Before we get started...

- Last Time:
  - Geometric Constraints
    - Covered four Basic and one Intermediate GCon’s
    - Recall that more complex GCon’s are obtained by combining the four basic ones: DP1, DP2, D, CD

- Today:
  - Finish Geometric Constraints
  - Talk about their partial derivatives

- HW5 – due on Feb. 25
  - Posted online

- Reading: Example 9.4.7
Intermediate GCon: ⊥1
[Perpendicular 1]
Intermediate GCon: \( \perp 1 \)

[Cntd.]

- Step 1. GCon \( \Phi_{\perp 1} \) reflects the fact that motion is such that a vector \( \mathbf{c}_j \) on body \( j \) is perpendicular on a plane of body \( i \). This plane is defined by specifying two noncolinear vectors \( \mathbf{a}_i \) and \( \mathbf{b}_i \) that are contained in that plane. Another way to state GCon \( \Phi_{\perp 1} \) is to say that \( \mathbf{c}_j \) is parallel to the normal of the said plane. This GCon is built using GCon-DP1 twice. As such, it introduces two ACEs and therefore removes two DOFs.

- Step 2. GCon \( \Phi_{\perp 1} \) has the following attributes:
  - Body \( i \) and the associated L-RF\( i \). The vectors \( \bar{\mathbf{a}}_i \) and \( \bar{\mathbf{b}}_i \).
  - Body \( j \) and the associated L-RF\( j \). The vector \( \bar{\mathbf{c}}_j \).

- Step 3. The \( \perp 1 \)-ACE asserts that:

\[
\Phi_{\perp 1}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j) = \begin{bmatrix}
\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{c}}_j, 0) \\
\Phi^{DP1}(i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j, 0)
\end{bmatrix} = \begin{bmatrix}
\bar{\mathbf{a}}_i^T \bar{\mathbf{A}}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j \\
\bar{\mathbf{b}}_i^T \bar{\mathbf{A}}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

- Steps 4, 5, and 6: see discussion for GCon DP1.
Intermediate GCon: \( \perp \perp 2 \) 
[Perpendicular 2] 

\[ \Phi^{\perp 2}: d_{ij} \perp \mathcal{P} (a_i, b_i) \]
Intermediate GCon: ⊥2
[Perpendicular 2] [Cntd.]

- Step 1. GCon $\Phi^{-2}$ reflects the fact that motion is such that a vector $\overrightarrow{P_iQ_j}$ from body $i$ to body $j$ remains perpendicular to a plane defined by two vectors $\vec{a}_i$ and $\vec{b}_i$. The GCon is derived by applying twice the basic GCon-DP2. As such, this GCon leads to two ACEs and it removes two DOFs.

- Step 2. GCon $\perp 2$ has the following attributes:
  - Body $i$ and the associated L-RF$_i$. On that body we need to know (1) the location $s^P_i$ of the point $P_i$, (2) the algebraic vector $\vec{a}_i$, and (3) the algebraic vector $\vec{b}_i$.
  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $s^Q_j$ of the point $Q_j$.

- Step 3. The $\perp 2$-ACE asserts that:

$$
\Phi^{-2}(i, \vec{a}_i, \vec{b}_i, s^P_i, j, s^Q_j) = 
\begin{bmatrix}
\Phi^{DP2}(i, \vec{a}_i, s^P_i, j, s^Q_j, 0) \\
\Phi^{DP2}(i, \vec{b}_i, s^P_i, j, s^Q_j, 0)
\end{bmatrix} = 
\begin{bmatrix}
\vec{a}_i^T A_i^T d_{ij} \\
\vec{b}_i^T A_i^T d_{ij}
\end{bmatrix} = 0
$$

- Steps 4, 5, and 6: see discussion for GCon D2.
High Level GCon: SJ [Spherical Joint]

\[
\Phi^{SJ} = \begin{bmatrix}
\Phi^{CD}(i, i, s_i^P, j, s_j^Q, 0) \\
\Phi^{CD}(j, i, s_i^P, j, s_j^Q, 0) \\
\Phi^{CD}(k, i, s_i^P, j, s_j^Q, 0) 
\end{bmatrix}
\]
High Level GCon: Spherical Joint

- Step 1. GCon $\Phi^{SJ}$ reflects the fact that motion is such that point $P$ on body $i$ and point $Q$ on body $j$ coincide at all times. This is equivalent to saying the the difference between the $x$, $y$, and $z$ coordinates of these two points is zero. The GCon therefore is implemented by a set of three GCon-CD and as such it removes three DOFs.

