# ME751
# Advanced Computational Multibody Dynamics

Section 9.3

February 16, 2010



"Courage is fear holding on a minute longer."
General George S. Patton

# Before we get started…

- Last Time:
  - Concerned with describing the orientation of a body using Euler Parameters

- Today:
  - Finish Euler Parameters discussion (angular velocity)
  - 3D Kinematics of a Rigid Body
  - Kinematics Analysis
  - Kinematic constraints

- HW4
  - Due on Th – make sure you don't make it harder than it needs to be

- Formatting issues cropped up in web ppt material
  - Situation will be corrected shortly (you can fix now, install MathType on your machine)
- Asking for **Feedback** – Tu, Feb. 23: Provide anonymously on a printed page two concerns and/or things that I can do to improve ME751

## $\dot{\mathbf{p}}$ given; $\quad \omega = ?$

$$\tilde{\bar{\omega}} = \mathbf{A}^T\dot{\mathbf{A}} = 2\mathbf{G}\mathbf{E}^T\mathbf{E}\dot{\mathbf{G}}^T$$

$$\Downarrow$$

$$\tilde{\bar{\omega}} = 2\mathbf{G}\dot{\mathbf{G}}^T$$

$$\Downarrow$$

$$\tilde{\bar{\omega}} = 2(\widetilde{\mathbf{G}\dot{\mathbf{p}}})$$

$$\Downarrow$$

$$\bar{\omega} = 2(\mathbf{G}\dot{\mathbf{p}})$$

$$\omega = \mathbf{A}\bar{\omega} = 2\mathbf{E}\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$

$$\Downarrow$$

$$\omega = 2\mathbf{E}\dot{\mathbf{p}}$$

pp.344, Haug's book

$\omega$ given; $\quad \dot{\mathbf{p}} = ?$

$$\bar{\omega} = 2\mathbf{G}\dot{\mathbf{p}}$$

$$\mathbf{G}^T\bar{\omega} = 2\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$

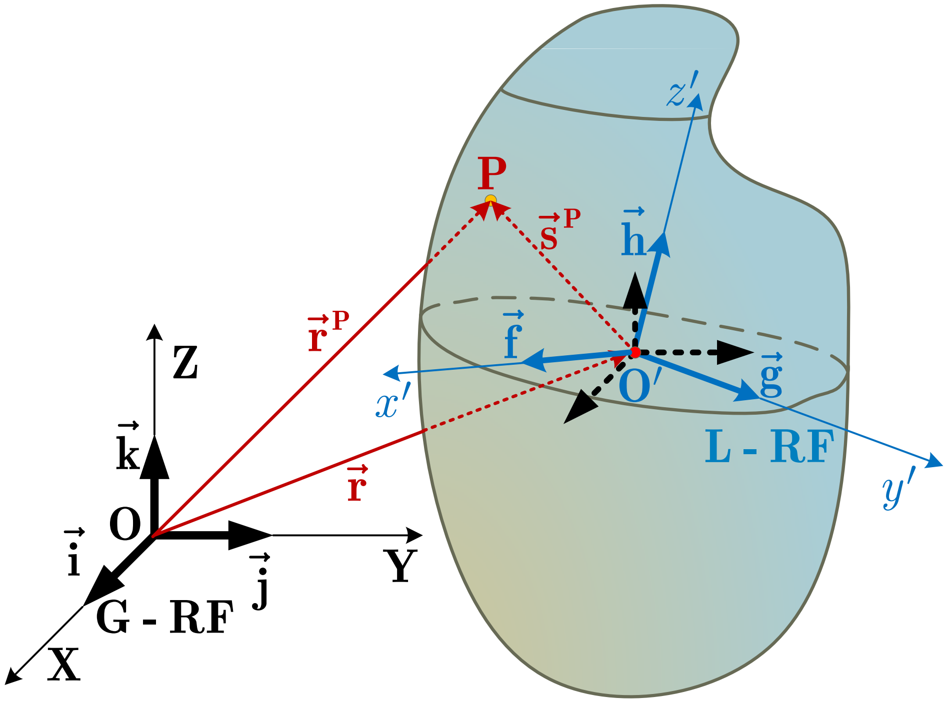
$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\bar{\omega} \longrightarrow \dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^T\omega$$

pp.345, Haug's book

# Full 3D Kinematics of Rigid Bodies

- So far, we focused on the rotation of a rigid body
- Scenario used: the body was connected to ground through a spherical joint that allowed it to experience an arbitrary rotation
- Yet bodies are in general experiencing both translation and rotation

- Framework and Notation Conventions:
  - A L-RF is attached to the rigid body at some location denoted by $O'$
  - Relative to the G-RF, point $O'$ is located by vector $\vec{\mathbf{r}}$
  - L-RF defined by vectors $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, $\vec{\mathbf{h}}$
  - An arbitrary point $P$ of the rigid body is considered. Its location relative to the L-RF is provided through the vector $\vec{\mathbf{s}}^P$



