ME751
Advanced Computational Multibody Dynamics

Section 9.3

February 18, 2010

“Discontent is the first necessity of progress.”
Thomas A. Edison

Before we get started…

- Last Time:
  - Euler Parameters – connection between their time derivative and angular velocity
  - 3D Kinematics of a Rigid Body
  - Kinematics Analysis

- Today:
  - Geometric Constraints
    - Basic, Intermediate, High Level

- HW5 – due on Feb. 25
  - Posted online later today

- Asking for your **feedback** – Tu, Feb. 23: Provide anonymously a printed page with two concerns and/or things that I can do to improve ME751

Position Analysis

- How do you get the position configuration of the mechanism?
  - Kinematic Analysis key observation: The number of constraints (kinematic and driving) is equal to the number of generalized coordinates: m=nc
    - This is a prerequisite for Kinematic Analysis

$$\mathbf{\Phi}(\mathbf{q},t) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q},t) \end{bmatrix}_{nc\times 1} = \mathbf{0}$$

$$\mathbf{q} \in \mathbb{R}^{nc} \qquad \mathbf{\Phi} : \mathbb{R}^{nc+1} \rightarrow \mathbb{R}^{nc}$$

IMPORTANT: This is a nonlinear systems with *nc* equations and *nc* unknowns that you must solve to find **q**

- The solution of the nonlinear system is found by using the so called "Newton-Raphson" algorithm
  - We'll elaborate on this later, for now just assume that you have a way to solve the above nonlinear system to find the solution **q**(t)

Velocity Analysis

- Take one time derivative of constraints Φ(q,t) to obtain the **velocity equation**:

$$\frac{d}{dt}\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi_q}\dot{\mathbf{q}} = \underbrace{-\mathbf{\Phi}_t}_{\nu}$$

- The Jacobian has as many rows (m) as it has columns (nc) since for Kinematics Analysis, NDOF=nc-m =0

- Therefore, you have a linear system that you need to solve to recover $\dot{\mathbf{q}}$

$$\mathbf{\Phi_q}\dot{\mathbf{q}} = \nu$$

Acceleration Analysis

- Take yet one more time derivative to obtain the **acceleration equation:**

$$\ddot{\mathbf{\Phi}} = \frac{d^2}{dt^2}\mathbf{\Phi}(\mathbf{q},t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi_q}\ddot{\mathbf{q}} = \underbrace{-(\mathbf{\Phi_q}\dot{\mathbf{q}})_\mathbf{q}\,\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\,\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}}_{\gamma}$$

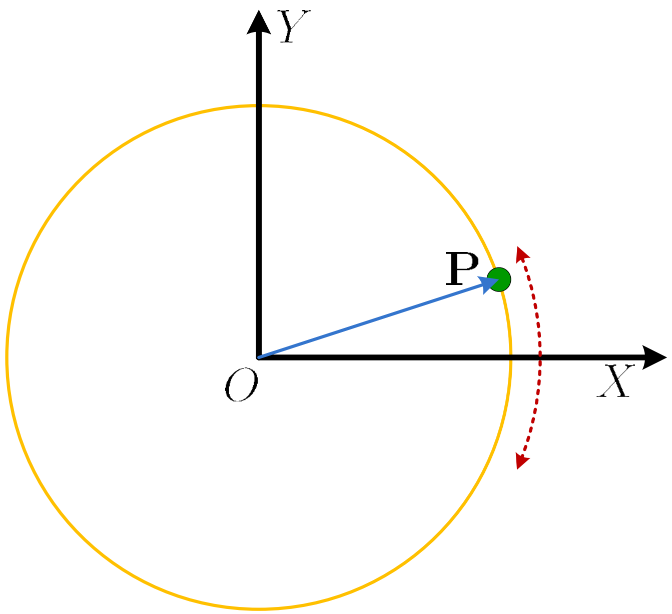
- NOTE: Getting right-hand side of acceleration equation is tedious
  - One observation that simplifies the computation: note that the right side of the above equation is made up of everything in the expression of $\ddot{\mathbf{\Phi}}$ that does *not* depend on second time derivatives (accelerations)

- Just like we pointed out for the velocity analysis, you also have to solve a linear system to retrieve the acceleration $\ddot{\mathbf{q}}$

$$\mathbf{\Phi_q}\ddot{\mathbf{q}} = \gamma$$

Exercise: Kinematic Analysis

- A particle moves on a circle of radius 1
- The generalized coordinates used are $\mathbf{q} = [x\ ,\ y]^T$
- The y coordinate has a prescribed motion: $y(t) = 0.1\sin(50\pi t)$
- Carry our Position Analysis for the given one particle system



