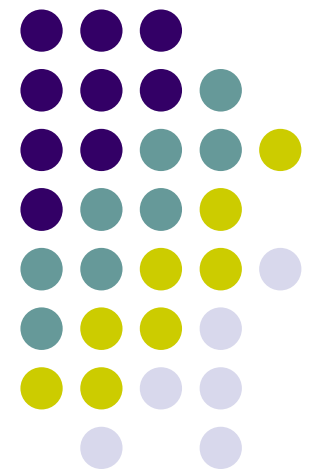


# ME751

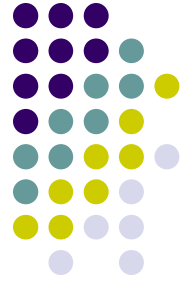
## Advanced Computational Multibody Dynamics

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March 4, 2010

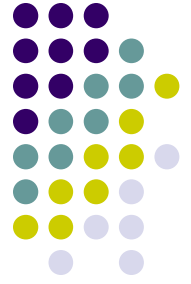


# Before we get started...



- Last Time:
  - Discussed Virtual Displacement and Variation of a Function
  - Covered the  $\delta \mathbf{r}$  ,  $\delta \pi$  case
- Today:
  - Discuss about how to visualize the time evolution of a mechanical system
  - Virtual Displacement: cover the  $\delta \mathbf{r}$  ,  $\delta \mathbf{p}$  case
  - Start discussion on how to obtain the EOM (the Dynamics problem)
- HW7 – due on March 11: posted online later today
  - This HW will require you to decide on a Final Project
- Forum up and running: <http://sbel.wisc.edu/Forum/index.php?board=4.0>
  - Use the forum to post questions about your HW, class notes, etc
  - I plan to monitor the discussion threads and contribute as much as I can
  - Started a discussion topic on course notes mistakes
    - Everyone who finds/posts a valid mistake first gets \$1 from me

# The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta\mathbf{p}$ Virtual Rotation



- Framework: assume you have a vector quantity that depends on  $\mathbf{p}$ . Assume that the value of  $\mathbf{p}$  changes to  $\mathbf{p} + \delta\mathbf{p}$ . What is the variation in the quantity that depends on  $\mathbf{p}$  due to the said change?
- Specifically, assume the vector quantity of interest is  $\mathbf{u}$ , and  $\mathbf{u}$  depends on  $\mathbf{p}$  and possibly time  $t$ :

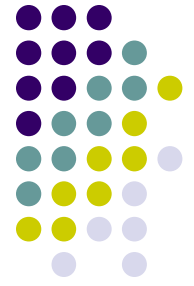
$$\mathbf{u} = \mathbf{u}(\mathbf{p}, t)$$

- I am interested at a fixed time  $t$  in the  $\delta\mathbf{u}$  below given  $\mathbf{p}$ ,  $\delta\mathbf{p}$ , and the expression of  $\mathbf{u}(\mathbf{p})$ :

$$\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p}, t) \quad \mathbf{p} + \delta\mathbf{p} \longrightarrow \mathbf{u}(\mathbf{p} + \delta\mathbf{p}, t) = \mathbf{u}(\mathbf{p}, t) + \delta\mathbf{u}$$

$$\delta\mathbf{u} = ?$$

# The Variation of a Function due to a Virtual Change of Orientation Induced by a $\delta \mathbf{p}$ Virtual Rotation



**[Cntd.]:**

- The answer to question of interest,  $\delta \mathbf{u}(\mathbf{p}) = ?$ , is obtained using a Taylor series expansion:

$$\begin{aligned}\mathbf{u}(\mathbf{p} + \delta \mathbf{p}, t) &= \mathbf{u}(\mathbf{p}, t) + \mathbf{u}_{\mathbf{p}} \delta \mathbf{p} + \dots \\ &\approx \mathbf{u}(\mathbf{p}, t) + \mathbf{u}_{\mathbf{p}} \delta \mathbf{p}\end{aligned}$$

- Then

$$\delta \mathbf{u}(\mathbf{p}) = \mathbf{u}(\mathbf{p} + \delta \mathbf{p}, t) - \mathbf{u}(\mathbf{p}, t) = \mathbf{u}_{\mathbf{p}} \delta \mathbf{p}$$

