Frictional Contact: Differential Variational Inequality Approaches

Final Thoughts, ME751
April 20, 2010

“All you need is ignorance and confidence and the success is sure.”
Mark Twain
Before we get started...

- Last Time:
  - Wrapped up DEM
  - Started DVI (last topic discussed in ME751)

- Today:
  - Finish the DVI approach
  - Final thoughts, ME751

- HW
  - Posted on class website, due on April 29, at the time of midterm exam

- What comes next
  - Talks from industry: NVIDIA (Apr 22), ADAMS (Apr 27), NVIDIA (May 6)
  - Midterm exam (Apr 29, 7:15-9:15 pm) – Room TBA
  - Trip to John Deere and Iowa (May 3-4)
The DVI Problem:
The EOM, in Fine Granularity Form

- The time evolution of the dynamical system is governed by the following differential variational inequality (DVI)

\[ B = 1, \ldots, nb : \quad m_B \ddot{\mathbf{r}}_B = \sum_{i \in B(B)} \left[ \Psi_{rB}^{(i)} \right]^T \dot{\gamma}_{i,b} + \mathbf{f}_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} (\dot{\gamma}_{i,n} \mathbf{n}_i + \dot{\gamma}_{i,u} \mathbf{u}_i + \dot{\gamma}_{i,w} \mathbf{w}_i) \]

\[ B = 1, \ldots, nb : \quad \ddot{\mathbf{J}}_B \dot{\mathbf{w}}_B = \sum_{i \in B(B)} \prod_B^T (\Psi^{(i)}) \dot{\gamma}_{i,b} + \tau_B(t, \mathbf{q}, \mathbf{v}) + \sum_{i \in A(B)} \mathbf{s}_{i,B} \mathbf{A}_B^T (\dot{\gamma}_{i,n} \mathbf{n}_i + \dot{\gamma}_{i,u} \mathbf{u}_i + \dot{\gamma}_{i,w} \mathbf{w}_i) \]

\[ B = 1, \ldots, nb : \quad \dot{\mathbf{p}}_B = \frac{1}{2} \mathbf{G}^T (p_B) \ddot{\mathbf{w}}_B \]

\[ \quad i \in B : \quad \Psi_i(\mathbf{q}, t) = 0 \]

\[ \quad i \in A : \quad 0 \leq \dot{\gamma}_{i,n} \perp \Phi_i(\mathbf{q}) \geq 0, \]

\[ \quad i \in A : \quad (\dot{\gamma}_{i,u}, \dot{\gamma}_{i,w}) = \arg\min_{\mu_i \dot{\gamma}_{i,n} \geq \sqrt{x^2 + y^2}} \mathbf{v}^T (x \mathbf{D}_{i,u} + y \mathbf{D}_{i,w}) \]
Notation Conventions

[1/2]

- To stick with the presentation in the paper of Anitescu and Tasora, we’ll use the following notation

  - Point 1: Instead of sticking with transpose of Jacobians, we’ll use gradients, which are defined precisely as the transpose of the Jacobians. Specifically,
    \[ \nabla_q \Psi_i = \Psi_{i,q}^T = [\partial \Psi_i / \partial q]^T \quad \text{and} \quad \nabla_q \Phi_i = \Phi_{i,q}^T = [\partial \Phi_i / \partial q]^T \nabla_q \]

  - Point 2: We’ll use the transformation matrix \( L(q) \) to link the time derivative of the level zero unknowns in the \( r - p \) formulation to the level one unknowns in the \( r - \omega \) formulation:
    \[ \dot{q} = L(q) \nu \]

  - Point 3: To keep the notation simpler (and probably confuse you), we’ll group the translational and rotational equations of motion in one big matrix-vector equation (nothing changed, except the notation) in order to have less symbols and equations to deal with