- Step 2. The GCon has the following attributes (inherited from $\Phi^{CD}$):
  - Body $i$ and the associated L-RF$_i$. On that body we need to know the location $\bar{s}_i^P$ of the point $P$.
  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $\bar{s}_j^Q$ of the point $Q$.

- Step 3. The SJ-ACE asserts that:

\[
\Phi^{SJ}(i, \bar{s}_i^P, j, \bar{s}_j^Q) = \begin{bmatrix}
\Phi^{CD}(i, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \\
\Phi^{CD}(j, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0) \\
\Phi^{CD}(k, i, \bar{s}_i^P, j, \bar{s}_j^Q, 0)
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

- Steps 4, 5, and 6: draw on the corresponding steps for $\Phi^{CD}$. 
High Level GCon: UJ [Universal Joint]

\[ \Phi_{UJ} = \begin{bmatrix} \Phi_{SJ}(i, s_i^P, j, s_j^Q) \\ \Phi_{DP1}(i, \bar{a}_i, j, \bar{a}_j, 0) \end{bmatrix} \]

\[ \phi = \pi/2 \]
\[ \theta_2 < \pi/2 \]

**Figure 9.4.15**  Singular behavior of universal joint.
High Level GCon: UJ [Universal Joint]

- Step 1. GCon $\Phi^{UJ}$ reflects the motion associated with a universal joint. According to the figure, if body $i$ is rotated, body $j$ starts rotated as well. The joint is defined as follows: first, note that this is very similar to a spherical joint, in that points $P_i$ and $Q_j$ coincide at all times. However, we need to capture that a rotation of $i$ induces a rotation of $j$. This is enforced by a requirement that vectors $\mathbf{a}_i$ and $\mathbf{a}_j$, which together define the cross of the joint, should stay at all times orthogonal. Note when body $i$ is held fixed, body $j$ can undertake two rotations around each of the axes $\mathbf{a}_i$ and $\mathbf{a}_j$ (the cross) of the UJ. Therefore, body $j$ has two DOFs relative to body $i$.

- Step 2. The GCon has the following attributes (inherited from $\Phi^{DP1}$ and $\Phi^{SJ}$):
  - Body $i$ and the associated L-RF$_i$. On that body we need to know the location $\bar{s}^P_i$ of the point $P$, and the direction $\bar{a}_i$, which defines half of the cross
  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $\bar{s}^Q_j$ of the point $Q$, and the direction $\bar{a}_j$, which defines the other half of the cross

- Step 3. The UJ-ACE asserts that:

$$
\Phi^{UJ}(i, \bar{s}^P_i, \bar{a}_i, j, \bar{s}^Q_j, \bar{a}_j) =
\begin{bmatrix}
\Phi^{CD}(i, i, s^P_i, j, s^Q_j, 0) \\
\Phi^{CD}(j, i, s^P_i, j, s^Q_j, 0) \\
\Phi^{CD}(k, i, s^P_i, j, s^Q_j, 0) \\
\Phi^{DP1}(i, \bar{a}_i, j, \bar{a}_j, 0)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
$$

- Steps 4, 5, and 6: draw on the corresponding steps for $\Phi^{CD}$ and $\Phi^{DP1}$. 
High Level GCon: CJ [Cylindrical Joint]
High Level GCon: CJ

- Step 1. GCon $\Phi^{CJ}$ reflects the motion associated with a cylindrical joint, which geometrically requires that two vectors on two bodies should stay collinear at all times. According to the figure, if body $j$ is fixed, body $i$ can slide and rotate around a specified axis and therefore has two DOFs.