- In the argument above, we rely on the fact that the virtual rotations, that is, the perturbations  $\delta \mathbf{p}$ , are small and therefore higher order terms that contain entries of  $\delta \mathbf{p}$ , that is,  $\delta e_0$ ,  $\delta e_1$ ,  $\delta e_2$ , or  $\delta e_3$ , can be safely approximated to be zero.
- Important observation: note that the time does not play a role in figuring out what the variation in  $\mathbf{u}$  is. In other words, looking into the variation of  $\mathbf{u}$  is an exercise that is carried out at a certain time  $t$ , and time is held fixed.
- Note that the same argument applies if  $u$  is a scalar function that depends on  $\mathbf{p}$ . In that case,

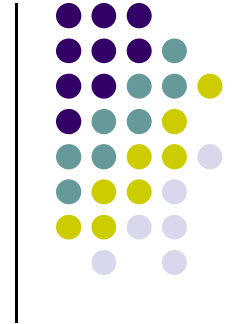
$$\delta u(\mathbf{p}, t) = u_{\mathbf{p}} \delta \mathbf{p}$$

# Exercise



- Calculate the variation of the function  $\mathbf{u}(\mathbf{p}) = \mathbf{A}(\mathbf{p})\bar{\mathbf{s}}$  due to a variation  $\delta\mathbf{p}$  in the Euler Parameters. The vector  $\bar{\mathbf{s}}$  does not depend on  $\mathbf{p}$ .

# Exercise



- Calculate the variation of the function  $u(\mathbf{p}) = \mathbf{p}^T \mathbf{p} - 1$  due to a variation  $\delta \mathbf{p}$  in the Euler Parameters

# Quick Question



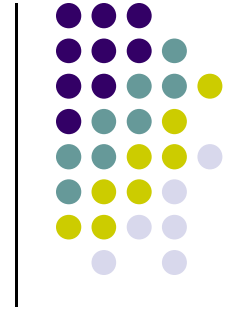
- Note that when interested in variations as induced by virtual rotations of the  $\delta\mathbf{p}$  flavor (as opposed to the  $\delta\bar{\pi}$  flavor), it is very straightforward to produce the quantity of interest:

$$\delta\mathbf{u}(\mathbf{p}) = \mathbf{u}_p\delta\mathbf{p}$$

- Why did not we take the same approach for the  $\delta\bar{\pi}$ ?
  - We couldn't do this direct approach for the the same reason we couldn't find a set of three variables whose time derivative is the angular velocity  $\bar{\omega}$
  - Specifically, there is no concept of partial derivative  $\mathbf{u}_{\bar{\pi}}$  to work with, and therefore we have to resort to the process that in the end expresses the variation  $\delta\mathbf{u}$  or the time derivative  $\dot{\mathbf{u}}$  using  $\bar{\mathbf{\Pi}}(\mathbf{u})$  and  $\delta\bar{\pi}$ , or  $\bar{\mathbf{\Pi}}(\mathbf{u})$  and  $\bar{\omega}$ , respectively

# Virtual Variation, Basic GCons: $\Phi^{DP1}$

## [The $\delta \mathbf{p}$ Flavor]



- Recall that

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = \mathbf{a}_i^T \mathbf{a}_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_i} = \mathbf{0}_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} = \mathbf{a}_j^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i)$$

$$\frac{\partial \Phi^{DP1}}{\partial \mathbf{r}_j} = \mathbf{0}_{1 \times 3} \quad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} = \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{a}}_j)$$

- Putting it all together,  $\delta \Phi^{DP1} = \Phi_{\mathbf{q}}^{DP1} \delta \mathbf{q}$ , where,

$$\Phi_{\mathbf{q}}^{DP1} = \left[ \mathbf{0}_{1 \times 3} \dots \mathbf{0}_{1 \times 3} \dots \mathbf{0}_{1 \times 3} \dots \mathbf{0}_{1 \times 4} \quad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_i} \quad \mathbf{0}_{1 \times 4} \dots \mathbf{0}_{1 \times 4} \quad \frac{\partial \Phi^{DP1}}{\partial \mathbf{p}_j} \quad \mathbf{0}_{1 \times 4} \dots \mathbf{0}_{1 \times 4} \right]$$