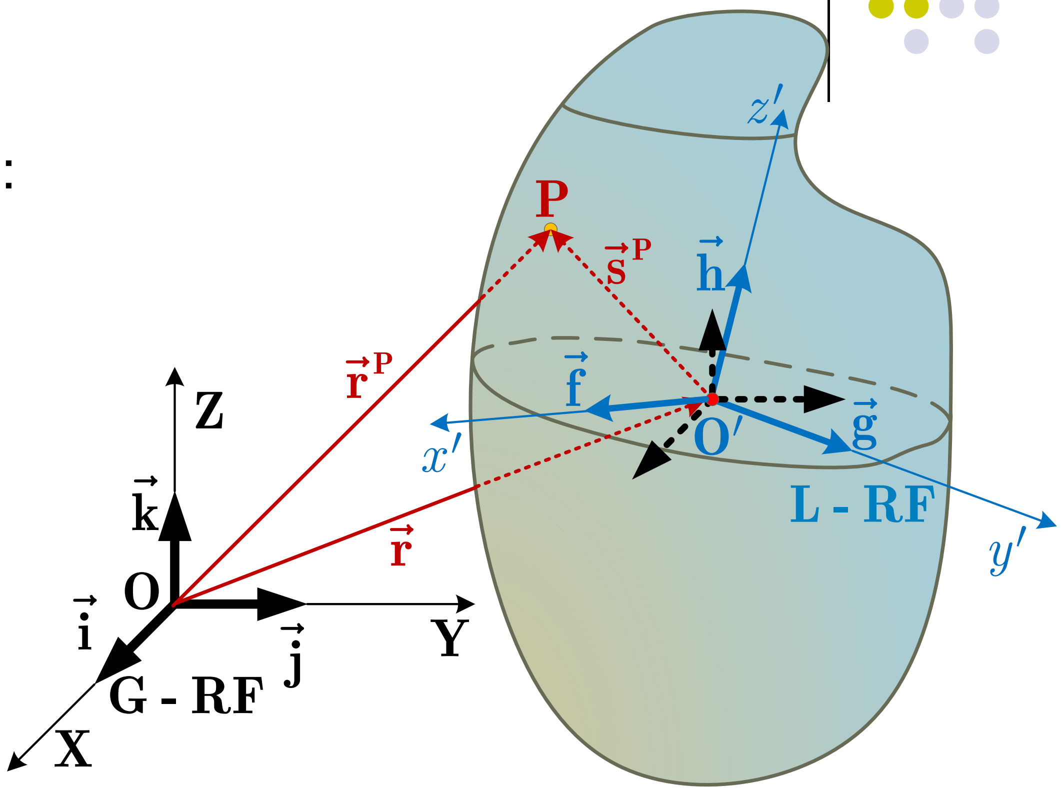
# 3D Rigid Body Kinematics:
## Determining Position of Arbitrary Point P
### [Very Important to Understand]

- In the Geometric Vector world:

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$

$$\Downarrow$$

$$\vec{\mathbf{r}}^P = \vec{\mathbf{r}} + \vec{\mathbf{s}}^P$$




- Algebraic Vector world:

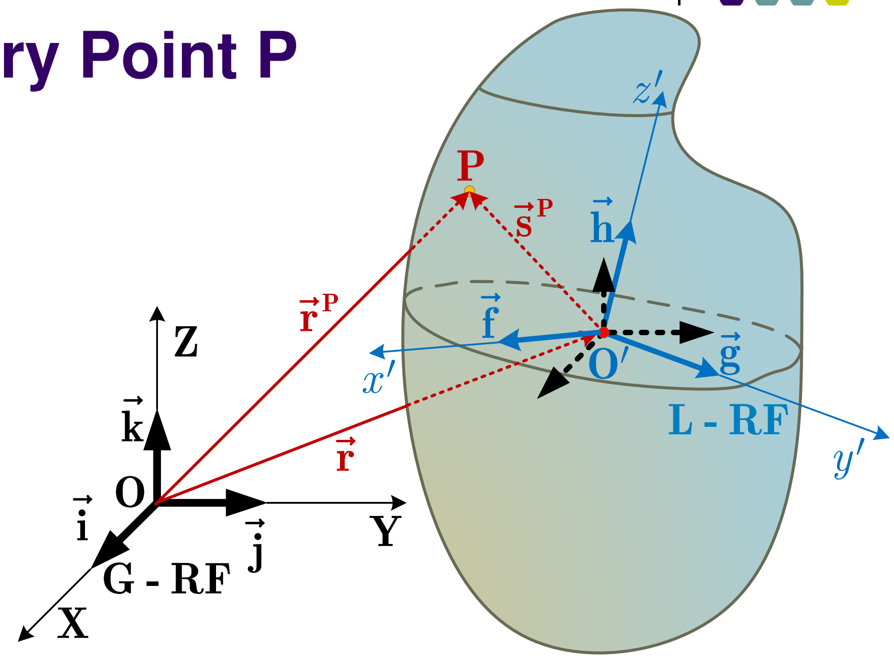
$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\bar{\mathbf{s}}^P$$

- Important observation:
  - The vector $\bar{\mathbf{s}}^P$ that provides the location of $P$ in the L-RF is a constant vector
    - True because the body is assumed to be rigid

# 3D Rigid Body Kinematics:
## Determining Velocity of Arbitrary Point P

- In the Geometric Vector world, by definition:

$$\vec{\mathbf{v}}^P = \frac{d\vec{\mathbf{r}}^P}{dt} = \dot{\vec{\mathbf{r}}} + \dot{\vec{\mathbf{s}}}^P = \dot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\mathbf{s}}^P$$