- Step 2. The rotation/sliding axis is defined on body $j$ by $\bar{c}_j$. On body $i$, it is defined by the normal to the plane defined by two vectors $\bar{a}_i$ and $\bar{b}_i$. Additionally, consider point $Q_j$ as being the origin of $\bar{c}_j$. This point together with $P_i$ defines the axis of the cylindrical joint. Without any loss of generality, $P_i$ will be considered the origin of $\bar{a}_i$ and $\bar{b}_i$. The GCon has the following attributes (inherited from $\Phi^{11}$ and $\Phi^{12}$):
  - Body $i$ and the associated L-RF$_i$. On that body we need to know the location $\bar{s}_i^P$ of the point $P$, and the directions $\bar{a}_i$ and $\bar{b}_i$.
  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $\bar{s}_j^Q$, the origin of the vector $\bar{c}_j$.

- Step 3. The CJ-ACE asserts that:

$$\Phi^{CJ}(i, \bar{s}_i^P, \bar{a}_i, \bar{b}_i, j, \bar{s}_j^Q, \bar{c}_j) = \begin{bmatrix} \Phi^{11}(i, \bar{a}_i, \bar{b}_i, j, \bar{c}_j) \\ \Phi^{12}(i, \bar{a}_i, \bar{b}_i, \bar{s}_i^P, j, \bar{s}_j^Q) \end{bmatrix} = \begin{bmatrix} 0_2 \\ 0_2 \end{bmatrix}$$

- Steps 4, 5, and 6: draw on the corresponding steps for $\Phi^{11}$ and $\Phi^{12}$. 


High Level GCon: RJ [Revolute Joint]

\[ \Phi^{RJ} = \begin{bmatrix} \Phi^{SJ}(i, s^P_i, j, s^Q_j) \\ \Phi^{11}(i, a_i, \bar{b}_i, j, \bar{c}_j) \end{bmatrix} \]
High Level GCon: RJ

- Step 1. GCon $\Phi^{RJ}$ reflects the motion associated with a revolute joint (also called hinge or rotational joint). According to the figure, if body $i$ is fixed, body $j$ can rotated around the hinge axis. Note that body $j$ has one DOF relative to body $i$.

- Step 2. The joint is defined as follows: first, note that this is very similar to a spherical joint, in that two points $P_i$ and $Q_j$ coincide at all times. Additionally, a vector $\mathbf{c}_j$ of origin $Q_j$ remains perpendicular on a plane $\mathcal{P}(\mathbf{a}_i, \mathbf{b}_i)$. Without loss of generality, we can assume that $P_i$ is the origin of $\mathbf{a}_i$ and $\mathbf{b}_i$. The GCon has the following attributes (inherited from $\Phi^{11}$ and $\Phi^{SJ}$):

  - Body $i$ and the associated L-RF$_i$. On that body we need to know the location $\bar{s}^P_i$ of the point $P$, and $\bar{a}_i$ and $\bar{b}_i$. The normal to the plane $\mathcal{P}(\mathbf{a}_i, \mathbf{b}_i)$ defines the direction of the axis of rotation on body $i$.

  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $\bar{s}^Q_j$ of the point $Q$, and the direction $\bar{c}_j$. The latter defines the direction of the axis of rotation on body $j$.

- Step 3. The RJ-ACE asserts that:

  $$
  \Phi^{RJ}(i, \bar{s}^P_i, \bar{a}_i, \bar{b}_i, j, \bar{s}^Q_j, \bar{c}_j) = \begin{bmatrix}
  \Phi^{SJ}(i, \bar{s}^P_i, j, \bar{s}^Q_j) \\
  \Phi^{11}(i, \bar{a}_i, \bar{b}_i, j, \bar{c}_j)
  \end{bmatrix} = \begin{bmatrix}
  0_3 \\
  0_2
  \end{bmatrix}
  $$