  - Point 4: We’ll use the following notation (\( h \) is the integration step-size)
    \[ \gamma_{i,n} = h \gamma_{i,n} \quad \gamma_{i,u} = h \gamma_{i,u} \quad \gamma_{i,w} = h \gamma_{i,w} \quad \gamma_{i,b} = h \gamma_{i,b} \]
    * Recall that time \( \times \) force (like in \( \gamma_{i,n} = h \gamma_{i,n} \)) is impulse, and it’s impulse that changes the momentum of a body
• Define the transformation matrix $A_{i \rightarrow G}$ that given the representation of a geometric vector in the contact reference frame associated with contact $i$ is used to generate its representation in the GRF:

$$A_{i \rightarrow G} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix}$$

- Note that the frictional contact force at contact $i$ as felt by body $A$ is simply

$$F^f_{i,A} = n_i \hat{\gamma}_{i,n} + u_i \hat{\gamma}_{i,u} + w_i \hat{\gamma}_{i,w} = \begin{bmatrix} n_i & u_i & w_i \end{bmatrix} \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix} = A_{i \rightarrow G} \cdot \hat{\gamma}_i \text{ where } \hat{\gamma}_i = \begin{bmatrix} \hat{\gamma}_{i,n} \\ \hat{\gamma}_{i,u} \\ \hat{\gamma}_{i,w} \end{bmatrix}$$

• A projection matrix $D_i$ is defined for each contact $i \in A$ to project the contact forces onto the equations of motion, both for translation and rotation. If we assume that contact $i$ acts between body $A$ and body $B$,

$$D_i = \begin{bmatrix} 0 \\ \vdots \\ A_{i \rightarrow G} \\ \tilde{s}_{i,A} A_{A}^T A_{i \rightarrow G} \\ 0 \\ \vdots \\ 0 \\ -A_{i \rightarrow G} \\ -\tilde{s}_{i,B} A_{B}^T A_{i \rightarrow G} \\ 0 \end{bmatrix}_{6nb \times 3}$$

- Notation used in expression of $D_i$: the vectors $\tilde{s}_{i,A}$ and $\tilde{s}_{i,B}$ represent the location of the contact point in the local reference frame of body $A$ and $B$, respectively

• The columns of $D_i$ are denoted by $D_{i,n}$, $D_{i,u}$, $D_{i,w}$ and are each vectors of dimension $6nb$:

$$D_i = \begin{bmatrix} D_{i,n} & D_{i,u} & D_{i,w} \end{bmatrix}_{6nb \times 3}$$
The Problem Setup

- The resulting problem that we have to deal with now looks like this

\[
\begin{align*}
\dot{q} &= L(q)v \\
M\dot{v} &= f(t, q, v) + \sum_{i \in \mathcal{B}} \hat{\gamma}_{i,b} \nabla \Psi_i + \sum_{i \in \mathcal{A}} (\hat{\gamma}_{i,n} D_{i,n} + \hat{\gamma}_{i,u} D_{i,u} + \hat{\gamma}_{i,w} D_{i,w}) \\
i \in \mathcal{B} :& \quad \Psi_i(q, t) = 0 \\
i \in \mathcal{A} :& \quad 0 \leq \hat{\gamma}_{i,n} \perp \Phi_i(q) \geq 0,
\end{align*}
\]

\[
(\hat{\gamma}_{i,u}, \hat{\gamma}_{i,w}) = \arg\min_{\mu \geq \sqrt{x^2 + y^2}} v^T (x D_{i,u} + y D_{i,w})
\]
The Discretization Process

- Recall that in our Ford F-150 direction solution, we used the Newton-Euler form of the equations of motion in conjunction with the level zero constraints (the position constraint equations).

- In the proposed approach to find the time evolution of the multi-body system we’ll use instead the level one constraints (velocity level constraints).