Kinematics Analysis: Comments on the Three Stages

- The three stages of Kinematics Analysis: position analysis, velocity analysis, and acceleration analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity, and acceleration, respectively

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\mathbf{\Phi_q}$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\mathbf{\Phi_q \, x = b}$$

- WHAT IS *DIFFERENT* BETWEEN THE THREE STAGES IS THE EXPRESSION OF THE RIGHT-SIDE OF THE LINEAR EQUATION, “**b**”

The Drill…

- Step 1: Identify the geometry of the motion whenever a constraint is limiting the absolute or relative motion of a body
- Step 2: Identify the attributes needed to fully describe the geometric constraint
- Step 3: Formulate the algebraic constraint equations $\Phi(\mathbf{q},t)=\mathbf{0}$, that capture the effect of the geometric constraint
- Step 4: Compute the Jacobian (or the sensitivity matrix) $\Phi_{\mathbf{q}}$
- Step 5: Compute ν, the right side of the velocity equation
- Step 6: Compute γ, the right side of the acceleration equation (tedious…)

Nomenclature & Notation Conventions

- Geometric Constraint (GCon): a real world geometric feature of the motion of the mechanical system
  - Examples:
    - Particle moves around point (1,2,3) on a sphere of radius 2.0
    - A unit vector $\mathbf{u}_6$ on body 6 is perpendicular on a certain unit vector $\mathbf{u}_9$ on body 9
    - The y coordinate of point Q on body 8 is 14.5

- Algebraic Constraint Equations (ACEs): in the virtual world, a collection of one or more algebraic constraints, involving the generalized coordinates of the mechanism and possibly time t, that capture the geometry of the motion as induced by a certain Geometric Constraint
  - Examples:
    - $(x-1)^2 + (y-2)^2 + (z-3)^2 - 4 = 0$
    - $\mathbf{u}_6^T \cdot \mathbf{u}_9 = 0$
    - $[0\ 1\ 0] \cdot \mathbf{r}_8^Q - 14.5 = 0$

- Modeling: the process that starts with the idealization of the real world to yield a GCon and continues with the GCon abstracting into a set of ACEs

The GCon Zoo: Basic GCons

- We have four basic GCons:
  - DP1: the dot product of two vectors on two bodies is specified
  - DP2: the dot product of a vector of on a body and a vector between two bodies is specified
  - D: the distance between two points on two different bodies is specified
  - CD: the difference between the coordinates of two bodies is specified

- Note:
  - DP1 stands for Dot Product 1
  - DP2 stands for Dot Product 2
  - D stands for distance
  - CD stands for coordinate difference

The GCon Zoo: Intermediate + High Level GCons

- We have two Intermediate GCons:
  - $\perp 1$: a vector is $\perp$ on a plane belonging to a different body
  - $\perp 2$: a vector between two bodies is $\perp$ on a plane belonging to the different body

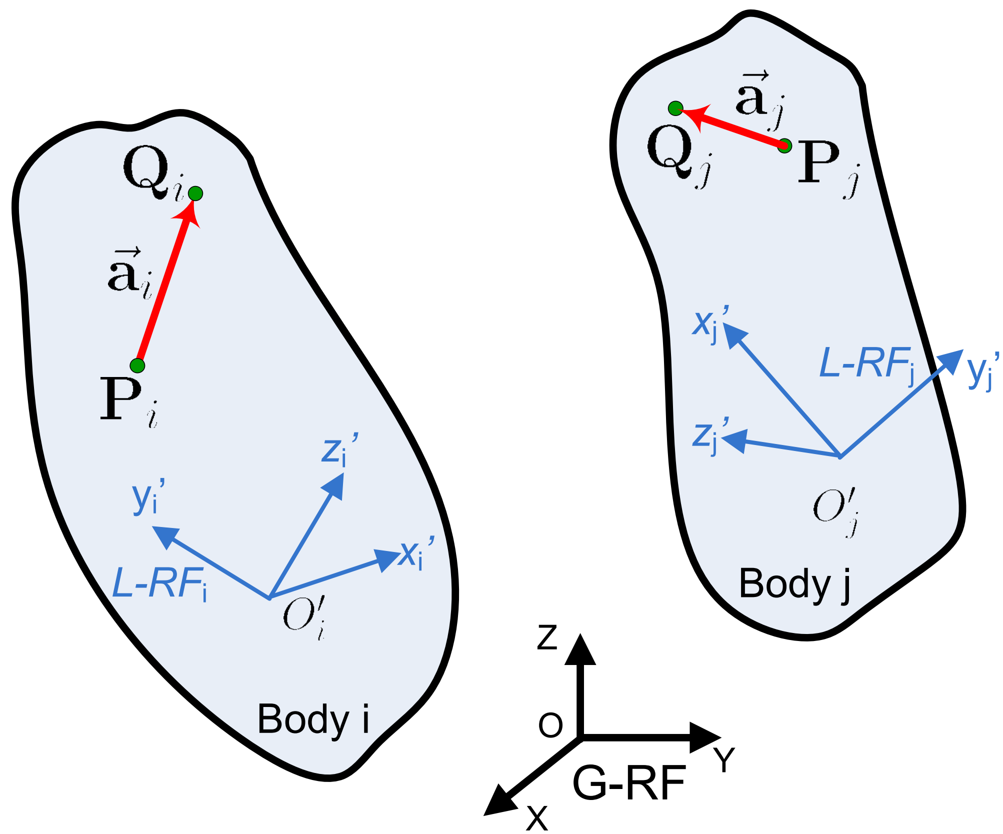
- We have a large number of High Level GCons (joints):
  - Spherical Joint (SJ)
  - Universal Joint (UJ)
  - Cylindrical Joint (CJ)
  - Revolute Joint (RJ)
  - Translational Joint (TJ)
  - Other composite joints (spherical-spherical, translational-revolute, etc.)