- Steps 4, 5, and 6: draw on the corresponding steps for $\Phi^{SJ}$ and $\Phi^{11}$.
High Level GCon: TJ
[Translational Joint]

\[
\Phi_{TJ} = \begin{bmatrix}
\Phi_{CJ}(i, \hat{s}_i^P, a_i, b_i, j, \hat{s}_j^Q, c_j) \\
\Phi_{DP1}(i, a_i, j, a_j, \text{const.})
\end{bmatrix}
\]
High Level GCon: TJ

- Step 1. GCon $\Phi^{TJ}$ reflects the motion associated with a translational joint, which geometrically is similar to a cylindrical joint with the caveat that the rotational degree of freedom of the latter is suppressed. According to the figure, if body $j$ is fixed, body $i$ can slide up and down. Therefore, the joint allows one DOF of relative motion.

- Step 2. The sliding axis is defined on body $j$ by $\vec{c}_j$. On body $i$, it is defined by the normal to the plane defined by two vectors $\vec{a}_i$ and $\vec{b}_i$. Additionally, consider the points $P_i$ and $Q_j$ as defining the axis of translation. Since the relative rotation about the translation direction is suppressed, a vector $\vec{a}_j$ on body $j$ parallel to $P(a_i, b_i)$ enters the definition of the joint. Without any loss of generality, $P_i$ can be considered the origin of $a_i, b_i$, while $Q_j$ the origin of $a_j$ and $c_j$. In the end, the GCon has the following attributes (inherited from $\Phi^{CJ}$ and $\Phi^{DP1}$):
  - Body $i$ and the associated L-RF$_i$. On that body we need to know the location $\vec{s}^P_i$ of the point $P$, and the directions $\vec{a}_i$ and $\vec{b}_i$, both with origin at $P$.
  - Body $j$ and the associated L-RF$_j$. On that body, we need to know the location $\vec{s}^Q_j$, the origin of the vectors $\vec{a}_j$ and $\vec{c}_j$.

- Step 3. The TJ-ACE asserts that:

$$
\Phi^{TJ}(i, \vec{s}^P_i, \vec{a}_i, \vec{b}_i, j, \vec{s}^Q_j, \vec{a}_j, \vec{c}_j) = 
\begin{bmatrix}
\Phi^{CJ}(i, \vec{s}^P_i, \vec{a}_i, \vec{b}_i, j, \vec{s}^Q_j, \vec{c}_j) \\
\Phi^{DP1}(i, \vec{a}_i, j, \vec{a}_j, \text{const.})
\end{bmatrix} = 
\begin{bmatrix}
0_4 \\
0
\end{bmatrix}
$$

- Steps 4, 5, and 6: draw on the corresponding steps for $\Phi^{DP1}$ and $\Phi^{CJ}$. 

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GCon’s: Concluding Remarks

- All the basic geometric constraints are scalar conditions

- If you want to implement your own Kinematics solver, with only four basic constraints you can cover 80% of the most used GCons

- The approach outlined for defining various GCons is not unique
  - Don’t have to use the four building blocks, you can have each GCon be defined in its own specific way
    - If you do this, you lose generality, hard to write software to generate a library of GCons

- Unfinished business: We still have to produce the Jacobian matrices for the four basic geometric constraints
Kinematics Analysis:
Comments on the Three Stages

- The three stages of Kinematics Analysis: position analysis, velocity analysis, and acceleration analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity, and acceleration, respectively.

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\Phi_q$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\Phi_q \ x = b$$

The $p - \omega$ Fork

“When You Come to a Fork in the Road, Take It!”
Yogi Berra
The $p - \omega$ Fork

- This could have been equally well called the $p - \bar{\omega}$ fork

- The fork:
  
  - When carrying out Velocity Analysis, should you try to solve a linear system for $\dot{p}$ and recover $\bar{\omega}$, or the other way around?
    
    * Recall that $\bar{\omega} = 2G\dot{p}$, but equally well, $\dot{p} = \frac{1}{2}G^T\bar{\omega}$

  - Similarly, when carrying out Acceleration Analysis, should you try to solve a linear system for $\ddot{p}$ and recover $\dot{\bar{\omega}}$, or the other way around?