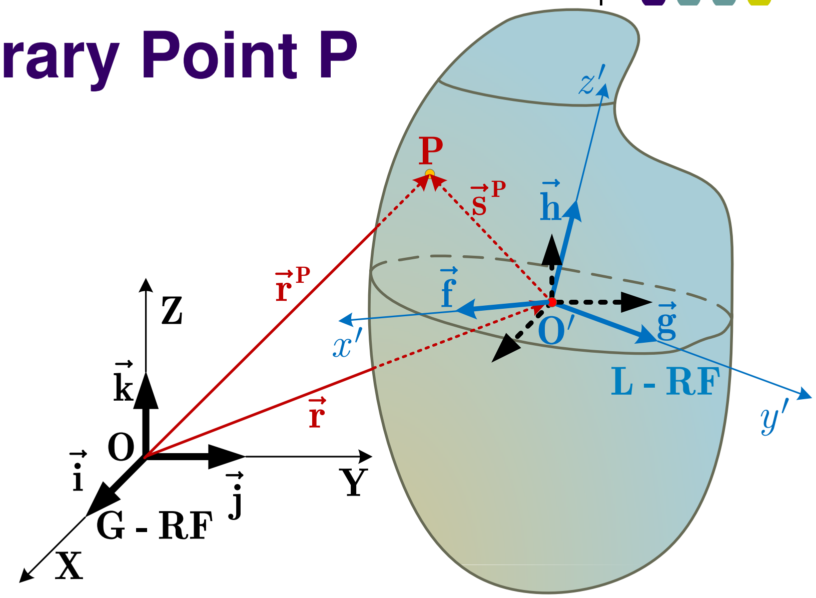



- Using the Algebraic Vector representation:

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{s}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{s}^P$$

- In plain words: the velocity $\dot{\mathbf{r}}^P$ of a point P is equal to the sum of the velocity $\dot{\mathbf{r}}$ of the point where the L-RF is located and the velocity $\tilde{\omega}\mathbf{s}^P$ due to the rotation with angular velocity $\omega$ of the rigid body

# 3D Rigid Body Kinematics:
## Determining Acceleration of Arbitrary Point P




- In the Geometric Vector world, by definition:

$$\vec{\mathbf{a}}^P \equiv \frac{d^2\vec{\mathbf{r}}^P}{dt^2} = \ddot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\omega} \times \vec{\mathbf{s}}^P + \dot{\vec{\omega}} \times \vec{\mathbf{s}}^P$$

- Using the Algebraic Vector representation:

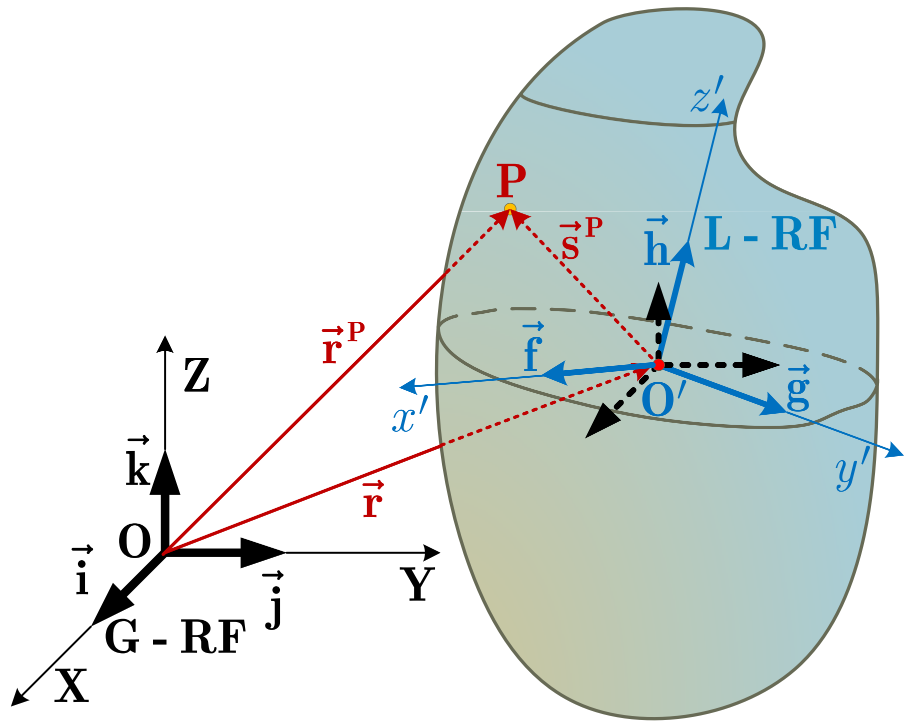
$$\mathbf{a}^P \equiv \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P + \tilde{\dot{\omega}}\mathbf{A}\bar{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{s}^P + \tilde{\dot{\omega}}\mathbf{s}^P$$

# 3D Translation and Rotation OK. Now What?

- Given an arbitrary point P on a rigid body, we know how to
  - Capture its position with respect to both a G-RF and a L-RF
  - Compute its velocity
  - Compute its acceleration

- This will become important when we discuss how to express in *mathematical terms* the fact that the motion of a body is constrained by the presence of joints that limit its relative motion to ground or to other bodies in a mechanical system

  - We start with a geometric perspective on the relative motion between two bodies and then formulate a set of equations in terms of algebraic vectors that enforce the kinematics; that is, capture the effect of the joint connecting the two bodies