The GCon Zoo: Overview

	$DP1$	$DP2$	D	CD
$\perp 1$	$\times\times$			
$\perp 2$		$\times\times$		
SJ				$\times\ \times\ \times$
UJ	$\times$			$[\times\ \times\ \times]_{SJ}$
CJ	$[\times\times]_{\perp 1}$	$[\times\times]_{\perp 2}$		
RJ	$[\times\times]_{\perp 1}$			$[\times\ \times\ \times]_{SJ}$
TJ	$\times\,[\times\times]_{\perp 1}$	$[\times\times]_{\perp 2}$		

- Note that there are other GCons that are used, but they see less mileage

Basic GCon: DP1



Basic GCon: DP1
[Cntd.]

- Step 1. GCon Φ^{DP1} reflects the fact that motion is such that the dot product between a vector on body i and a second vector on body j assumes a specified value.
- Step 2. We have the following attributes (quantities required to properly define the GCon above):
  - Body i and the associated L-RF$_i$. On that body, we need to know the algebraic vector $\bar{\mathbf{a}}_i$
  - Body j and the associated L-RF$_j$. On that body, we need to know the algebraic vector $\bar{\mathbf{a}}_j$
  - The prescribed value that the dot product should assume. This prescribed value *does not depend* on GCs, but might depend on time. The prescribed value is specified through the function $f(t)$.
    - Most often, $f(t) = 0$, which indicates that the two vectors are orthogonal.
    - If $f(t)$ actually depends on time, this leads to Φ^{DP1} being a driving (rheonomic) constraint.
- Step 3. The ACE asserts that:

$$\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = 0$$

Basic GCon: DP1
[Cntd.]

- Step 4. The Jacobian for DP1: discussed next lecture
- Step 5. The ν term that enters the righ-hand side of the velocity equation assumes the form:

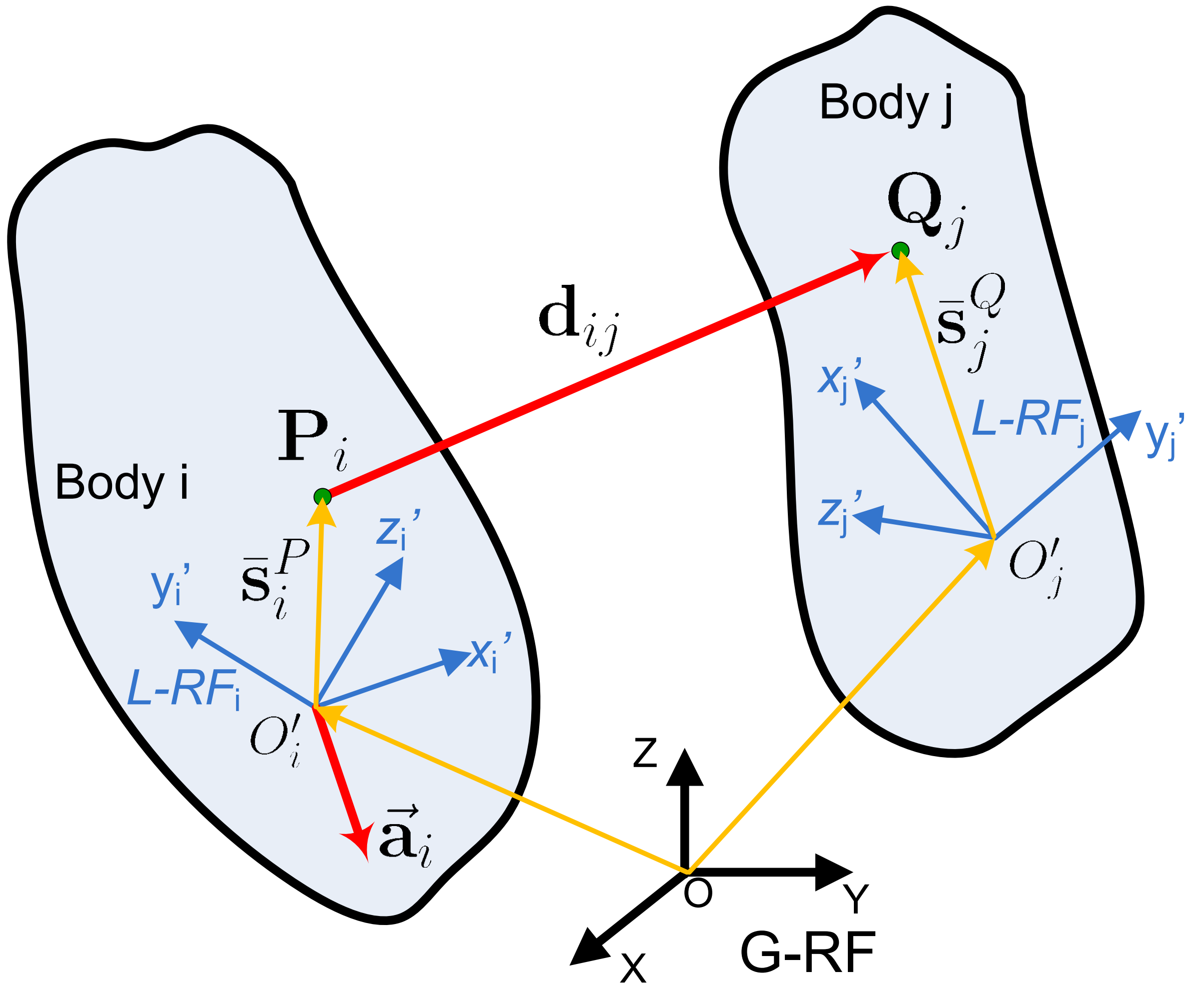
$$\nu^{DP1} = -\frac{\partial \Phi^{DP1}}{\partial t} = \frac{\partial f}{\partial t}$$

- Step 6. The γ term that enters the righ-hand side of the acceleration equation assumes the form:

$$\gamma^{DP1} = -\bar{\mathbf{a}}_j^T(\mathbf{A}_j^T\mathbf{A}_i\tilde{\bar{\omega}}_i\tilde{\bar{\omega}}_i + \tilde{\bar{\omega}}_j\tilde{\bar{\omega}}_j\mathbf{A}_j^T\mathbf{A}_i)\bar{\mathbf{a}}_i + 2\bar{\omega}_j^T\tilde{\bar{\mathbf{a}}}_j\mathbf{A}_j^T\mathbf{A}_i\tilde{\bar{\mathbf{a}}}_i\bar{\omega}_i + \frac{\partial^2 f}{\partial t^2}$$

- Note: The γ term only depends on position and velocity information - Important since it is used to compute the acceleration and therefore it should not depend on acceleration (to prevent a circular argument)
- GCon-DP1 imposes one ACE and removes one DOF

Basic GCon: DP2



Basic GCon: DP2
[Cntd.]

- Step 1. GCon Φ^{DP2} reflects the fact that motion is such that the dot product between a vector $\vec{\mathbf{a}}_i$ on body i and a second vector $\overrightarrow{P_iQ_j}$ from body i to body j assumes a specified value.
- Step 2. We have the following attributes (quantities required to properly define the GCon above):
  - Body i and the associated L-RF$_i$. On that body we need to know (*1*) the algebraic vector $\bar{\mathbf{a}}_i$, and (*2*) the location $\bar{\mathbf{s}}_i^P$ of the point P_i.
  - Body j and the associated L-RF$_j$. On that body, we need to know the location $\bar{\mathbf{s}}_j^Q$ of the point Q_j
  - The prescribed value that the dot product should assume. This prescribed value *does not depend* on GCs, but might depend on time. The prescribed value is specified through the function $f(t)$.
    - Most often, $f(t) = 0$, which indicates that $\vec{\mathbf{a}}_i$ and $\overrightarrow{P_iQ_j}$ are orthogonal.
    - If $f(t)$ actually depends on time, this leads to Φ^{DP2} being a driving (rheonomic) constraint.
- Step 3. The DP2-ACE asserts that:

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

Basic GCon: DP2

[Cntd.]