- Implications:
  - Since the level zero constraints are not enforced, there will be drift in the solution.
  - Stabilization terms, that penalize the violation of the level zero constraints, are added to the level one bilateral and unilateral constraints.
  - The bilateral and unilateral constraints are massaged into the following (note that a superscript \((l)\) denotes the time step; used to be a subscript, typically \(n\), yet \(n\) stands for ’normal’ now):

\[
\begin{align*}
\forall i \in \mathcal{B} & : \quad \frac{1}{h} \Psi_i(q^{(l)}_{i}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0 \\
\forall i \in \mathcal{A} & : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0
\end{align*}
\]

* The above reformulation becomes the Second Key Twist (out of three).
* There is much to be said here, modifying the expression of the constraints and working with level one constraints are two issues that demand more attention than what we pay here.
The discretized equations look like this:

\[
M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,h} \nabla \Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
\]

\[
i \in A : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \text{argmin}_{\mu, \gamma_{i,n} \geq \sqrt{x^2 + y^2}} v^T (x D_{i,u} + y D_{i,w})
\]

\[
q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}.
\]

The first four of the equations above together combine for an optimization problem with equilibrium constraints.

Why an optimization problem?
- Because the way the Coulomb friction model is posed

What type of optimization problem?
- This represents a nonlinear optimization problem
- Can be linearized if the friction cone is discretized and represented as a multifaceted pyramid (bad idea: problem size increases, anisotropy creeps in)

What are the 'equilibrium constraints'?
- Your typical optimization problem might display algebraic equality or inequality constraints
- Above, we are solving an optimization problem for which the constraints represent the discretization of a set of differential equations
● Left: November 1925 issue of TIME magazine with Painleve on the cover
● Right: Discretization of the friction cone with a multifaceted pyramid (bad idea)
Given: matrix \( M \) of dimension \( n \times n \), and vector \( q \) of dimension \( n \)

Find: vector[s] \( z \) of dimension \( n \) such that

\[
\begin{align*}
  z &\geq 0 \\
  Mz + q &\geq 0 \\
  &\quad \& \\
  z^T (Mz + q) &= 0
\end{align*}
\]

The last condition above sometimes stated as

\( z \perp (Mz + q) \)

Seminal contributions: Lemke, Cottle, Mangasarian, J.S. Pang
Example, Problem Leading to LCP

- Assume $M$ is a symmetric matrix and $q$ is a vector, both of dimension $n$

- Solve the minimization problem

\[
\begin{align*}
\min_x & \quad \frac{1}{2}x^T M x + q^T x \\
\text{subject to} & \quad x \geq 0
\end{align*}
\]

- NOTE: This optimization problem leads to precisely the LCP stated on the previous slide

- There are well established methods used to solve LCPs (Lemke’s method, for instance)
Example 1

- Solve the LCP defined by

\[
M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -3 \\ 2 \end{bmatrix}
\]

- Note:

\[
z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Mz + q = \begin{bmatrix} 0 \\ 3 \end{bmatrix}
\]
Example 2

- Solve the LCP defined by

\[ M = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \]

- Note:

\[ z = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad Mz + q = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ Mz + q = 0_{3 \times 1} \]
The NCP → CCP Metamorphosis

The Third Key Twist (out of three)

- Dealing with some generic nonlinear optimization problem like the one above is daunting
- Can we do some trick and recast it as a simpler optimization problem for which (i) we are guaranteed that a solution exists (ideally, it would be unique, in some sense), and (ii) there are tailored algorithms that we can use to efficiently find the solution
- Coming from the left field, it turns out that if you introduce a relaxation over the complementarity constraints the problem that you have to solve can be posed as a cone complementarity problem (CCP). To this end, rather than working with

\[ i \in A : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} \geq 0 \]

Work with this:

\[ i \in A : 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0 \]
The Cone Complementarity Problem