## End: Kinematics of a Rigid Body in 3D

## Begin: Kinematics Analysis of Mech. System

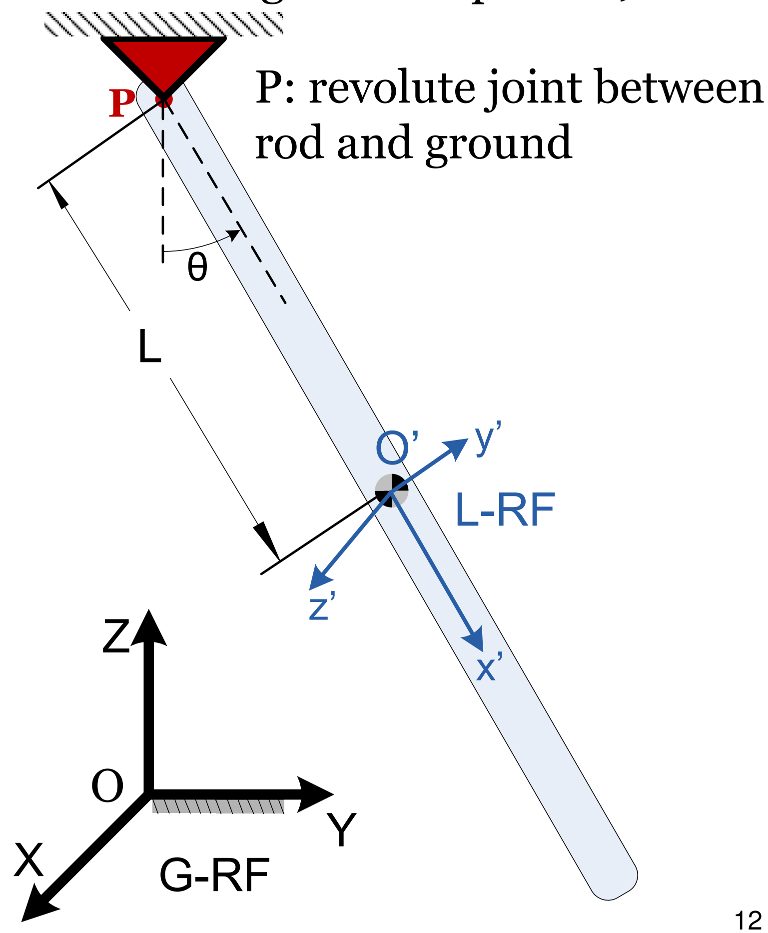
# Kinematics Analysis: Definition

- Kinematics Analysis – the process of computing the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanical system independent of the forces that produce its motion

# Motivating Example: Motion of Simple Pendulum

- A revolute (hinge) joint present at point **P**
- A motion $\theta(t)=4t^2$ is applied to the pendulum
- Find the time evolution of this pendulum

Simple 3D Pendulum
(connected to ground at point P)



# Kinematic Analysis Stages

- Position Analysis Stage
  - Challenging
- Velocity Analysis Stage
  - Simple
- Acceleration Analysis Stage
  - OK

- To take care of all these stages, ONE step is critical:
  - Write down the constraint equations associated with the joints present in your mechanism
- Once you have the constraints, the rest is boilerplate

# Why is Kinematics Important?

- It can be an end in itself…
  - *Kinematic Analysis* - Interested how components of a certain mechanism move when motion[s] are applied
  - *Kinematic Synthesis* – Interested in finding how to design a mechanism to perform a certain operation in a certain way
    - NOTE: ME751 only covers Kinematic Analysis

- It is important to understand Kinematics since the building blocks of the infrastructure here will be recycled when assembling the infrastructure for the Kinetic problem (“Dynamics Analysis”, discussed in Chapter 11)

- In general, the Dynamic Analysis sees more mileage compared to the Kinematic Analysis of mechanism

# Nomenclature & Conventions
**[1st out of 2]**

- We are dealing with rigid bodies only

- Recall this: L-RF – is a <u>body fixed Reference Frame</u> used to describe the position and orientation of a rigid body in the 3D space

- Body Cartesian generalized coordinates (GCs)
  - Used to define the position and orientation of the L-RF mentioned above
  - We'll use Euler Parameters (less headaches than Euler Angles…)
  - For body "i" the GCs that we'll work with are

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{p}_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ e_{0,i} \\ e_{1,i} \\ e_{2,i} \\ e_{3,i} \end{bmatrix}$$

$x_i, y_i, z_i$: Tell us where the body is located

$e_{0,i}, e_{1,i}, e_{2,i}, e_{3,i}$: Tell us how the body is oriented

Note: $\mathbf{q}_i \in \mathbb{R}^7$

# Nomenclature & Conventions
**[2/2]**

- The <u>system GSs</u> – the array of all bodies GCs

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \cdots \\ \mathbf{q}_{nb} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ \vdots \\ e_{2,nb} \\ e_{3,nb} \end{bmatrix} \equiv \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_{nc} \end{bmatrix} \in \mathbb{R}^{nc}$$

- NOTE: for a mechanism with *nb* bodies, the number *nc* of Cartesian generalized coordinates is

$$nc = 7 \cdot nb$$

  - "nc" stands for "number of coordinates"

- Recall we have a number of *nb* "Euler Parameter normalization constraints":

$$\mathbf{p}_i^T \cdot \mathbf{p}_i = 1 \; , \qquad i = 1, 2, \ldots, nb$$

# Putting Things in Perspective

- Before getting lost in the details of the Kinematics Analysis:
  - Recall that we presented a collection of terms that will help understand the "language" of Kinematics
  - We are about to give a 30,000 feet perspective of things to come to justify the need for the material presented over the next two lectures
    - Among the concepts introduced today, here are the more important ones:
      - Constraint equations (as a means to specifying the geometry associated with the motion of a mechanism)
      - Jacobian matrix (or simply, the Jacobian)

