Backward Euler

Applied to solve a Nonlinear IVP.

IVP:

$$\begin{cases} \dot{x} = \alpha - x - \dfrac{4xy}{1+x^2} \\[2ex] \dot{y} = \beta x \left(1 - \dfrac{y}{1+x^2}\right) \end{cases}$$

$x(0) = 0$

$y(0) = 2$

$t \in [0, 20]$

α, β – two constants given to you

Backward Euler:

$$\begin{cases} x_n = x_{n-1} + h\,\dot{x}_n \\ y_n = y_{n-1} + h\,\dot{y}_n \end{cases} \qquad \text{Eq. } (*)$$

Based on the IVP problem, we have:

$$\begin{cases} \dot{x}_n = \alpha - x_n - \dfrac{4x_n y_n}{1+x_n^2} \\[2ex] \dot{y}_n = \beta x_n \left(1 - \dfrac{y_n}{1+x_n^2}\right) \end{cases} \qquad \text{Eq. } (**)$$

substitute Eq (**) into Eq. (*) :

$$\begin{cases} x_n = x_{n-1} + h\left[\alpha - x_n - \dfrac{4x_n y_n}{1+x_n^2}\right] \\[2ex] y_n = y_{n-1} + h\beta x_n \left(1 - \dfrac{y_n}{1+x_n^2}\right) \end{cases}$$

This is a nonlinear system in x_n and y_n. Apply Newton Raphson to solve it.

Rewrite nonlinear system like:

$$\begin{bmatrix} x_n(1+h) + \dfrac{4h\,x_n y_n}{1+x_n^2} - x_{n-1} - h\alpha \\ \\ -h\beta x_n + y_n + \dfrac{h\beta\, x_n y_n}{1+x_n^2} - y_{n-1} \end{bmatrix} \equiv g(x_n, y_n) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Jacobian:

$$J(x_n, y_n) = \begin{bmatrix} 1+h+4hy_n \dfrac{1-x_n^2}{(1+x_n^2)^2} & \dfrac{4h\,x_n}{1+x_n^2} \\ \\ -h\beta + h\beta \cdot \dfrac{1-x_n^2}{(1+x_n^2)^2} & 1+\beta h \cdot \dfrac{x_n}{1+x_n^2} \end{bmatrix}_{2\times 2}$$

Then, at each time step t_n, apply Newton-Raphson like this

$$\Delta^{(\nu)} = \begin{bmatrix} \Delta^{(\nu)} x \\ \Delta^{(\nu)} y \end{bmatrix}$$

0: Starting point : $y_n^{(0)} = y_{n-1}$

1: Compute correction $\Delta^{(\nu)}$: $J(x_n^{(\nu)}, y_n^{(\nu)}) \cdot \Delta^{(\nu)} = -g(x_n^{(\nu)}, y_n^{(\nu)})$

2: Apply correction $x_n^{(\nu+1)} = x_n^{(\nu)} + \Delta^{(\nu)} x$

$$y_n^{(\nu+1)} = y_n^{(\nu)} + \Delta^{(\nu)} y$$

3: compute norm residual $\|\Delta^{(\nu)}\|$

4: If $\|\Delta^{(\nu)}\| < \varepsilon$, break; otherwise set $\nu = \nu+1$ and go to step 1

5: set $x_n = x_n^{(\nu)}$ & $y_n = y_n^{(\nu)}$.