ME751 – Assignment 1

Problem 1. Prove that if $\mathbf{A} \in \mathbb{R}^{m\times p}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$, then $(\mathbf{AC})^T = \mathbf{C}^T\mathbf{A}^T$.

Problem 2. Prove that if the matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{C} \in \mathbb{R}^{m\times m}$ are both invertible, then their product is invertible and $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$.

Problem 3. Prove that if the matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$ is invertible, $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Problem 4. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$. Without using MATLAB, compute $||\, \mathbf{A} \,||_p$ for $p = 1, 2, \infty$ (pen and paper problem).

Problem 5. Use MATLAB to compute the condition number associated with the Hilbert matrix of order 10, $\mathbf{H}_{10}$. Below, I provided as an example the Hilbert matrix of order 5, $\mathbf{H}_5$. Note that in general, $H_{ij} = \dfrac{1}{i+j-1}$. What do you think, would a linear system like $\mathbf{H}_{10}\mathbf{x} = \mathbf{b}$ be ill-conditioned or not?

$$\mathbf{H}_5 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

Problem 6. Assume that k is a scalar, $\mathbf{C} \in \mathbb{R}^{n\times n}$ is a constant matrix, $\mathbf{y} \in \mathbb{R}^{n}$ is a vector that does not depend on $\mathbf{x} \in \mathbb{R}^{n}$, and $\mathbf{p}^T = [p_1, \ldots, p_n]$ and $\mathbf{q}^T = [q_1, \ldots, q_n]$ are two vectors that change in time. Prove the following relations ($\mathbf{I}_n$ is the identity matrix of dimension n):

$$\frac{\partial}{\partial \mathbf{q}}(k\mathbf{q}) = k\mathbf{I}_n$$

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{C}\mathbf{q}) = \mathbf{C}$$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^T\mathbf{C}\mathbf{y}) = \mathbf{y}^T\mathbf{C}^T$$

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{C}\mathbf{q}) = \dot{\mathbf{p}}^T\mathbf{C}\mathbf{q} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$