# Joints (Physical System) vs. Constraint Equations (Virtual System)

- Physical Mechanical System:
  - A mechanical system (mechanism) uses joints to connect bodies
  - Moreover, some of its components are driven in a predefined fashion

- Virtual System:
  - A set of constraint equations needs to be specified to capture the effect of the joints present in the physical model
  - Some of these equations will be time dependent to capture motions

- Constraint Equations, taxonomy
  - Holonomic vs. Nonholonomic constraint
    - Holonomic: only depends on generalized coordinates, not on their time derivative
  - Scleronomic ("Kinematic") vs. Rheonomic ("Driving") constraints

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\mathbf{\Phi}^K(\mathbf{q}) = \mathbf{0} \qquad\qquad \mathbf{\Phi}^D(\mathbf{q}, t) = \mathbf{0}$$

# Kinematic Constraints, $\mathbf{\Phi}^K(\mathbf{q}) = \mathbf{0}$

- What are they, and what role do they play?
  - A collection of equations that, if satisfied, coerce the bodies in the model to move like the bodies of the mechanism
    - They enforce the geometry of the motion

- Most important thing in relation to constraints:
  - For each joint in the model, the equations of constraint that you use must imply the relative motion allowed by the joint
    - This is where we'll spend a lecture
  - Keep in mind: the way you **model** should resemble the **physical system** (the geometry of the motion)

- Notation: We'll use $m_K$ to denote the number of kinematic (or scleronomic) constraints present in the model:

$$\mathbf{\Phi}^K(\mathbf{q}) \in \mathbb{R}^{m_K}$$

# Driving Constraints, $\mathbf{\Phi}^D(\mathbf{q},t) = \mathbf{0}$

- What are they, and what role do they play?
  - An equation, such as $\theta$(t)=$\pi$ t + $\pi$/2, that specifies how a generalized coordinate or a relation that depends on model GCs changes in time
    - They define (prescribe) the motion

- Notation:
  - We'll use $m_D$ to denote the number of driving (or rheonomic) constraints present in the model

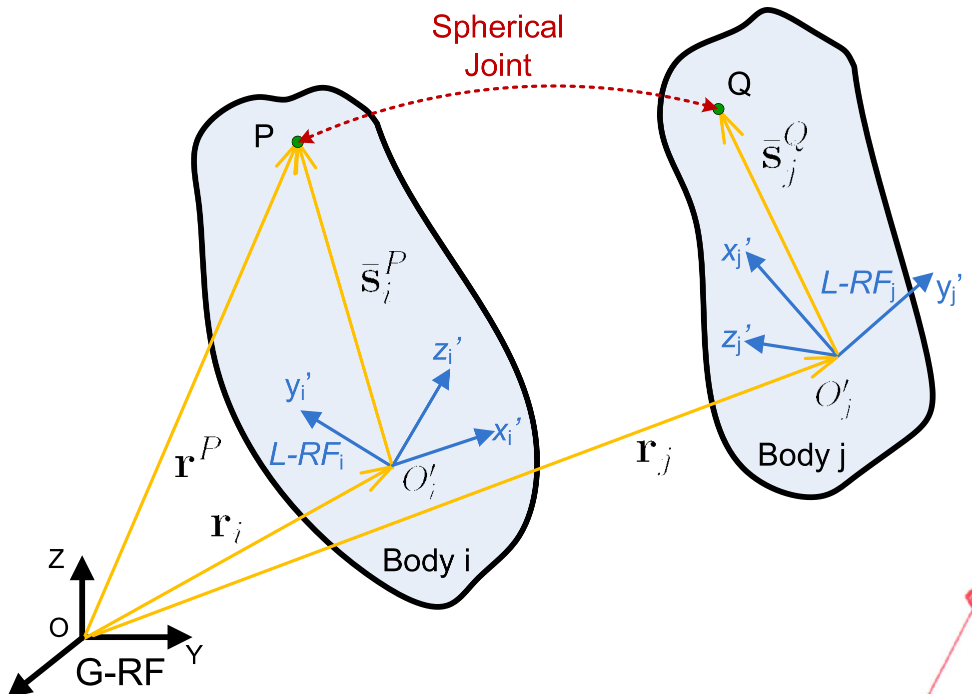
$$\mathbf{\Phi}^D(\mathbf{q},t) \in \mathbb{R}^{m_D}$$

  - We'll use m to denote the total number of constraints (kinematic and driving) present in the model:

$$m = m_K + m_D$$

# Example:
# Handling a Spherical Joint

- Define a set of Kinematic Constraints that reflect the existence of a spherical joint between points P on body i and Q on body j




$$\mathbf{r}^P = \mathbf{r}_i + \mathbf{s}_i^P = \mathbf{r}_i + \mathbf{A}_i(\mathbf{p}_i)\bar{\mathbf{s}}_i^P$$

$$\mathbf{r}^Q = \mathbf{r}_j + \mathbf{s}_j^Q = \mathbf{r}_j + \mathbf{A}_j(\mathbf{p}_j)\bar{\mathbf{s}}_j^Q$$