Partials with respect to  $\mathbf{r}$  ← → Partials with respect to  $\mathbf{p}$

$$= \left[ \begin{array}{cccccccccccc} \mathbf{0}_{1 \times 3} & \dots & \mathbf{0}_{1 \times 3} & \dots & \mathbf{0}_{1 \times 3} & \dots & \mathbf{0}_{1 \times 4} & \mathbf{a}_j^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i) & \mathbf{0}_{1 \times 4} & \dots & \mathbf{0}_{1 \times 4} & \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{a}}_j) & \mathbf{0}_{1 \times 4} & \dots & \mathbf{0}_{1 \times 4} \end{array} \right]$$

↑ Body 1,  $\mathbf{r}$    
 ↑ Body  $i$ ,  $\mathbf{r}$    
 ↑ Body  $j$ ,  $\mathbf{r}$    
 ↑ Body  $i-1$ ,  $\mathbf{p}$    
 ↑ Body  $i$ ,  $\mathbf{p}$    
 ↑ Body  $i+1$ ,  $\mathbf{p}$    
 ↑ Body  $j-1$ ,  $\mathbf{p}$    
 ↑ Body  $j$ ,  $\mathbf{p}$    
 ↑ Body  $j+1$ ,  $\mathbf{p}$    
 ↑ Body  $nb$ ,  $\mathbf{p}$



# [Short Detour]: Computing $\delta \mathbf{d}_{ij}$



- Recall that

$$\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P = \mathbf{r}_j + \mathbf{s}_j^Q - \mathbf{r}_i - \mathbf{s}_i^P$$

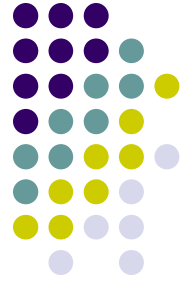
- Recall also that

$$\begin{aligned} [\mathbf{d}_{ij}]_{\mathbf{q}_i, \mathbf{q}_j} &= \begin{bmatrix} -\mathbf{I}_3 & -(\mathbf{s}_i^P)_{\mathbf{p}_i} & \mathbf{I}_3 & (\mathbf{s}_j^Q)_{\mathbf{p}_j} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{I}_3 & -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{I}_3 & \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \end{aligned}$$

- It follows that

$$\delta \mathbf{d}_{ij} = \begin{bmatrix} -\mathbf{I}_3 & -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{I}_3 & \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r}_i \\ \delta \mathbf{p}_i \\ \delta \mathbf{r}_j \\ \delta \mathbf{p}_j \end{bmatrix} = [\mathbf{d}_{ij}]_{\mathbf{q}} \cdot \delta \mathbf{q}$$

# Virtual Variation, Basic GCons: $\Phi^{DP2}$ [The $\delta \mathbf{p}$ Flavor]



- Recall that

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \mathbf{a}_i^T \mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

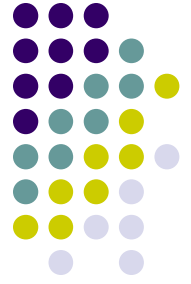
$$\Phi_{\mathbf{q}_i, \mathbf{q}_j}^{DP2}(\mathbf{a}_i, \mathbf{d}_{ij}) = \left[ \begin{array}{cc} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \\ \mathbf{a}_i^T & \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{array} \right]$$

- It follows that

$$\delta \Phi^{DP2} = \left[ \begin{array}{cc} -\mathbf{a}_i^T & \mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) - \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \\ \mathbf{a}_i^T & \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{array} \right] \cdot \left[ \begin{array}{c} \delta \mathbf{r}_i \\ \delta \mathbf{p}_i \\ \delta \mathbf{r}_j \\ \delta \mathbf{p}_j \end{array} \right] = \Phi_{\mathbf{q}}^{DP2} \cdot \delta \mathbf{q}$$

