ME751
Advanced Computational Multibody Dynamics

March 4, 2010

"Without music to decorate it, time is just a bunch of boring production deadlines or dates by which bills must be paid."
Frank Zappa

Before we get started…

- Last Time:
  - Discussed Virtual Displacement and Variation of a Function
  - Covered the $\delta\mathbf{r}$, $\delta\pi$ case

- Today:
  - Discuss about how to visualize the time evolution of a mechanical system
  - Virtual Displacement: cover the $\delta\mathbf{r}$, $\delta\mathbf{p}$ case
  - Start discussion on how to obtain the EOM (the Dynamics problem)

- HW7 – due on March 11: posted online later today
    - This HW will require you to decide on a Final Project

- Forum up and running: http://sbel.wisc.edu/Forum/index.php?board=4.0
  - Use the forum to post questions about your HW, class notes, etc
  - I plan to monitor the discussion threads and contribute as much as I can
  - Started a discussion topic on course notes mistakes
    - Everyone who finds/posts a valid mistake first gets $1 from me

The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta\mathbf{p}$ Virtual Rotation

- Framework: assume you have a vector quantity that depends on $\mathbf{p}$. Assume that the value of $\mathbf{p}$ changes to $\mathbf{p} + \delta\mathbf{p}$. What is the variation in the quantity that depends on $\mathbf{p}$ due to the said change?
- Specifically, assume the vector quantity of interest is $\mathbf{u}$, and $\mathbf{u}$ depends on $\mathbf{p}$ and possibly time t:

$$\mathbf{u} = \mathbf{u}(\mathbf{p}, t)$$

- I am interested at a fixed time t in the $\delta\mathbf{u}$ below given $\mathbf{p}$, $\delta\mathbf{p}$, and the expression of $\mathbf{u}(\mathbf{p})$:

$$\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p}, t) \qquad \mathbf{p} + \delta\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p} + \delta\mathbf{p}, t) = \mathbf{u}(\mathbf{p}, t) + \delta\mathbf{u}$$

$$\delta\mathbf{u} = ?$$

The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta\mathbf{p}$ Virtual Rotation

[Cntd.]:

- The answer to question of interest, $\delta\mathbf{u}(\mathbf{p}) =?$, is obtained using a Taylor series expansion:

$$\begin{aligned} \mathbf{u}(\mathbf{p}+\delta\mathbf{p},t) &= \mathbf{u}(\mathbf{p},t) + \mathbf{u}_{\mathbf{p}}\delta\mathbf{p} + \dots \\ &\approx \mathbf{u}(\mathbf{p},t) + \mathbf{u}_{\mathbf{p}}\delta\mathbf{p} \end{aligned}$$

- Then

$$\delta\mathbf{u}(\mathbf{p}) = \mathbf{u}(\mathbf{p}+\delta\mathbf{p},t) - \mathbf{u}(\mathbf{p},t) = \mathbf{u}_{\mathbf{p}}\delta\mathbf{p}$$

- In the argument above, we rely on the fact that the virtual rotations, that is, the perturbations $\delta\mathbf{p}$, are small and therefore higher order terms that contain entries of $\delta\mathbf{p}$, that is, δe_0, δe_1, δe_2, or δe_3, can be safely approximated to be zero.

- Important observation: note that the time does note play a role in figuring out what the variation in $\mathbf{u}$ is. In other words, looking into the variation of $\mathbf{u}$ is an exercise that is carried out at a certain time t, and time is held fixed.

- Note that the same argument applies if u is a scalar function that depends on $\mathbf{p}$. In that case,

$$\delta u(\mathbf{p},t) = u_{\mathbf{p}}\delta\mathbf{p}$$

Exercise

- Calculate the variation of the function $\mathbf{u}(\mathbf{p}) = \mathbf{A}(\mathbf{p})\bar{\mathbf{s}}$ due to a variation $\delta\mathbf{p}$ in the Euler Parameters. The vector $\bar{\mathbf{s}}$ does not depend on $\mathbf{p}$.

Exercise

- Calculate the variation of the function $u(\mathbf{p}) = \mathbf{p}^T \mathbf{p} - 1$ due to a variation $\delta \mathbf{p}$ in the Euler Parameters

Quick Question

- Note that when interested in variations as induced by virtual rotations of the $\delta\mathbf{p}$ flavor (as opposed to the $\delta\bar{\pi}$ flavor), it is very straightforward to produce the quantity of interest:

$$\delta\mathbf{u}(\mathbf{p}) = \mathbf{u}_{\mathbf{p}}\delta\mathbf{p}$$

- Why did not we take the same approach for the $\delta\bar{\pi}$?
  - We couldn't do this direct approach for the the same reason we couldn't find a set of three variables whose time derivative is the angular velocity $\bar{\omega}$
  - Specifically, there is no concept of partial derivative $\mathbf{u}_{\bar{\pi}}$ to work with, and therefore we have to resort to the process that in the end expresses the variation $\delta\mathbf{u}$ or the time derivative $\dot{\mathbf{u}}$ using $\bar{\mathbf{\Pi}}(\mathbf{u})$ and $\delta\bar{\pi}$, or $\bar{\mathbf{\Pi}}(\mathbf{u})$ and $\bar{\omega}$, respectively

