“Without music to decorate it, time is just a bunch of boring production deadlines or dates by which bills must be paid.”
Frank Zappa
Before we get started…

- **Last Time:**
  - Discussed Virtual Displacement and Variation of a Function
  - Covered the $\delta r$, $\delta \pi$ case

- **Today:**
  - Discuss about how to visualize the time evolution of a mechanical system
  - Virtual Displacement: cover the $\delta r$, $\delta p$ case
  - Start discussion on how to obtain the EOM (the Dynamics problem)

- **HW7 – due on March 11: posted online later today**
  - This HW will require you to decide on a Final Project

- **Forum up and running:** [http://sbel.wisc.edu/Forum/index.php?board=4.0](http://sbel.wisc.edu/Forum/index.php?board=4.0)
  - Use the forum to post questions about your HW, class notes, etc
  - I plan to monitor the discussion threads and contribute as much as I can
  - Started a discussion topic on course notes mistakes
    - Everyone who finds/posts a valid mistake first gets $1 from me
The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta p$ Virtual Rotation

- Framework: assume you have a vector quantity that depends on $p$. Assume that the value of $p$ changes to $p + \delta p$. What is the variation in the quantity that depends on $p$ due to the said change?

- Specifically, assume the vector quantity of interest is $u$, and $u$ depends on $p$ and possibly time $t$:

$$u = u(p, t)$$

- I am interested at a fixed time $t$ in the $\delta u$ below given $p$, $\delta p$, and the expression of $u(p)$:

$$p \rightarrow u(p, t) \quad p + \delta p \rightarrow u(p + \delta p, t) = u(p, t) + \delta u$$

$$\delta u = ?$$
The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta p$ Virtual Rotation

[Cntd.]

- The answer to question of interest, $\delta u(p) = ?$, is obtained using a Taylor series expansion:

  $$u(p + \delta p, t) = u(p, t) + u_p \delta p + \ldots$$

  $$\approx u(p, t) + u_p \delta p$$

- Then

  $$\delta u(p) = u(p + \delta p, t) - u(p, t) = u_p \delta p$$

- In the argument above, we rely on the fact that the virtual rotations, that is, the perturbations $\delta p$, are small and therefore higher order terms that contain entries of $\delta p$, that is, $\delta e_0$, $\delta e_1$, $\delta e_2$, or $\delta e_3$, can be safely approximated to be zero.

- Important observation: note that the time does not play a role in figuring out what the variation in $u$ is. In other words, looking into the variation of $u$ is an exercise that is carried out at a certain time $t$, and time is held fixed.

- Note that the same argument applies if $u$ is a scalar function that depends on $p$. In that case,

  $$\delta u(p, t) = u_p \delta p$$
Exercise

- Calculate the variation of the function $u(p) = A(p)\bar{s}$ due to a variation $\delta p$ in the Euler Parameters. The vector $\bar{s}$ does not depend on $p$. 
Exercise

- Calculate the variation of the function $u(p) = p^T p - 1$ due to a variation $\delta p$ in the Euler Parameters
Quick Question

- Note that when interested in variations as induced by virtual rotations of the $\delta p$ flavor (as opposed to the $\delta \bar{\pi}$ flavor), it is very straightforward to produce the quantity of interest:

$$\delta u(p) = u_p \delta p$$

- Why did not we take the same approach for the $\delta \bar{\pi}$?

  We couldn’t do this direct approach for the the same reason we couldn’t find a set of three variables whose time derivative is the angular velocity $\bar{\omega}$

  - Specifically, there is no concept of partial derivative $u_{\bar{\pi}}$ to work with, and therefore we have to resort to the process that in the end expresses the variation $\delta u$ or the time derivative $\dot{u}$ using $\Pi(u)$ and $\delta \bar{\pi}$, or $\Pi(u)$ and $\bar{\omega}$, respectively
Virtual Variation, Basic GCons: $\Phi^{DP1}$
[The $\delta p$ Flavor]

- Recall that

$$\Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, f(t)) = a_i^T A_i^T A_j \bar{a}_j - f(t) = a_i^T a_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^{DP1}}{\partial r_i} = 0_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial p_i} = a_j^T B(p_i, \bar{a}_i)$$

$$\frac{\partial \Phi^{DP1}}{\partial r_j} = 0_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial p_j} = a_i^T B(p_j, \bar{a}_j)$$

- Putting it all together, $\delta \Phi^{DP1} = \Phi_q^{DP1} \delta q$, where,

$$\Phi_q^{DP1} = \begin{bmatrix} 0_{1 \times 3} \ldots 0_{1 \times 3} \ldots 0_{1 \times 3} \ldots \ldots 0_{1 \times 4} & \frac{\partial \Phi^{DP1}}{\partial p_i} & 0_{1 \times 4} \ldots 0_{1 \times 4} & \frac{\partial \Phi^{DP1}}{\partial p_j} & 0_{1 \times 4} \ldots 0_{1 \times 4} \end{bmatrix}$$