- Step 4. The Jacobian for DP2: discussed next lecture
- Step 5. The ν term that enters the righ-hand side of the velocity equation assumes the form:

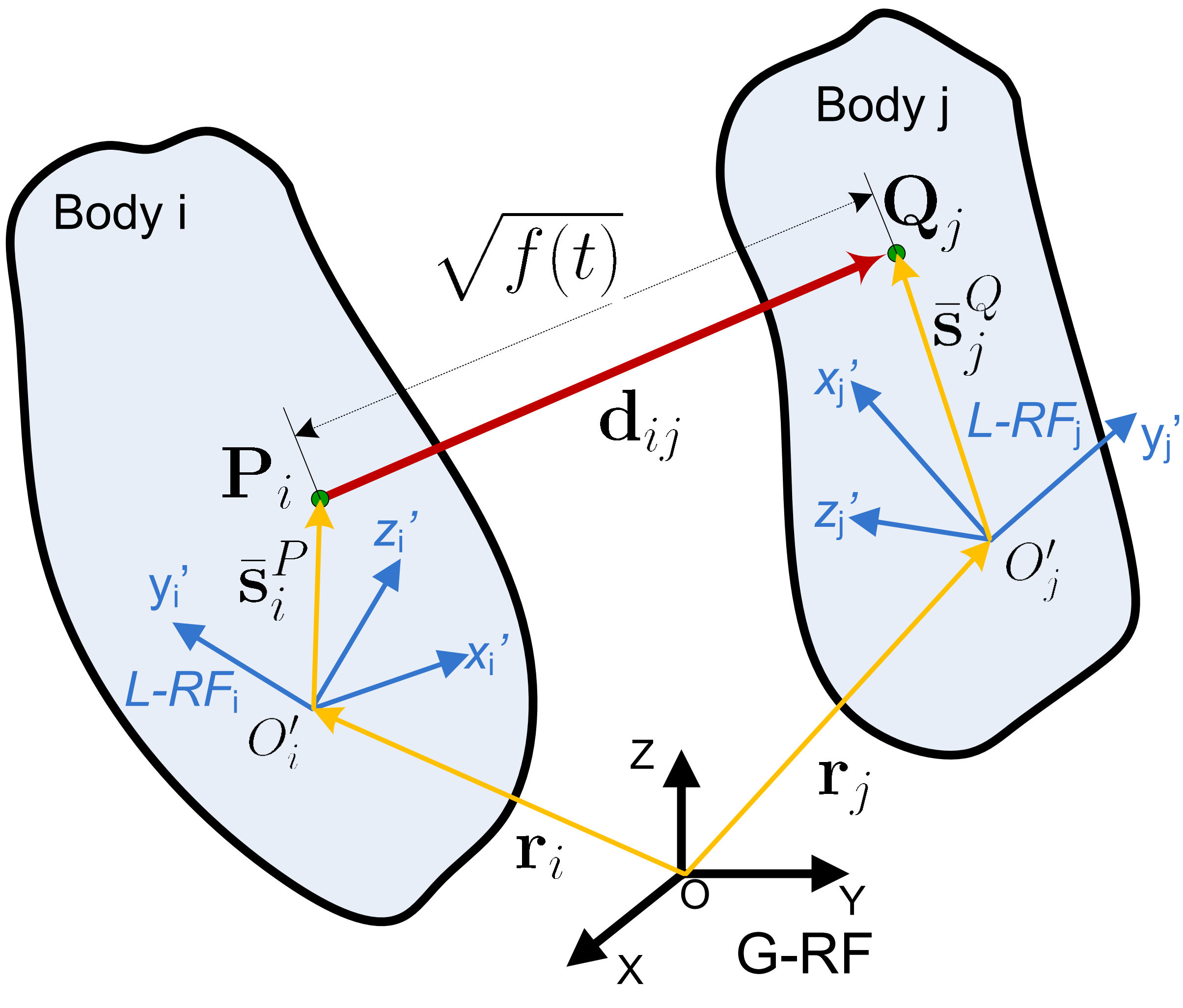
$$\nu^{DP2} = -\frac{\partial \Phi^{DP2}}{\partial t} = \frac{\partial f}{\partial t}$$

- Step 6. The γ term that enters the righ-hand side of the acceleration equation assumes the form:

$$\begin{aligned}\gamma^{DP2} \quad &= \quad 2\bar{\omega}_i^T \tilde{\bar{\mathbf{a}}}_i \mathbf{A}_i^T (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) + 2[\bar{\mathbf{s}}_j^Q]^T \tilde{\bar{\omega}}_j \mathbf{A}_j^T \mathbf{A}_i \tilde{\bar{\omega}}_i \bar{\mathbf{a}}_i - [\bar{\mathbf{s}}_i^P]^T \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{a}}_i \\ &\quad -[\bar{\mathbf{s}}_j^Q]^T \tilde{\bar{\omega}}_j \tilde{\bar{\omega}}_j \mathbf{A}_j^T \mathbf{A}_i \bar{\mathbf{a}}_i - \mathbf{d}_{ij}^T \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{a}}_i + \frac{\partial^2 f}{\partial t^2}\end{aligned}$$

- GCon-DP2 imposes one ACE and as such removes one DOF

Basic GCon: D [Distance]



Basic GCon: D [Distance]
[Cntd.]

