Problem 1. Prove that if $A \in \mathbb{R}^{m \times p}$ and $C \in \mathbb{R}^{p \times n}$, then $(AC)^T = C^T A^T$.

Problem 2. Prove that if the matrices $A \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{m \times m}$ are both invertible, then their product is invertible and $(AC)^{-1} = C^{-1} A^{-1}$.

Problem 3. Prove that if the matrices $A \in \mathbb{R}^{m \times m}$ is invertible, $(A^{-1})^{-1} = A$.

Problem 4. Consider the matrix $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$. Without using MATLAB, compute $\|A\|_p$ for $p = 1, 2, \infty$ (pen and paper problem).

Problem 5. Use MATLAB to compute the condition number associated with the Hilbert matrix of order 10, $H_{10}$. Below, I provided as an example the Hilbert matrix of order 5, $H_5$. Note that in general, $H_{ij} = \frac{1}{i + j - 1}$. What do you think, would a linear system like $H_{10} x = b$ be ill-conditioned or not?

$$H_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$
Problem 6. Assume that $k$ is a scalar, $C \in \mathbb{R}^{n \times n}$ is a constant matrix, $y \in \mathbb{R}^n$ is a vector that does not depend on $x \in \mathbb{R}^n$, and $p^T = [p_1, \ldots, p_n]$ and $q^T = [q_1, \ldots, q_n]$ are two vectors that change in time. Prove the following relations ($I_n$ is the identity matrix of dimension $n$):

$$\frac{\partial}{\partial q}(kq) = kI_n$$

$$\frac{\partial}{\partial q}(Cq) = C$$

$$\frac{\partial}{\partial x}(x^T Cy) = y^T C^T$$

$$\frac{d}{dt}(p^T Cq) = p^T Cq + p^T Cq$$