Partials with respect to $r$  Partials with respect to $p$

$$= \begin{bmatrix} 0_{1 \times 3} \ldots 0_{1 \times 3} \ldots 0_{1 \times 3} \ldots \ldots 0_{1 \times 4} & a_j^T B(p_i, \bar{a}_i) & 0_{1 \times 4} \ldots 0_{1 \times 4} & a_i^T B(p_j, \bar{a}_j) & 0_{1 \times 4} \ldots 0_{1 \times 4} \end{bmatrix}$$

Body 1, $r$  Body $i$, $r$  Body $j$, $r$  Body $i-1$, $p$  Body $i$, $p$  Body $i+1$, $p$  Body $j-1$, $p$  Body $j$, $p$  Body $j+1$, $p$  Body nb, $p$
[Short Detour]:

Computing $\delta d_{ij}$

- Recall that
  \[ d_{ij} = r_j + A_j \bar{s}^Q_j - r_i - A_i \bar{s}^P_i = r_j + s^Q_j - r_i - s^P_i \]

- Recall also that
  \[
  [d_{ij}]_{q_i, q_j} = [-I_3, -(s^P_i)_{p_i}, I_3, (s^Q_j)_{p_j}]
  \]
  \[
  = [-I_3, -B(p_i, s^P_i), I_3, B(p_j, s^Q_j)]
  \]

- It follows that
  \[
  \delta d_{ij} = [-I_3, -B(p_i, s^P_i), I_3, B(p_j, s^Q_j)] \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = [d_{ij}]_q \cdot \delta q
  \]
Virtual Variation, Basic GCons: $\Phi^{DP2}$

[The $\delta p$ Flavor]

- Recall that

$$\Phi^{DP2}(i, a_i, s^p_i, j, s^Q_j, f(t)) = a_i^T A_i^T d_{ij} - f(t) = a_i^T d_{ij} - f(t) = 0$$

- Recall also that

$$\Phi^{DP2}_{q_i, q_j}(a_i, d_{ij}) = [-a_i^T \quad d_{ij}^T B(p_i, s^p_i) - a_i^T B(p_i, s^p_i) \quad a_i^T \quad a_i^T B(p_j, s^Q_j)]$$

- It follows that

$$\delta \Phi^{DP2} = [a_i^T \quad d_{ij}^T B(p_i, s^p_i) \quad a_i^T B(p_i, s^p_i) \quad a_i^T \quad a_i^T B(p_j, s^Q_j)] \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = \Phi^{DP2}_{q_i, q_j} \cdot \delta q$$
Virtual Variation, Basic GCons: $\Phi^D$
[The $\delta\mathbf{p}$ Flavor]

- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = d_{ij}^T d_{ij} - f(t) = 0$$

- It also that

$$\Phi^D_{q_i, q_j} = [ -2d_{ij}^T, -2d_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{s}_i^P), 2d_{ij}^T, 2d_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{s}_j^Q) ]$$

- It follows that

$$\delta \Phi^D = [ -2d_{ij}^T, -2d_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{s}_i^P), 2d_{ij}^T, 2d_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{s}_j^Q) ] \cdot \begin{bmatrix} \delta \mathbf{r}_i \\ \delta \mathbf{p}_i \\ \delta \mathbf{r}_j \\ \delta \mathbf{p}_j \end{bmatrix} = D_{q} \cdot \delta \mathbf{q}$$
Virtual Variation, Basic GCons: $\Phi^{CD}$
[The $\delta p$ Flavor]

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(c, i, \bar{s}_i^P, j, \bar{s}_j^Q, f(t)) = c^T d_{i,j} - f(t) = 0$$

- Recall also that

$$\Phi_{q_i, q_j}^{CD} = \begin{bmatrix} -c^T & -c^T B(p_i, \bar{s}_i^P) & c^T & c^T B(p_j, \bar{s}_j^Q) \end{bmatrix}$$

- It follows that

$$\delta \Phi^{CD} = \begin{bmatrix} -c^T & -c^T B(p_i, \bar{s}_i^P) & c^T & c^T B(p_j, \bar{s}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta r_i \\ \delta p_i \\ \delta r_j \\ \delta p_j \end{bmatrix} = \Phi_q^{CD} \cdot \delta q$$
Virtual Variation, Euler Parameter Normalization Constraint: $\Phi^p$