# Virtual Variation, Basic GCons: $\Phi^D$

## [The $\delta\mathbf{p}$ Flavor]



- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = 0$$

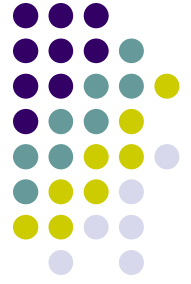
- It also that

$$\Phi_{\mathbf{q}_i, \mathbf{q}_j}^D = [ -2\mathbf{d}_{ij}^T \quad -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad 2\mathbf{d}_{ij}^T \quad 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) ]$$

- It follows that

$$\delta\Phi^D = [ -2\mathbf{d}_{ij}^T \quad -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad 2\mathbf{d}_{ij}^T \quad 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) ] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi_{\mathbf{q}}^D \cdot \delta\mathbf{q}$$

# Virtual Variation, Basic GCons: $\Phi^{CD}$ [The $\delta\mathbf{p}$ Flavor]



- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = 0$$

- Recall also that

$$\Phi_{\mathbf{q}_i, \mathbf{q}_j}^{CD} = [ -\mathbf{c}^T \quad -\mathbf{c}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{c}^T \quad \mathbf{c}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) ]$$

- It follows that

$$\delta\Phi^{CD} = [ -\mathbf{c}^T \quad -\mathbf{c}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{c}^T \quad \mathbf{c}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) ] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \\ \delta\mathbf{r}_j \\ \delta\mathbf{p}_j \end{bmatrix} = \Phi_{\mathbf{q}}^{CD} \cdot \delta\mathbf{q}$$

# Virtual Variation, Euler Parameter Normalization Constraint: $\Phi^{\mathbf{P}}$



- Recall that the Euler Parameter normalization constraint assumes the expression

$$\Phi_i^{\mathbf{P}} = \mathbf{p}_i^T \mathbf{p}_i - 1 = 0$$

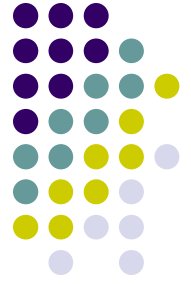
- Recall also that

$$(\Phi_i^{\mathbf{P}})_{\mathbf{q}_i} = [ \mathbf{0}_{1 \times 3} \quad 2\mathbf{p}_i^T ]$$

- It follows that

$$\delta\Phi_i^{\mathbf{P}} = [ \mathbf{0}_{1 \times 3} \quad 2\mathbf{p}_i^T ] \cdot \begin{bmatrix} \delta\mathbf{r}_i \\ \delta\mathbf{p}_i \end{bmatrix} = (\Phi_i^{\mathbf{P}})_{\mathbf{q}} \cdot \delta\mathbf{q}$$

# Comments [1/2]



- In the next lectures we'll express the virtual variation of a function that depends on the position and orientation of one or more bodies in the system using one of two sets of virtual displacements:  $\delta \mathbf{r}$  and  $\delta \bar{\pi}$ , or  $\delta \mathbf{r}$  and  $\delta \mathbf{p}$
- At the end of the day, a virtual displacement of the bodies in the system will lead to a virtual variation of a generic constraint  $\Phi^\alpha$ ,  $\alpha \in \{DP1, DP2, D, CD, \mathbf{p}\}$ :

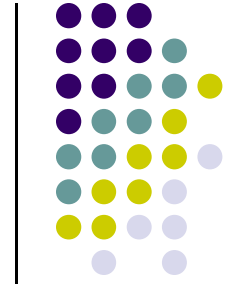
$$\delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \Phi_{\mathbf{r}}^\alpha \delta \mathbf{r} + \bar{\Pi}(\Phi^\alpha) \delta \bar{\pi} = \Phi_{\mathbf{r}}^\alpha \delta \mathbf{r} + \Phi_{\mathbf{p}}^\alpha \delta \mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_{\mathbf{r}}^\alpha & \bar{\Pi}(\Phi^\alpha) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi^\alpha) \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix}$$

$$\delta \Phi^\alpha(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_{\mathbf{r}}^\alpha & \Phi_{\mathbf{p}}^\alpha \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \mathbf{p} \end{bmatrix} = \Phi_{\mathbf{q}}^\alpha \cdot \delta \mathbf{q}$$