- Step 1. GCon Φ^D reflects the fact that motion is such that the distance between point P on body i and point Q on body j assumes a specified value greater than zero.
- Step 2. GCon attributes (quantities required to properly define the GCon above):
  - Body i and the associated L-RF$_i$. On that body we need to know the location $\bar{\mathbf{s}}_i^P$ of the point P.
  - Body j and the associated L-RF$_j$. On that body, we need to know the location $\bar{\mathbf{s}}_j^Q$ of the point Q
  - The prescribed value that the distance between the two points assumes. This prescribed value *does not depend* on GCs, but might depend on time. The prescribed value is specified through the function $f(t)$.
    - Most often, $f(t) = C^2 > 0$, which defines a kinematic constraint (the power 2 emphasizes that the constant function assumes a positive value).
    - If $f(t)$ actually depends on time, this leads to Φ^D being a driving (rheonomic) constraint.
- Step 3. The D-ACE asserts that:

$$
\begin{aligned}
\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) \\
&= (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P)^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0
\end{aligned}
$$

Basic GCon: D [Distance]
[Cntd.]

- Step 4. The Jacobian for D: discussed next lecture
- Step 5. The ν term that enters the righ-hand side of the velocity equation assumes the form:

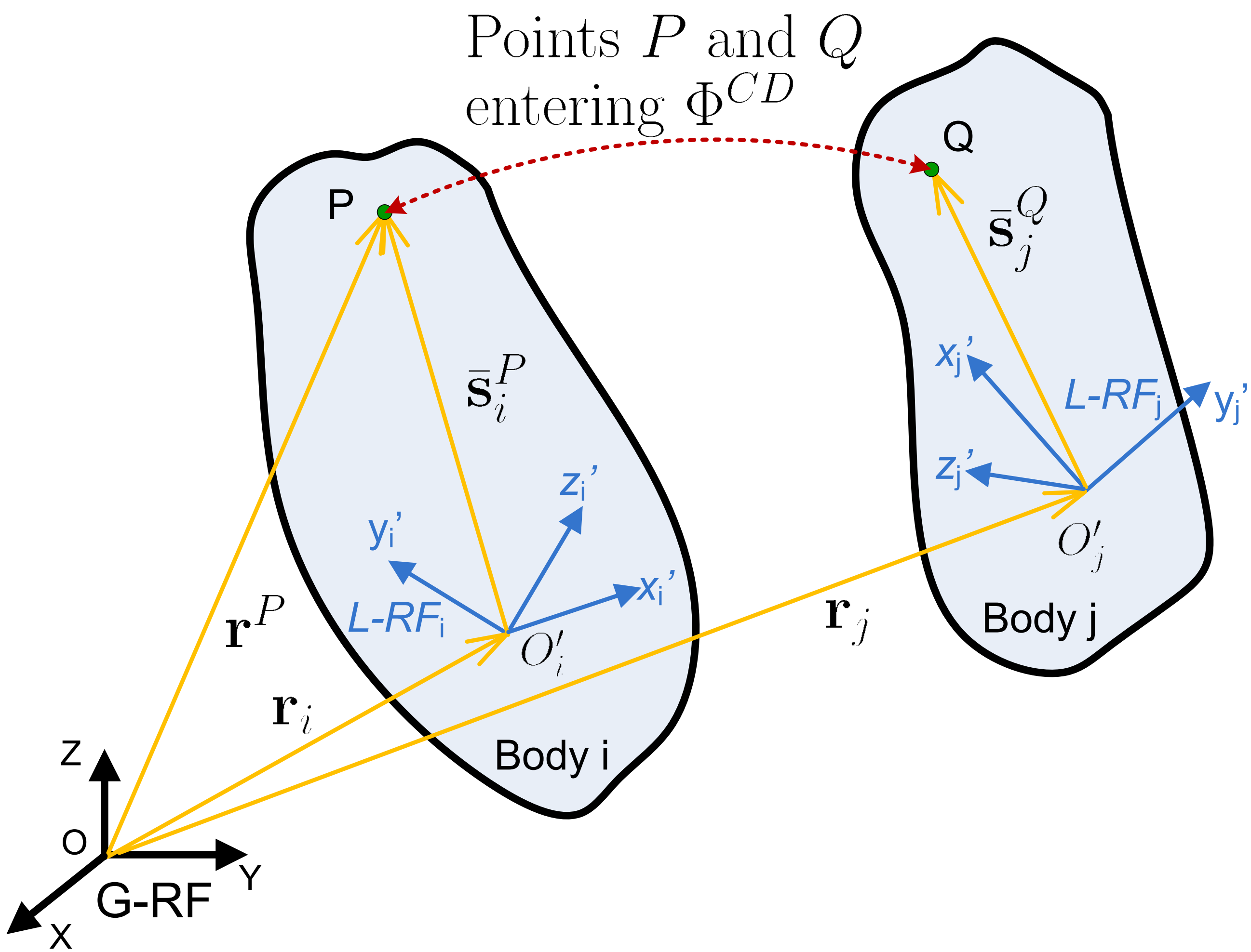
$$\nu^D = -\frac{\partial \Phi^D}{\partial t} = \frac{\partial f}{\partial t}$$