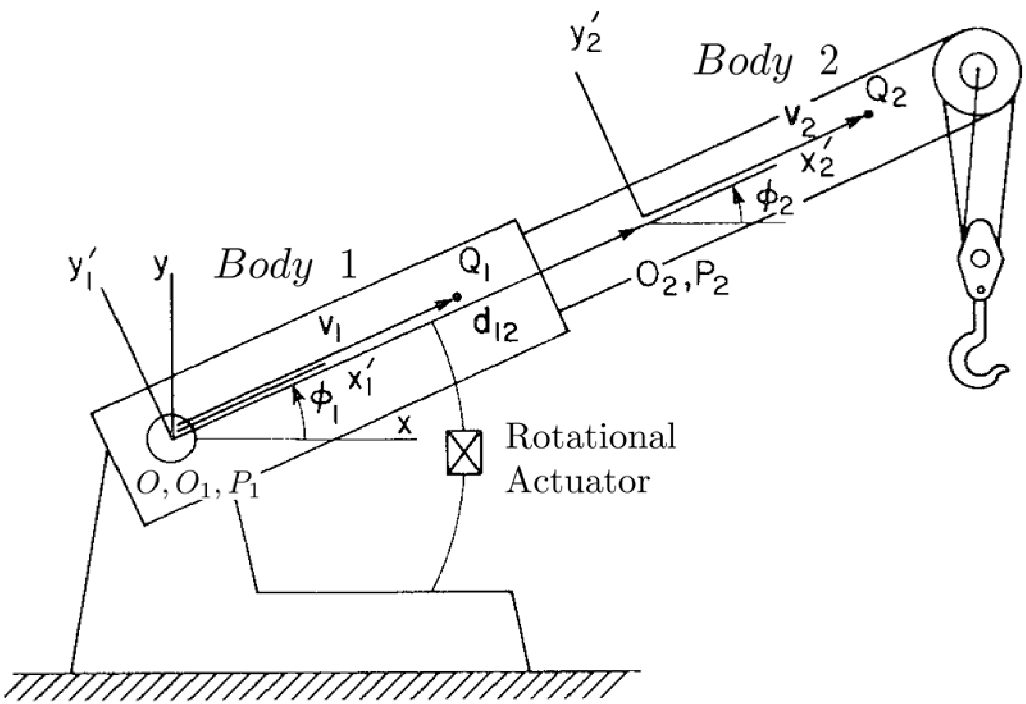
$$\boxed{\mathbf{r}^P = \mathbf{r}^Q} \quad \Rightarrow \quad \mathbf{r}_i + \mathbf{A}_i\bar{\mathbf{s}}_i^P = \mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q$$

$$\mathbf{\Phi}(\mathbf{q}_i, \mathbf{q}_j) = \mathbf{r}_i + \mathbf{A}_i\bar{\mathbf{s}}_i^P - \mathbf{r}_j - \mathbf{A}_j\bar{\mathbf{s}}_j^Q = \mathbf{0}_3$$

- Note that the spherical joint condition is enforced by requiring that the points **P** and **Q** coincide at all times

# Example: Specifying Motions

- Wrecker boom with two motions prescribed (ME451 example)




- Prescribed motions:

$$||P_1 P_2|| = 3 + 0.1t$$

$$\phi_1(t) = 0.1t$$

# Degrees of Freedom

- Number of degrees of freedom (NDOF, ndof) is equal to total number of generalized coordinates minus the number of constraints that these coordinates must satisfy
  - Sometimes also called “Gruebler Count”

$$NDOF = nc - m_K - m_D$$

- Quick Remarks:
  - NDOF is an attribute of the model, and it is independent of the set of generalized coordinates used to represent the motion of the mechanism
  - When using Euler Parameters for body orientation, $m_K$ should also include the set of nb normalization constraints

- In general, for carrying out Kinematic Analysis, NDOF=0
  - For Dynamics Analysis, NDOF $\geq$ 0

# Motion: Causes

- How can one set a mechanical system in motion?
  - Approach leading to Kinematic Analysis
    - Prescribe motions for various components of the mechanical system until NDOF=0
    - For a well posed problem, you'll be able to uniquely determine **q**(t) as the solution of an algebraic problem

(Forget this for now…)

  - Approach leading to Dynamics Analysis
    - Apply a set of forces upon the mechanism and specify a number of motions, but when doing the latter make sure you end up with NDOF $\geq$ 0
    - For a well posed problem, **q**(t) found as the solution of a differential problem

# Position Analysis

- How do you get the position configuration of the mechanism?
  - Kinematic Analysis key observation: The number of constraints (kinematic and driving) is equal to the number of generalized coordinates: m=nc
    - This is a prerequisite for Kinematic Analysis

$$\mathbf{\Phi}(\mathbf{q}, t) = \begin{bmatrix} \mathbf{\Phi}^K(\mathbf{q}) \\ \mathbf{\Phi}^D(\mathbf{q}, t) \end{bmatrix}_{nc \times 1} = \mathbf{0}$$