- The resulting problem that we have to deal with now looks like this

\[
M(v^{(l+1)} - v^{(l)}) = hf(t^{(l)}, q^{(l)}, v^{(l)}) + \sum_{i \in B} \gamma_{i,b} \nabla \Psi_i + \sum_{i \in A} (\gamma_{i,n} D_{i,n} + \gamma_{i,u} D_{i,u} + \gamma_{i,w} D_{i,w})
\]

\[
i \in B : \quad \frac{1}{h} \Psi_i(q^{(l)}, t) + \nabla \Psi_i^T v^{(l+1)} + \frac{\partial \Psi_i}{\partial t} = 0
\]

\[
i \in A : \quad 0 \leq \gamma_{i,n} \perp \frac{1}{h} \Phi_i(q^{(l)}) + D_{i,n}^T v^{(l+1)} - \mu_i \sqrt{(v^T D_{i,u})^2 + (v^T D_{i,w})^2} \geq 0
\]

\[
(\gamma_{i,u}, \gamma_{i,w}) = \arg\min_{\mu_i \gamma_{i,n} \geq \sqrt{x^2+y^2}} v^T (x D_{i,u} + y D_{i,w})
\]

\[
q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}.
\]
Cone Complementarity Problem (CCP)

- The new discretization that draws on the modified complementarity condition converges when \( h \rightarrow 0 \) to the solution of the original discretization.

- After some more massaging, with the exception of the very last one, the equations on the previous slide combine to lead to the following CCP (for the intermediates steps that you need to get to the CCP see paper of Anitescu and Tasora or come and talk to me):

  - Introduce the convex hypercone...
    \[
    \Upsilon = \left( \bigoplus_{i \in A(q)} \mathcal{F}^i \right) \oplus \left( \bigoplus_{i \in B(q)} \mathcal{B}^i \right)
    \]
    where \( \mathcal{F}^i \) is the \( i \)-th friction cone, \( \mathcal{B}^i \) is \( \mathbb{R} \).

  - ... and its polar hypercone
    \[
    \Upsilon^o = \left( \bigoplus_{i \in A(q)} \mathcal{F}^{i_o} \right) \oplus \left( \bigoplus_{i \in B(q)} \mathcal{B}^{i_o} \right)
    \]

  - The CCP that needs to be solved at each time step is as follows (note that matrix \( N \) and vector \( d \) are computed based on state information at time-step \( t^{(l)} \)):
    * Find the Lagrange hyper-multiplier \( \gamma \) that satisfies:
      \[
      \gamma \in \Upsilon \perp -(N \gamma + d) \in \Upsilon^o
      \]
Visualization of convex hypercone $\Upsilon$ and its polar complement $\Upsilon^\circ$

$\text{(N}_i\gamma_i + d_i) = 0_{3 \times 1}$

- Sliding scenario
- Sticking scenario
Problem Solved At Each Time Step
(going from \( t^{(l)} \) to \( t^{(l+1)} \))

- It boils down to solving this:

\[
\begin{align*}
\gamma & \in \mathcal{X} \quad \perp \quad -(\mathbf{N}\gamma + \mathbf{d}) \in \mathcal{Y}^o \\
\mathbf{v}^{(l+1)} &= M^{-1} \left( \mathbf{k} + \mathbf{D}\gamma \right) \\
\mathbf{q}^{(l+1)} &= \mathbf{q}^{(l)} + h\mathbf{L}(\mathbf{q}^{(l)})\mathbf{v}^{(l+1)}
\end{align*}
\]
Implementation of Method

- The method outlined implemented using two loops
  - Outer loop – runs the time stepping
  - Inner loop – CCP Algorithm (solves CCP problem at each time step)
Outer Loop (Time-Stepping)

1. Set $t = 0$, step counter $l = 0$, provide initial values for $q^{(l)}$ and $v^{(l)}$.

2. Perform collision detection between bodies. For each contact $i$, compute $D_{i,n}$, $D_{i,u}$, $D_{i,w}$.

3. For each body, compute forces $f(t^{(l)}, q^{(l)}, v^{(l)})$.

4. Use CCP Algorithm to solve the cone complementarity problem and obtain unknown impulse $\gamma$ and velocity $v^{(l+1)}$.