- Recall that when solving for the position, you must stick with $p$ when solving the nonlinear system $\Phi(q, t) = 0_m$
The $\mathbf{p} - \omega$ Fork [Cntd.]

- Why do we have a fork, after all?
  
  - Focus discussion on the Velocity Analysis, same arguments carry over to Acceleration Analysis discussion

- Fork shows up do to the fact that $\dot{\Phi}(\mathbf{q}, t)$ is a linear function of $\dot{\mathbf{p}}$ if I stick with having the Euler Parameters as unknowns, but it is also a linear function of $\omega$ (or $\omega$), if I decide to make the angular velocity the unknown in the Velocity Analysis problem

- Specifically, using Haug book’s notation, we can express the time derivative $\dot{\Phi}(\mathbf{q}, t)$ in one of the following two ways (recall that $\mathbf{q} = \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}_{nc}$):

  $$\frac{d\Phi(r, p, t)}{dt} = \Phi_r \dot{r} + \Phi_p \dot{p} + \Phi_t = [\Phi_r \quad \Phi_p] \begin{bmatrix} \dot{r} \\ \dot{p} \end{bmatrix} + \Phi_t = 0_{7nb} \Rightarrow \Phi_q \dot{q} = \nu_{7nb}$$

  $$\frac{d\Phi(r, p, t)}{dt} = \Phi_r \dot{r} + \Phi_l \dot{\omega} + \Phi_t = [\Phi_r \quad \Phi_l] \begin{bmatrix} \dot{r} \\ \dot{\omega} \end{bmatrix} + \Phi_t = 0_{6nb} \Rightarrow \mathbf{J} \begin{bmatrix} \dot{r} \\ \dot{\omega} \end{bmatrix} = \nu_{6nb}$$

- Conclusion: depending on which path we take, to carry out Velocity Analysis, we need $\Phi_r$ and $\Phi_p$, or $\Phi_r$ and $\Phi_l$. 

The $p - \omega$ Fork \[\text{[Cntd.]}\]

- Note that we use a different notation than used in Haug’s book (table 9.4.1, pp. 357). Therein, the quantity $\Pi$ is called $\Phi_\pi$. In this class we reserve a subscript to denote a partial derivative with respect to that variable.

  - In this case, (i) $\pi$ is not a variable, and (ii) $\Phi_\pi$ is not a partial derivative.
  - Specifically, by definition, $\Pi$ is the coefficient matrix that multiplies $\omega$ in the expression of the time derivative $\dot{\Phi}(q, t)$

- Note that there is the $J = [\Phi_r \ \Pi]$ matrix that can be used to compute directly $\omega$ instead of $\bar{\omega}$:

$$\frac{d\Phi(r, p, t)}{dt} = \Phi_r \dot{r} + \Pi \omega + \Phi_t = [\Phi_r \ \Pi] \begin{bmatrix} \dot{r} \\ \omega \end{bmatrix} + \Phi_t = 0_{6nb} \Rightarrow J \begin{bmatrix} \dot{r} \\ \omega \end{bmatrix} = \nu_{6nb}$$

- Finally, note that for the Acceleration Analysis, one ends up with the same scenario, where the problem can be formulated in one of the following forms

$$\Phi_q \ddot{q} = \gamma_{7nb} \quad \text{or} \quad \bar{J} \begin{bmatrix} \ddot{r} \\ \dot{\omega} \end{bmatrix} = \gamma_{6nb} \quad \text{or} \quad J \begin{bmatrix} \ddot{r} \\ \dot{\omega} \end{bmatrix} = \gamma_{6nb}$$
The $\Phi_p$ versus $\Pi$ Issue

- $\Phi_r$, $\Phi_p$, and $\Pi$ are needed for both Kinematics and Dynamics Analysis
- The drill is as follows: first compute the partial derivatives $\Phi_r$ and $\Phi_p$ for the four basic constraints: DP1, DP2, D, CD
  