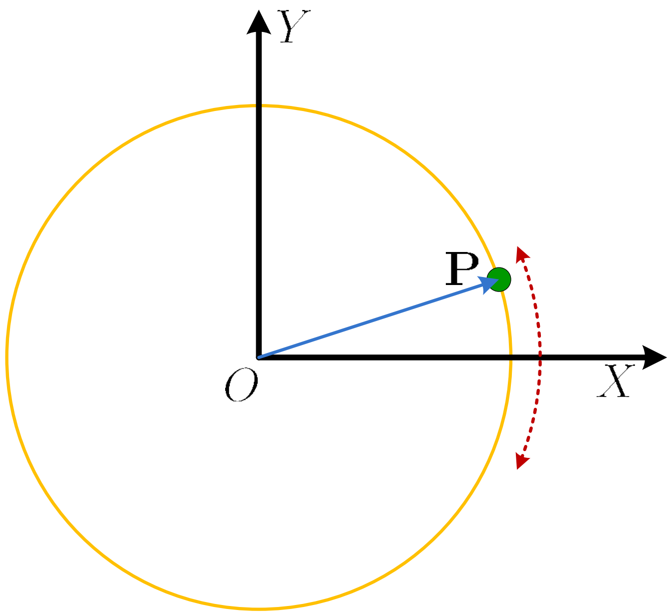
$$\mathbf{q} \in \mathbb{R}^{nc} \qquad \mathbf{\Phi} : \mathbb{R}^{nc+1} \rightarrow \mathbb{R}^{nc}$$

IMPORTANT: This is a nonlinear systems with *nc* equations and *nc* unknowns that you must solve to find **q**

- The solution of the nonlinear system is found by using the so called "Newton-Raphson" algorithm
  - We'll elaborate on this later, for now just assume that you have a way to solve the above nonlinear system to find the solution **q**(t)

# Exercise:
# Kinematic Analysis

- A particle moves on a circle of radius 1
- The generalized coordinates used are $\mathbf{q} = [x\ ,\ y]^T$
- The y coordinate has a prescribed motion: $y(t) = 0.1\sin(50\pi t)$
- Carry our Position Analysis for the given one particle system



# Velocity Analysis

- Take one time derivative of constraints $\Phi$(q,t) to obtain the **velocity equation**:

$$\frac{d}{dt}\mathbf{\Phi}(\mathbf{q},t)=\mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \underbrace{-\mathbf{\Phi}_t}_{\nu}$$

- The Jacobian has as many rows (m) as it has columns (nc) since for Kinematics Analysis, NDOF=nc-m =0

- Therefore, you have a linear system that you need to solve to recover $\dot{\mathbf{q}}$

$$\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \nu$$

# Acceleration Analysis

- Take yet one more time derivative to obtain the **acceleration equation:**

$$\ddot{\mathbf{\Phi}} = \frac{d^2}{dt^2}\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \underbrace{-\left(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}}_{\gamma}$$

- NOTE: Getting right-hand side of acceleration equation is tedious
  - One observation that simplifies the computation: note that the right side of the above equation is made up of everything in the expression of $\ddot{\mathbf{\Phi}}$ that does *not* depend on second time derivatives (accelerations)

- Just like we pointed out for the velocity analysis, you also have to solve a linear system of retrieve the acceleration $\ddot{\mathbf{q}}$

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \gamma$$

# Kinematics Analysis: Comments on the Three Stages

- The three stages of Kinematics Analysis: position analysis, velocity analysis, and acceleration analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity, and acceleration, respectively

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\mathbf{\Phi_q}$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\mathbf{\Phi_q}\,\mathbf{x} = \mathbf{b}$$

- WHAT IS *DIFFERENT* BETWEEN THE THREE STAGES IS THE EXPRESSION OF THE RIGHT-SIDE OF THE LINEAR EQUATION, “**b**”

# Formulating Kinematic Constraints

- What are we after?

- Derive kinematic constraints that specify that the location and/or attitude of a body wrt the global (or absolute) RF is constrained in a certain way
  - Sometimes called <u>absolute</u> constraints

- Derive kinematic constraints that couple the relative motion of two bodies
  - Sometimes called <u>relative</u> constraints

# The Drill…

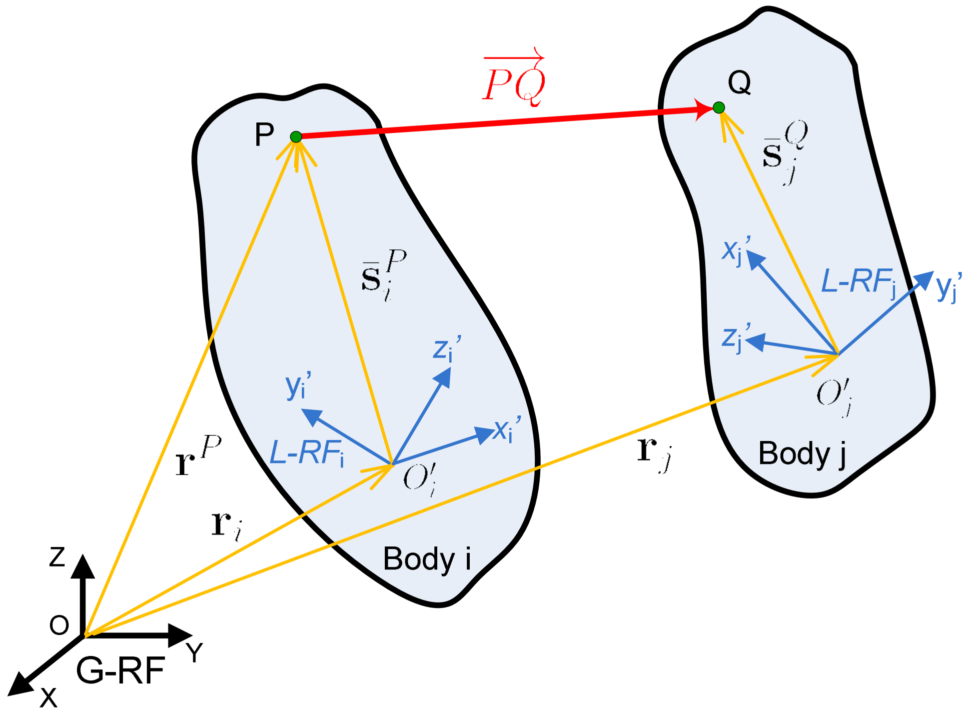
- Step 1: Identify a joint or joint primitive (revolute, translational, relative distance, etc.; i.e., the *physical* thing) acting between two components of a mechanism