Virtual Variation, Basic GCons: Φ^{DP1} [The $\delta\mathbf{p}$ Flavor]

- Recall that

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = \mathbf{a}_i{}^T \mathbf{a}_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_i} = \mathbf{0}_{1\times 3} \qquad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} = \mathbf{a}_j^T \mathbf{B}\left(\mathbf{p}_i, \bar{\mathbf{a}}_i\right)$$

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_j} = \mathbf{0}_{1\times 3} \qquad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} = \mathbf{a}_i^T \mathbf{B}\left(\mathbf{p}_j, \bar{\mathbf{a}}_j\right)$$

- Putting it all together, $\delta\Phi^{DP1} = \Phi^{DP1}_{\mathbf{q}} \delta\mathbf{q}$, where,

$$\Phi^{DP1}_{\mathbf{q}} = \left[\mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \ldots \mathbf{0}_{1\times 4} \;\; \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} \;\; \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4} \;\; \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} \;\; \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4}\right]$$

Partials with respect to $\mathbf{r}$ ← | → Partials with respect to $\mathbf{p}$

$$= \left[\mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \mathbf{0}_{1\times 3} \ldots \ldots \mathbf{0}_{1\times 4} \;\; \mathbf{a}_j^T \mathbf{B}\left(\mathbf{p}_i, \bar{\mathbf{a}}_i\right) \;\; \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4} \;\; \mathbf{a}_i^T \mathbf{B}\left(\mathbf{p}_j, \bar{\mathbf{a}}_j\right) \;\; \mathbf{0}_{1\times 4} \ldots \mathbf{0}_{1\times 4}\right]$$

Body 1, **r** | Body i, **r** | Body j, **r** | Body i-1, **p** | Body i, **p** | Body i+1, **p** | Body j-1, **p** | Body j, **p** | Body j+1, **p** | Body nb, **p**

[Short Detour]:

Computing $\delta\mathbf{d}_{ij}$

- Recall that

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P = \mathbf{r}_j + \mathbf{s}_j^Q - \mathbf{r}_i - \mathbf{s}_i^P$$

- Recall also that

$$\begin{aligned} [\mathbf{d}_{ij}]_{\mathbf{q}_i,\mathbf{q}_j} &= [-\mathbf{I}_3 \quad -(\mathbf{s}_i^P)_{\mathbf{p}_i} \quad \mathbf{I}_3 \quad (\mathbf{s}_j^Q)_{\mathbf{p}_j}] \\ &= [-\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)] \end{aligned}$$

- It follows that

$$\delta\mathbf{d}_{ij} = [-\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = [\mathbf{d}_{ij}]_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Basic GCons: Φ^{DP2} [The $\delta\mathbf{p}$ Flavor]

- Recall that

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \mathbf{a}_i{}^T \mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{DP2}_{\mathbf{q}_i,\mathbf{q}_j}(\mathbf{a}_i, \mathbf{d}_{ij}) = \begin{bmatrix} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{a}_i^T & \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta\Phi^{DP2} = \begin{bmatrix} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{a}_i^T & \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^{DP2}_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Basic GCons: Φ^D [The $\delta\mathbf{p}$ Flavor]

- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) \quad = \quad \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = 0$$

- It also that

$$\Phi^D_{\mathbf{q}_i, \mathbf{q}_j} = [\; -2\mathbf{d}_{ij}^T \qquad -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \qquad 2\mathbf{d}_{ij}^T \qquad 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \;]$$

- It follows that

$$\delta\Phi^D = [\; -2\mathbf{d}_{ij}^T \qquad -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \qquad 2\mathbf{d}_{ij}^T \qquad 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \;] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^D_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Basic GCons: Φ^{CD} [The $\delta\mathbf{p}$ Flavor]

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T\mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{CD}_{\mathbf{q}_i,\mathbf{q}_j} = \begin{bmatrix} -\mathbf{c}^T & -\mathbf{c}^T\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{c}^T & \mathbf{c}^T\mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta\Phi^{CD} = \begin{bmatrix} -\mathbf{c}^T & -\mathbf{c}^T\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{c}^T & \mathbf{c}^T\mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi^{CD}_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Virtual Variation, Euler Parameter Normalization Constraint: $\Phi^{\mathbf{p}}$

- Recall that the Euler Parameter normalization constraint assumes the expression

$$\Phi_i^{\mathbf{p}} = \mathbf{p}_i^T \mathbf{p}_i - 1 = 0$$

- Recall also that

$$(\Phi_i^{\mathbf{p}})_{\mathbf{q}_i} = \begin{bmatrix} \mathbf{0}_{1\times 3} & 2\mathbf{p}_i^T \end{bmatrix}$$