  - Recall that all the other intermediate and higher level GCon’s are obtained by stacking together partials of DP1, DP2, D, CD. Therefore, all we need is $\Phi_r$, $\Phi_p$, and $\Pi$ for these four basic GCon building blocks
  
  - Note that all basic GCon’s depend on the generalized coordinates of two bodies: $i$, and $j$. Therefore, when taking the partial derivative with respect to $r = \begin{bmatrix} r_1 \\ \vdots \\ r_{nb} \end{bmatrix}$ or $p = \begin{bmatrix} p_1 \\ \vdots \\ p_{nb} \end{bmatrix}$, at most two sets of non-zero entries are obtained in $\Phi_r$ and $\Phi_p$, respectively.

- At the end, we will also look into how to compute $\Pi$ (these are provided in Haug’s book)
[Short Detour]

Computing \([A(p) \bar{a}]_p\)

- Note that,

\[
[A(p) \bar{a}]_p = \left[(e_0^2 - e^T e) \bar{a} + 2(\epsilon e^T + e_0 \epsilon) \bar{a}\right]_p
\]

\[
= \begin{bmatrix}
2e_0 \bar{a} + 2 \epsilon \bar{a} & -2 \epsilon e^T + 2 e^T \epsilon \bar{a} I_3 + 2 \epsilon \bar{a}^T - 2 e_0 \bar{a}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2e_0 \bar{a} + 2 \epsilon \bar{a} & -2 \epsilon \bar{a} + 2 \epsilon \bar{a}^T - 2 e_0 \bar{a}
\end{bmatrix}
\]

\[
= 2 \begin{bmatrix}
(e_0 I_3 + \epsilon) \bar{a} & \epsilon \bar{a}^T - (e_0 I_3 + \epsilon) \bar{a}
\end{bmatrix}
\]

- The following identities were used to obtain the result above:

\[
(e^T e \bar{a})_e = (\epsilon e^T e)_e = \bar{a} (e^T e)_e = 2 \epsilon \bar{a}^T \quad \& \quad (e e^T \bar{a})_e = (e (\bar{a}^T e))_e = \epsilon \bar{a}^T + (\bar{a}^T e) I_3
\]

- We define a matrix \(B \in \mathbb{R}^{3 \times 4}\) given two vectors \(p \in \mathbb{R}^4\) and \(\bar{a} \in \mathbb{R}^3\) as

\[
B(p, \bar{a}) \equiv 2 \begin{bmatrix}
(e_0 I_3 + \epsilon) \bar{a} & \epsilon \bar{a}^T - (e_0 I_3 + \epsilon) \bar{a}
\end{bmatrix}
\]

- Then, the partial of interest is obtained as

\[
[A(p) \bar{a}]_p - a_p - B(p, \bar{a})
\]
Basic GCon DP1: $\Phi^D_{p}^{P1}$ and $\Phi^D_{p}^{P1}$

- Recall that

$$\Phi^D_{p}^{P1}(i, \tilde{a}_i, j, \tilde{a}_j, f(t)) = a_i^T A_i^T A_j \tilde{a}_j - f(t) = a_i^T a_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial \Phi^D_{p}^{P1}}{\partial r_i} = 0_{1\times 3} \quad \frac{\partial \Phi^D_{p}^{P1}}{\partial p_i} = a_j^T B(p_i, \tilde{a}_i)$$

$$\frac{\partial \Phi^D_{p}^{P1}}{\partial r_j} = 0_{1\times 3} \quad \frac{\partial \Phi^D_{p}^{P1}}{\partial p_j} = a_i^T B(p_j, \tilde{a}_j)$$

- Putting it all together (note that $\Phi^D_{q}^{P1} \in \mathbb{R}^{1\times 7nb}$),

$$\Phi^D_{q}^{P1} = \begin{bmatrix}
0_{1\times 3} & \cdots & 0_{1\times 3} & \cdots & 0_{1\times 3} & \partial \Phi^D_{p}^{P1} & 0_{1\times 3} & \cdots & 0_{1\times 3} & \partial \Phi^D_{p}^{P1} & 0_{1\times 3} & \cdots & 0_{1\times 3}
\end{bmatrix}$$