- Step 2: Formulate the algebraic equations $\Phi(\mathbf{q})=\mathbf{0}$, that capture the effect of the joint
  - This is called “modeling”

- Step 3: Compute the Jacobian (or the sensitivity matrix) $\Phi_{\mathbf{q}}$

- Step 4: Compute $\nu$, the right side of the velocity equation

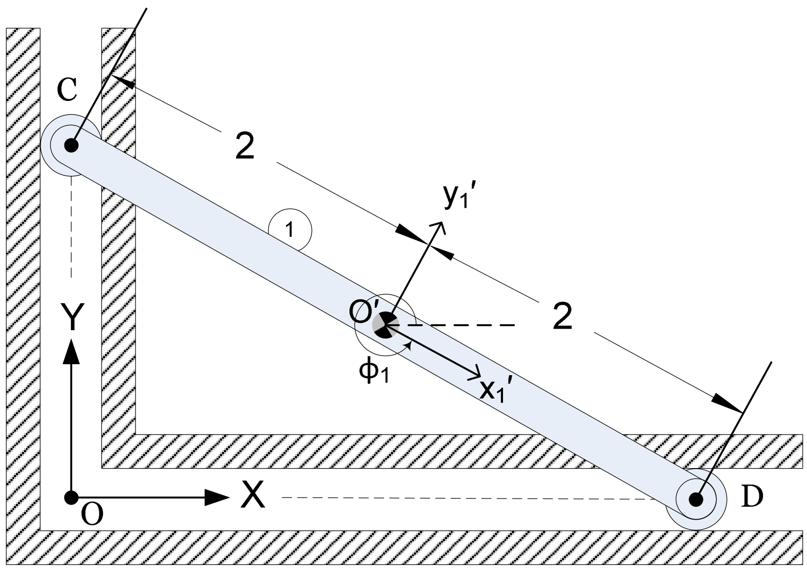
- Step 5: Compute $\gamma$, the right side of the acceleration equation (tedious…)



$\overrightarrow{PQ}$
P
Q
$\bar{\mathbf{s}}_i^P$
$\bar{\mathbf{s}}_j^Q$
$x_j$'
L-RF$_j$
$y_j$'
$z_j$'
$O'_j$
$z_i$'
$y_i$'
$x_i$'
L-RF$_i$
$O'_i$
$\mathbf{r}^P$
$\mathbf{r}_i$
$\mathbf{r}_j$
Body i
Body j
Z
O
Y
X
G-RF

# Example

- The location of point O' in the OXY global RF is $[x,y]^T$. The orientation of the bar is described by the angle $\phi_1$. Find the location of C and D expressed in the global reference frame as functions of x, y, and $\phi_1$.



# Example 2.4.3: Slider Crank

- Based on information provided in figure (b), derive the position vector associated with point P (that is, find position of point P in the global reference frame OXY)

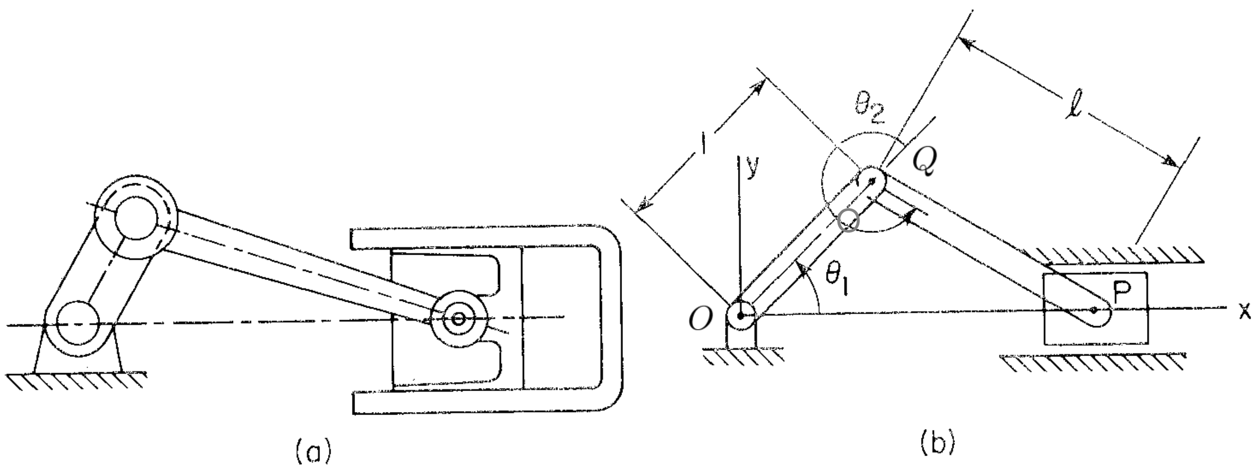



**Figure 2.4.6** Slider–crank mechanism. (a) Physical system. (b) Kinematic model.