  - Stuelpnagel, J., “On the parametrization of the three-dimensional rotation group”, SIAM Review, 1964. 6: p. 422-430.
  - Paper available on the class website
- Note that this does not mean that you cannot express **A** as a function of three GCs ***locally***
  - This is in fact guaranteed by last lecture’s proof that drew on the Implicit Function Theorem

Expressing the Angular Velocity when Using Euler Angles

Quick Review, Angular Velocity

- Note that since **A** is orthonormal,

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}$$

Identity matrix

- Take a time derivative to eventually get

$$\dot{\mathbf{A}}\mathbf{A}^T = -\mathbf{A}\dot{\mathbf{A}}^T = -\left(\dot{\mathbf{A}}\mathbf{A}^T\right)^T$$

- This means that the matrix product at the left is a skew-symmetric matrix
  - Therefore there is a generator vector $\boldsymbol{\omega}$ that can be used to represent this skew-symmetric matrix:

$$\dot{\mathbf{A}}\mathbf{A}^T = \tilde{\omega}$$

  - **By definition, the generator vector $\boldsymbol{\omega}$ is called the angular velocity of the body on which the L-RF is attached**
  - ***NOTE*: $\boldsymbol{\omega}$ is an attribute of the body, and not of the L-RF attached to it**

Angular Velocity for 3-1-3 Euler Sequence

- When using the 3-1-3 Euler angles, based on the expression of **A**, one gets:

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T = (\dot{\mathbf{A}}_1\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_1\dot{\mathbf{A}}_2\mathbf{A}_3 + \mathbf{A}_1\mathbf{A}_2\dot{\mathbf{A}}_3)\mathbf{A}_3^T\mathbf{A}_2^T\mathbf{A}_1^T$$

- What you additionally know is that:

$$\left\{\begin{array}{l} \dot{\mathbf{A}}_1 = \dot{\phi}\begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \\ \dot{\mathbf{A}}_2 = \dot{\theta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \\ \\ \dot{\mathbf{A}}_3 = \dot{\psi}\begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}\right.$$

Angular Velocity for 3-1-3 Euler Sequence

- If you carry out the horrific matrix multiplications you end up with the following expression for $\boldsymbol{\omega}$:

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} + \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix} \dot{\psi}$$

- Equivalently, you can write this in matrix form like

$$\omega = \begin{bmatrix} 0 & \cos\phi & \sin\theta\sin\phi \\ 0 & \sin\phi & -\sin\theta\cos\phi \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \equiv \mathbf{B}\dot{\epsilon}$$

…where the array of generalized coordinates ε is defined as

$$\epsilon = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

Angular Velocity for 3-1-3 Euler Sequence

- There is a simpler way to get $ though, and it draws on the following identity that we proved last time:

$$\widetilde{(\mathbf{A}\bar{\mathbf{s}})} = \mathbf{A}\tilde{\bar{\mathbf{s}}}\mathbf{A}^T$$

- The above identity is used in conjunction with :

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T = (\dot{\mathbf{A}}_1\mathbf{A}_2\mathbf{A}_3 + \mathbf{A}_1\dot{\mathbf{A}}_2\mathbf{A}_3 + \mathbf{A}_1\mathbf{A}_2\dot{\mathbf{A}}_3)\mathbf{A}_3^T\mathbf{A}_2^T\mathbf{A}_1^T$$

- Open up the parenthesis above to get that

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\phi} + \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}\dot{\theta} + \begin{bmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{bmatrix}\dot{\psi}$$

Angular Velocity for 3-1-3 Euler Sequence

- The previous slide says that you can compute the time derivatives of the Euler angles given **ω** by solving the linear system

$$\mathbf{B}\dot{\epsilon} = \omega$$

- Note that det(**B**)=sinθ, which yet again indicates that when the second rotation (the "1" in the 3-1-3 sequence) is „ =kπ, the matrix **B** is singular and you are in trouble (singular configuration)

- To avoid this situation people introduced a set of four "Euler parameters" which have less issues albeit at a slight increase in abstraction
  - You have some redundancy in information (three parameters are necessary, yet you have four Euler parameters ， leads to one additional constraint equation)