Partials with respect to $r$  
Partials with respect to $p$

$$= \begin{bmatrix}
0_{1\times 3} & \cdots & 0_{1\times 3} & \cdots & 0_{1\times 3} & a_j^T B(p_i, \tilde{a}_i) & 0_{1\times 3} & \cdots & 0_{1\times 3} & a_i^T B(p_j, \tilde{a}_j) & 0_{1\times 3} & \cdots & 0_{1\times 3}
\end{bmatrix}$$

Body 1, $r$  
Body $i$, $r$  
Body $j$, $r$  
Body $i$-1, $p$  
Body $i$, $p$  
Body $i$+1, $p$  
Body $j$-1, $p$  
Body $j$, $p$  
Body $j$+1, $p$  
Body nb, $p$
[Short Detour]:
Computing $[d_{ij}]_q$

- Recall that

$$d_{ij} = r_j + A_j \bar{s}_j^Q - r_i - A_i \bar{s}_i^P = r_j + s_j^Q - r_i - s_i^P$$

- It follows that

$$[d_{ij}]_{q_i,q_j} = [-I_3 \quad -(s_i^P)_{p_i} \quad I_3 \quad (s_j^Q)_{p_j}]$$

$$= [-I_3 \quad -B(p_i, \bar{s}_i^P) \quad I_3 \quad B(p_j, \bar{s}_j^Q)]$$
Basic GCon DP2: $\Phi_r^{DP2}$ and $\Phi_p^{DP1}$

- Recall that

$$\Phi^{DP2}(i, \bar{a}_i, \bar{s}^p_i, j, \bar{s}^Q_j, f(t)) = \bar{a}_i^T A_i^T d_{ij} - f(t) = a_i^T d_{ij} - f(t) = 0$$

- It follows that

$$\Phi_{q_i, q_j}^{DP2}(a_i, d_{ij}) = a_i^T (d_{ij})_{q_i, q_j} + d_{ij}^T (a_i)_{q_i, q_j}$$

$$- a_i^T [ -I_3 \quad -B(p_i, \bar{s}_i^p) \quad I_3 \quad B(p_j, \bar{s}_j^Q) ] + d_{ij}^T [ 0 \quad B(p_i, \bar{a}_i) \quad 0 \quad 0 ]$$

$$= [ -a_i^T \quad d_{ij}^T B(p_i, \bar{s}_i^p) - a_i^T B(p_i, \bar{s}_i^p) \quad a_i^T \quad a_i^T B(p_j, \bar{s}_j^Q) ]$$
Basic GCon D: $\Phi^D_r$ and $\Phi^D_p$

- Recall that the GCon-D assumes the expression
  \[
  \Phi^D(i, \bar{s}^P_i, j, \bar{s}^Q_j, f(t)) = d_{ij}^T d_{ij} - f(t) = 0
  \]

- It follows that
  \[
  \Phi^D_{q_i, q_j} = (d_{ij}^T d_{ij})_{q_i, q_j} = 2d_{ij}^T [d_{ij}]_{q_i, q_j}
  \]

  \[
  = 2d_{ij}^T [ -I_3 -B(p_i, \bar{s}^P_i) I_3 B(p_j, \bar{s}^Q_j) ]
  \]
Basic GCon CD: $\Phi^{CD}_r$ and $\Phi^{CD}_p$

- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(c, i, \bar{s}^P_i, j, \bar{s}^Q_j, f(t)) = c^T d_{ij} - f(t) = 0$$

- It follows that

$$\Phi^{D}_{q_i, q_j} = (c^T d_{ij})_{q_i, q_j} = c^T [d_{ij}]_{q_i, q_j}$$

$$= c^T [-I_3 \quad -B(p_i, \bar{s}^P_i) \quad I_3 \quad B(p_j, \bar{s}^Q_j) ]$$

$$= [ -c^T \quad -c^T B(p_i, \bar{s}^P_i) \quad c^T \quad c^T B(p_j, \bar{s}^Q_j) ]$$