- Step 6. The γ term that enters the righ-hand side of the acceleration equation assumes the form:

$$\begin{aligned}\gamma^D &= -2(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)^T(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i) + 2[\bar{\mathbf{s}}_j^Q]^T \tilde{\bar{\omega}}_j \tilde{\bar{\omega}}_j \bar{\mathbf{s}}_j^Q + 2[\bar{\mathbf{s}}_i^P]^T \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P - 4[\bar{\mathbf{s}}_j^Q]^T \tilde{\bar{\omega}}_j \mathbf{A}_j^T \mathbf{A}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P \\ &+4(\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i)^T(\mathbf{A}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P - \mathbf{A}_j \tilde{\bar{\omega}}_j \bar{\mathbf{s}}_j^Q) - 2\mathbf{d}_{ij}(\mathbf{A}_j \tilde{\bar{\omega}}_j \tilde{\bar{\omega}}_j \bar{\mathbf{s}}_j^Q - \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P) + \frac{\partial^2 f}{\partial t^2}\end{aligned}$$

- Note: GCon-D imposes one ACE and as such it removes one DOF

Basic GCon: CD [Coordinate Difference]
[Cntd.]



Basic GCon: CD
[Cntd.]

- Step 1. GCon Φ^{CD} reflects the fact that motion is such that the difference between the x (or y or z) coordinate of point P on body i and the x (or y or z) coordinate of point Q on body j assumes a specified value.

- Step 2. GCon attributes:
  - The coordinate $\mathbf{c}$ of interest: $\mathbf{c} \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
  - Body i and the associated L-RF$_i$. On that body we need to know the location $\bar{\mathbf{s}}_i^P$ of the point P.
  - Body j and the associated L-RF$_j$. On that body, we need to know the location $\bar{\mathbf{s}}_j^Q$ of the point Q
  - The prescribed value that the coordinate difference assumes. This prescribed value *does not depend* on GCs, but might depend on time. The prescribed value is specified through the function $f(t)$.
    - If $f(t) = \text{const.}$, Φ^{CD} defines a kinematic constraint. Otherwise, it defines a driving (rheonomic) constraint.
    - In many cases the second body j ends up being the ground. In this case, by convention, $j = 0$ (the G-RF is attached to body 0).

- Step 3. The CD-ACE asserts that:

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = \mathbf{c}^T(\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0$$

Basic GCon: CD
[Cntd.]

- Step 4. The Jacobian for CD: discussed next lecture
- Step 5. The ν term that enters the righ-hand side of the velocity equation assumes the form:

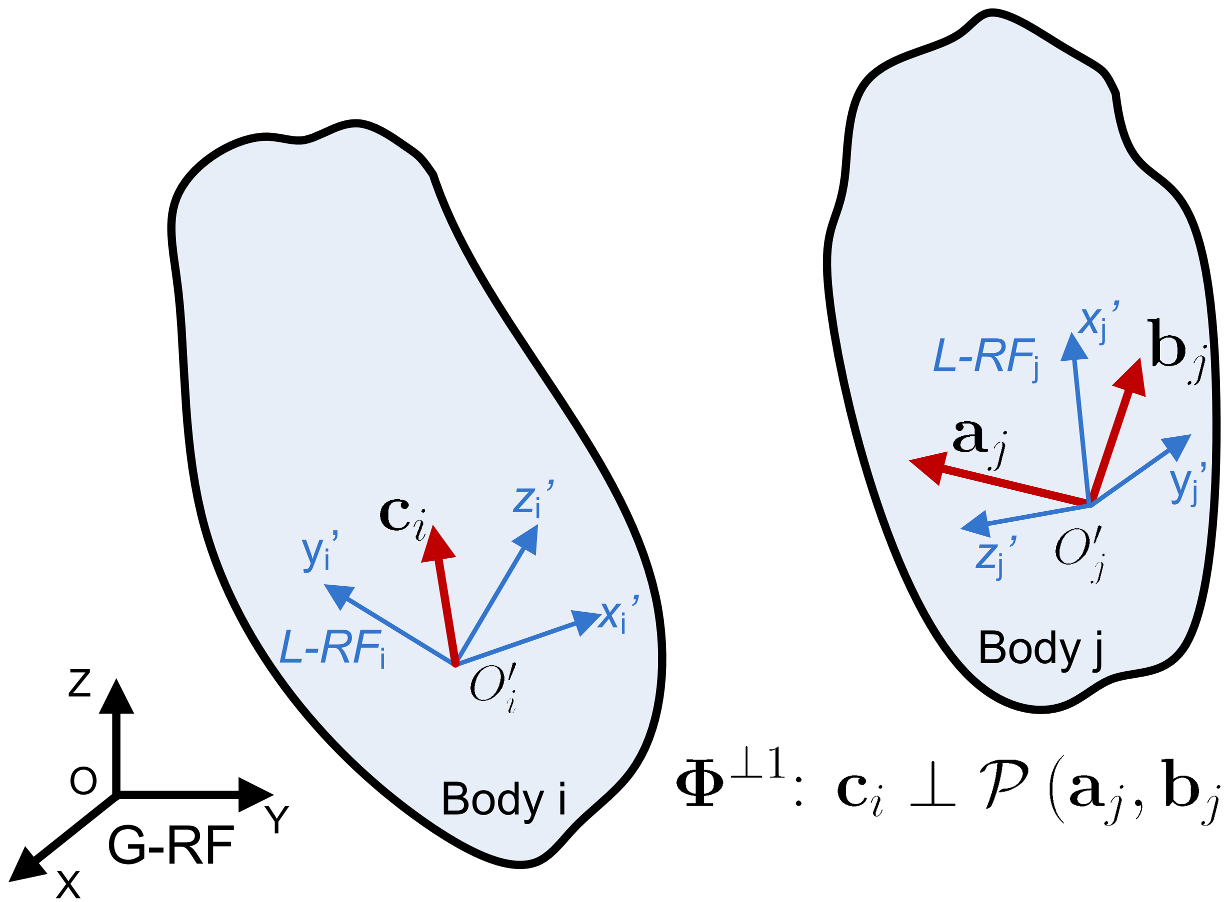
$$\nu^D = -\frac{\partial \Phi^D}{\partial t} = \frac{\partial f}{\partial t}$$

- Step 6. The γ term that enters the righ-hand side of the acceleration equation assumes the form:

$$\gamma^{CD} = \mathbf{c}^T(\mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P - \mathbf{A}_j \tilde{\bar{\omega}}_j \tilde{\bar{\omega}}_j \bar{\mathbf{s}}_j^Q)$$

- Note: GCon-CD imposes one ACE and as such it removes one DOF

Intermediate GCon: ⊥1 [Perpendicular 1]



$$\mathbf{\Phi}^{\perp 1}\text{: } \mathbf{c}_i \perp \mathcal{P}\left(\mathbf{a}_j, \mathbf{b}_j\right)$$

Intermediate GCon: $\perp 1$

[Cntd.]

- Step 1. GCon $\mathbf{\Phi}^{\perp 1}$ reflects the fact that motion is such that a vector $\mathbf{c}_i$ on body i is perpendicular on a plane of body j. This plane is defined by specifying two noncolinear vectors $\mathbf{a}_j$ and $\mathbf{b}_j$ that are contained in that plane. Another way to state GCon $\Phi^{\perp 1}$ is to say that $\mathbf{c}_i$ is parallel to the normal of the said plane. This GCon is built using GCon-DP1 twice. As such, it introduces two ACEs and therefore removes two DOFs.

- Step 2. GCon $\mathbf{\Phi}^{\perp 1}$ has the following attributes :

  – Body i and the associated L-RF$_i$. The vector $\bar{\mathbf{c}}_i$.

  – Body j and the associated L-RF$_j$. The vectors $\bar{\mathbf{a}}_j$ and $\bar{\mathbf{b}}_j$.

- Step 3. The $\perp 1$-ACE asserts that:

$$\mathbf{\Phi}^{\perp 1}(i, \bar{\mathbf{c}}_i, j, \bar{\mathbf{a}}_j, \bar{\mathbf{b}}_j) = \begin{bmatrix} \Phi^{DP1}(i, \bar{\mathbf{c}}_i, j, \bar{\mathbf{a}}_j, 0) \\ \Phi^{DP1}(i, \bar{\mathbf{c}}_i, j, \bar{\mathbf{b}}_j, 0) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j \\ \bar{\mathbf{c}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{b}}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Steps 4, 5, and 6: see discussion for GCon DP1.