5. Update positions using $q^{(l+1)} = q^{(l)} + hL(q^{(l)})v^{(l+1)}$.

6. Increment $t := t + h$, $l := l + 1$, and repeat from step 2 until $t > t_{\text{end}}$.

“Inner Loop”

Preprocessing

Post processing
Inner Loop (CCP Algorithm)

1. For each contact $i$, evaluate $\eta_i = 3 / \text{Trace}(D_i^T M^{-1} D_i)$.

2. If some initial guess $\gamma^*$ is available for multipliers, then set $\gamma^0 = \gamma^*$, otherwise $\gamma^0 = 0$.

3. Initialize velocities: $v^0 = \sum_i M^{-1} D_i \gamma_i^0 + M^{-1} \tilde{k}$.

4. For each contact $i$, compute changes in multipliers for contact constraints:

   $$\gamma_i^{r+1} = \lambda \Pi_{\gamma_i} (\gamma_i^r - \omega \eta_i (D_i^T v^r + b_i)) + (1 - \lambda) \gamma_i^r ;$$

   $$\Delta \gamma_i^{r+1} = \gamma_i^{r+1} - \gamma_i^r ;$$

   $$\Delta v_i = M^{-1} D_i \Delta \gamma_i^{r+1} .$$

5. Apply updates to the velocity vector:

   $$v^{r+1} = v^r + \sum_i \Delta v_i$$

6. $r := r + 1$. Repeat from 4 until convergence or $r > r_{max}$
Parallelism, Opportunities

1. Perform parallel collision detection
2. Copy contact, body, and constraint data structures to GPU
3. (Body parallel) Force kernel
4. (Contact parallel) Contact preprocessing kernel
5. (Contact parallel) CCP contact kernel
6. (Constraint parallel) CCP constraint kernel
7. (Reduction-slot parallel) Reduction kernel
8. (Body parallel) Body velocity update kernel
9. (Body parallel) Time integration kernel

Iteration loop
General Comments, DEM and DVI

- There is hand waving when it comes to handling friction and contact
  - More so with DEM than DVI

- Handling frictional contact is equally art and science
  - To get something to run robustly requires tweaking
  - Requires time to understand the landscape in this area

- Doing justice to this topic would probably require six more lectures

- It’s an active area of research that can answer some challenging engineering and science open questions


ME751
Putting Things in Perspective
ME751 – What did we learn?
[1/2]

- Learned about vectors and locating points attached to moving rigid bodies
- Learned how to describe the orientation of a body in 3D space
- Learned how to express geometric constraints associated with the relative motion of two bodies (four building blocks – DP1, DP2, D, CD)
- Learned about the Kinematics Analysis, carried out for zero DOF systems in which one or more motions are prescribed
  - Position Analysis (requires solution of nonlinear system)
  - Velocity Analysis (requires solution of linear system)
  - Acceleration Analysis (requires solution of linear system)
ME751 – What did we learn? [2/2]

- Learned how to formulate the equations of motion for a system of interconnected and mutually interacting rigid bodies

- Learned how to solve the equations that govern the time evolution of mechanical system
  - These equations are differential-algebraic equations (DAEs)
  - The workhorse was the BDF family of implicit integration formulas
  - We used the F-150 direct approach but more sophisticated approaches exist

- We discussed briefly about accounting for the friction and contact phenomena in mechanical systems
  - DEM approaches – simple, integration step-sizes are very short
  - DVI approaches – pretty complex, integration step-sizes are long though
ME751: Things overlooked

- We didn’t do justice to the frictional contact problem
  - Would be good to talk about this for more 7-8 lectures

- We didn’t cover at all multi-body systems with deformable bodies
  - Floating frame of reference formulation
  - Absolute nodal coordinate formulation (ANCF)
  - Structural finite element approaches
  - Linear & nonlinear isoparametric finite element method
  - Handling of friction and contact for deformable bodies

- Real-time simulation for gaming and virtual reality applications
  - Connection to SPH (Smoothed Particle Hydrodynamics)

- High Performance Computing (HPC) for Computational Multibody Dynamics applications