- It follows that

$$\delta\Phi_i^{\mathbf{p}} = \begin{bmatrix} \mathbf{0}_{1\times 3} & 2\mathbf{p}_i^T \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \end{bmatrix} = (\Phi_i^{\mathbf{p}})_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Comments [1/2]

- In the next lectures we'll express the virtual variation of a function that depends on the position and orientation of one or more bodies in the system using one of two sets of virtual displacements: $\delta\mathbf{r}$ and $\delta\bar{\pi}$, or $\delta\mathbf{r}$ and $\delta\mathbf{p}$

- At the end of the day, a virtual displacement of the bodies in the system will lead to a virtual variation of a generic constraint Φ^α, $\alpha \in \{DP1, DP2, D, CD, \mathbf{p}\}$:

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \Phi^\alpha_{\mathbf{r}}\delta\mathbf{r} + \bar{\Pi}(\Phi^\alpha)\delta\bar{\pi} = \Phi^\alpha_{\mathbf{r}}\delta\mathbf{r} + \Phi^\alpha_{\mathbf{p}}\delta\mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \begin{bmatrix} \Phi^\alpha_{\mathbf{r}} & \bar{\Pi}(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi^\alpha) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$$

$$\delta\Phi^\alpha(\mathbf{r},\mathbf{p}) = \begin{bmatrix} \Phi^\alpha_{\mathbf{r}} & \Phi^\alpha_{\mathbf{p}} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{p} \end{bmatrix} = \Phi^\alpha_{\mathbf{q}} \cdot \delta\mathbf{q}$$

Comments [2/2]

- Recall that we are supposed to collect *all* m constraints and stick them together in one big $\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \end{bmatrix}$, if we plan to work with virtual displacements expressed in terms of $\delta\mathbf{r}$ and $\delta\bar{\pi}$, or in $\mathbf{\Phi}^F = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \\ \mathbf{\Phi}^{\mathbf{P}}(\mathbf{p}) \end{bmatrix}$, if we plan to work with $\delta\mathbf{r}$ and $\delta\mathbf{p}$.

- Recall that any one of the constraints in $\mathbf{\Phi}$ is one of the four basic GCons that we introduced

- When interested in the variation of $\mathbf{\Phi}$, we simply stack together the variation of each of the GCons that enters in $\mathbf{\Phi}$. The situation for $\mathbf{\Phi}^F$ is similar, except that here you need to account explicitly for the Euler Parameter normalization constraints.

- A virtual displacement of the bodies in the system will lead to a virtual variation $\delta\mathbf{\Phi}$ that depends on the position and orientation of the bodies:

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_{\mathbf{r}}\delta\mathbf{r} + \bar{\mathbf{\Pi}}(\mathbf{\Phi})\delta\bar{\pi} \qquad \text{or} \qquad \delta\mathbf{\Phi}^F = \mathbf{\Phi}_{\mathbf{r}}^F\delta\mathbf{r} + \mathbf{\Phi}_{\mathbf{p}}^F\delta\mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta\mathbf{\Phi}(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$$

$$\delta\mathbf{\Phi}^F(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}}^F & \mathbf{\Phi}_{\mathbf{p}}^F \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{p} \end{bmatrix} = \mathbf{\Phi}_{\mathbf{q}}^F \cdot \delta\mathbf{q}$$

The Concept of Consistent Virtual Displacements

- Framework: assume that your $\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}$ is such that the constraints are satisfied; i.e., $\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0}$. Apply now a virtual displacement $\delta\mathbf{r}_i$ and $\delta\mathbf{p}_i$ to each body i in the system.

- Question: how should you choose the virtual displacements $\delta\mathbf{r}_i$ and $\delta\mathbf{p}_i$, $i = 1, \ldots, nb$ so that the new configuration is also consistent?

- I am interested in a healthy $\delta\mathbf{q}$:

$$\mathbf{q} \longrightarrow \mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0} \qquad \mathbf{q} + \delta\mathbf{q} \longrightarrow \mathbf{\Phi}^F(\mathbf{q} + \delta\mathbf{q}, t) = \mathbf{\Phi}^F(\mathbf{q}, t) + \delta\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0}$$

- It follows that

$$\delta\mathbf{\Phi}^F(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi}^F_{\mathbf{q}}\delta\mathbf{q} = \mathbf{0}$$

- *By definition*, a virtual displacement $\delta\mathbf{q}$ is said to be consistent with the set of constraints present in the system if $\mathbf{\Phi}^F_{\mathbf{q}}\delta\mathbf{q} = \mathbf{0}$ holds

- Note that a similar train of thought can be followed to define a $\begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$ consistent virtual displacement. The condition in that case reads

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{r}} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \mathbf{0}$$

Deriving the EOM