# Comments [2/2]



- Recall that we are supposed to collect *all*  $m$  constraints and stick them together in one big  $\Phi = \begin{bmatrix} \Phi^K(\mathbf{q}) \\ \Phi^D(\mathbf{q}, t) \end{bmatrix}$ ,  
if we plan to work with virtual displacements expressed in terms of  $\delta\mathbf{r}$  and  $\delta\bar{\pi}$ , or in  $\Phi^F = \begin{bmatrix} \Phi^K(\mathbf{q}) \\ \Phi^D(\mathbf{q}, t) \\ \Phi^P(\mathbf{p}) \end{bmatrix}$ , if  
we plan to work with  $\delta\mathbf{r}$  and  $\delta\mathbf{p}$ .
- Recall that any one of the constraints in  $\Phi$  is one of the four basic GCons that we introduced
- When interested in the variation of  $\Phi$ , we simply stack together the variation of each of the GCons that enters in  $\Phi$ . The situation for  $\Phi^F$  is similar, except that here you need to account explicitly for the Euler Parameter normalization constraints.
- A virtual displacement of the bodies in the system will lead to a virtual variation  $\delta\Phi$  that depends on the position and orientation of the bodies:

$$\delta\Phi = \Phi_{\mathbf{r}}\delta\mathbf{r} + \bar{\Pi}(\Phi)\delta\bar{\pi} \quad \text{or} \quad \delta\Phi^F = \Phi_{\mathbf{r}}^F\delta\mathbf{r} + \Phi_{\mathbf{p}}^F\delta\mathbf{p}$$

- In matrix form, we can express the above relations as

$$\delta\Phi(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_{\mathbf{r}} & \bar{\Pi}(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$$

$$\delta\Phi^F(\mathbf{r}, \mathbf{p}) = \begin{bmatrix} \Phi_{\mathbf{r}}^F & \Phi_{\mathbf{p}}^F \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\mathbf{p} \end{bmatrix} = \Phi_{\mathbf{q}}^F \cdot \delta\mathbf{q}$$

# The Concept of Consistent Virtual Displacements



- Framework: assume that your  $\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}$  is such that the constraints are satisfied; i.e.,  $\Phi^F(\mathbf{q}, t) = \mathbf{0}$ . Apply now a virtual displacement  $\delta\mathbf{r}_i$  and  $\delta\mathbf{p}_i$  to each body  $i$  in the system.
- Question: how should you choose the virtual displacements  $\delta\mathbf{r}_i$  and  $\delta\mathbf{p}_i$ ,  $i = 1, \dots, nb$  so that the new configuration is also consistent?
- I am interested in a healthy  $\delta\mathbf{q}$ :

$$\mathbf{q} \longrightarrow \Phi^F(\mathbf{q}, t) = \mathbf{0} \quad \mathbf{q} + \delta\mathbf{q} \longrightarrow \Phi^F(\mathbf{q} + \delta\mathbf{q}, t) = \Phi^F(\mathbf{q}, t) + \delta\Phi^F(\mathbf{q}, t) = \mathbf{0}$$

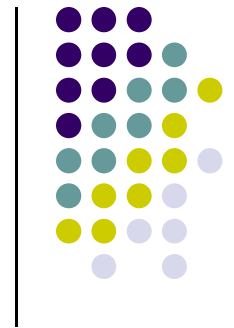
- It follows that

$$\delta\Phi^F(\mathbf{q}, t) = \mathbf{0} \quad \Rightarrow \quad \Phi_{\mathbf{q}}^F \delta\mathbf{q} = \mathbf{0}$$

- *By definition*, a virtual displacement  $\delta\mathbf{q}$  is said to be consistent with the set of constraints present in the system if  $\Phi_{\mathbf{q}}^F \delta\mathbf{q} = \mathbf{0}$  holds
- Note that a similar train of thought can be followed to define a  $\begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}$  consistent virtual displacement. The condition in that case reads

$$\begin{bmatrix} \Phi_{\mathbf{r}} & \bar{\Pi}(\Phi) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\Phi) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \mathbf{0}$$





# Deriving the EOM