- Recall that the Euler Parameter normalization constraint assumes the expression

$$\Phi^p_i = p^T_i p_i - 1 = 0$$

- Recall also that

$$(\Phi^p_i)_q = \begin{bmatrix} 0_{1\times 3} & 2p^T_i \end{bmatrix}$$

- It follows that

$$\delta\Phi^p_i = \begin{bmatrix} 0_{1\times 3} & 2p^T_i \end{bmatrix} \cdot \begin{bmatrix} \delta r_i \\ \delta p^T_i \end{bmatrix} = (\Phi^p_i)_q \cdot \delta q$$
**Comments [1/2]**

- In the next lectures we’ll express the virtual variation of a function that depends on the position and orientation of one or more bodies in the system using one of two sets of virtual displacements: $\delta \mathbf{r}$ and $\delta \mathbf{\pi}$, or $\delta \mathbf{r}$ and $\delta \mathbf{p}$.

- At the end of the day, a virtual displacement of the bodies in the system will lead to a virtual variation of a generic constraint $\Phi^\alpha$, $\alpha \in \{DP1, DP2, D, CD, p\}$:

  \[
  \delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \Phi_r^\alpha \delta \mathbf{r} + \Pi(\Phi^\alpha) \delta \mathbf{\pi} = \Phi_r^\alpha \delta \mathbf{r} + \Phi_p^\alpha \delta \mathbf{p}
  \]

- In matrix form, we can express the above relations as

  \[
  \delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_r^\alpha & \Pi(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \mathbf{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi^\alpha) \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \mathbf{\pi} \end{bmatrix}
  \]

  \[
  \delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_r^\alpha & \Phi_p^\alpha \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \mathbf{p} \end{bmatrix} = \Phi_q^\alpha \cdot \delta \mathbf{q}
  \]
Comments [2/2]

- Recall that we are supposed to collect all \( m \) constraints and stick them together in one big \( \Phi = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \\ \Phi^K(q) \\ \Phi^D(q, t) \\ \Phi^p(p) \end{bmatrix} \), if we plan to work with virtual displacements expressed in terms of \( \delta r \) and \( \delta \pi \), or in \( \Phi^F = \begin{bmatrix} \Phi^K(q) \\ \Phi^D(q, t) \end{bmatrix} \), if we plan to work with \( \delta r \) and \( \delta p \).

- Recall that any one of the constraints in \( \Phi \) is one of the four basic GCons that we introduced.

- When interested in the variation of \( \Phi \), we simply stack together the variation of each of the GCons that enters in \( \Phi \). The situation for \( \Phi^F \) is similar, except that here you need to account explicitly for the Euler Parameter normalization constraints.

- A virtual displacement of the bodies in the system will lead to a virtual variation \( \delta \Phi \) that depends on the position and orientation of the bodies:

\[
\delta \Phi = \Phi_r \delta r + \Pi(\Phi) \delta \pi \quad \text{or} \quad \delta \Phi^F = \Phi_r^F \delta r + \Phi_p^F \delta p
\]

- In matrix form, we can express the above relations as

\[
\delta \Phi(r, p) = \begin{bmatrix} \Phi_r & \Pi(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix} = \bar{R}(\Phi) \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix}
\]

\[
\delta \Phi^F(r, p) = \begin{bmatrix} \Phi_r^F & \Phi_p^F \end{bmatrix} \cdot \begin{bmatrix} \delta r \\ \delta p \end{bmatrix} = \Phi_q^F \cdot \delta q
\]
The Concept of Consistent Virtual Displacements

- Framework: assume that your $q = \begin{bmatrix} r \\ p \end{bmatrix}$ is such that the constraints are satisfied; i.e., $\Phi^F(q, t) = 0$. Apply now a virtual displacement $\delta r_i$ and $\delta p_i$ to each body $i$ in the system.

- Question: how should you choose the virtual displacements $\delta r_i$ and $\delta p_i$, $i = 1, \ldots, nb$ so that the new configuration is also consistent?

- I am interested in a healthy $\delta q$:

$$q \rightarrow \Phi^F(q, t) = 0 \quad \Rightarrow \quad q + \delta q \rightarrow \Phi^F(q + \delta q, t) = \Phi^F(q, t) + \delta \Phi^F(q, t) = 0$$

- It follows that

$$\delta \Phi^F(q, t) = 0 \quad \Rightarrow \quad \Phi_q^F \delta q = 0$$

- By definition, a virtual displacement $\delta q$ is said to be consistent with the set of constraints present in the system if $\Phi_q^F \delta q = 0$ holds.

- Note that a similar train of thought can be followed to define a $\begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix}$ consistent virtual displacement. The condition in that case reads

$$\begin{bmatrix} F_r \\ \Pi(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix} = \bar{R}(\Phi) \cdot \begin{bmatrix} \delta r \\ \delta \pi \end{bmatrix} = 0$$
Deriving the EOM