“Courage is fear holding on a minute longer.”
General George S. Patton
Before we get started…

- Last Time:
  - Concerned with describing the orientation of a body using Euler Parameters

- Today:
  - Finish Euler Parameters discussion (angular velocity)
  - 3D Kinematics of a Rigid Body
  - Kinematics Analysis
  - Kinematic constraints

- HW4
  - Due on Th – make sure you don’t make it harder than it needs to be

- Formatting issues cropped up in web ppt material
  - Situation will be corrected shortly (you can fix now, install MathType on your machine)

- Asking for **Feedback** – Tu, Feb. 23: Provide anonymously on a printed page two concerns and/or things that I can do to improve ME751
\( \dot{p} \text{ given; } \omega = ? \)

\[
\tilde{\omega} = A^T \dot{A} = 2GE^T E \dot{G}^T
\]

\[\downarrow\]

\[
\tilde{\omega} = 2G \dot{G}^T
\]

\[\downarrow\]

\[
\tilde{\omega} = 2(G \dot{p})
\]

\[\downarrow\]

\[
\bar{\omega} = 2(G \dot{p})
\]

\[
\omega = A \bar{\omega} = 2E G^T G \dot{p}
\]

\[\downarrow\]

\[
\omega = 2E \dot{p}
\]

pp.344, Haug's book
\[ \omega \text{ given; } \quad \dot{p} = ? \]

\[
\bar{\omega} = 2G\dot{p} \\
G^T \bar{\omega} = 2G^T G\dot{p} \\
\dot{p} = \frac{1}{2} G^T \bar{\omega} \quad \Rightarrow \quad \dot{p} = \frac{1}{2} E^T \omega
\]
So far, we focused on the rotation of a rigid body.
Scenario used: the body was connected to ground through a spherical joint that allowed it to experience an arbitrary rotation.
Yet bodies are in general experiencing both translation and rotation.

Framework and Notation Conventions:

- A L-RF is attached to the rigid body at some location denoted by \( O' \).
- Relative to the G-RF, point \( O' \) is located by vector \( \vec{r} \).
- L-RF defined by vectors \( \vec{f}, \vec{g}, \vec{h} \).
- An arbitrary point \( P \) of the rigid body is considered. Its location relative to the L-RF is provided through the vector \( \vec{s}_P \).
3D Rigid Body Kinematics:
Determining Position of Arbitrary Point P
[Very Important to Understand]

- In the Geometric Vector world:
  \[
  \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}
  \]
  \[
  \overrightarrow{r}^P = \overrightarrow{r} + \overrightarrow{s}^P
  \]

- Algebraic Vector world:
  \[
  \overrightarrow{r}^P = \overrightarrow{r} + \overrightarrow{s}^P = \overrightarrow{r} + A\overrightarrow{s}^P
  \]

- Important observation:
  - The vector \( \overrightarrow{s}^P \) that provides the location of \( P \) in the L-RF is a constant vector
    * True because the body is assumed to be rigid
3D Rigid Body Kinematics:
Determining Velocity of Arbitrary Point P

- In the Geometric Vector world, by definition:

\[ \vec{v}^P = \frac{d\vec{r}^P}{dt} = \vec{\dot{r}} + \vec{\dot{s}}^P = \vec{\dot{r}} + \vec{\omega} \times \vec{s}^P \]

- Using the Algebraic Vector representation:

\[ \vec{r}^P = \vec{\dot{r}} + \vec{\dot{s}}^P = \vec{\dot{r}} + \dot{A}\vec{s}^P = \vec{\dot{r}} + \tilde{\omega}A\vec{s}^P = \vec{\dot{r}} + \tilde{\omega}\vec{s}^P \]

- In plain words: the velocity \( \vec{r}^P \) of a point P is equal to the sum of the velocity \( \vec{\dot{r}} \) of the point where the L-RF is located and the velocity \( \tilde{\omega}\vec{s}^P \) due to the rotation with angular velocity \( \omega \) of the rigid body.
3D Rigid Body Kinematics: Determining Acceleration of Arbitrary Point P

- In the Geometric Vector world, by definition:

\[
\dddot{\mathbf{r}}^P \equiv \frac{d^2 \mathbf{r}^P}{dt^2} = \ddot{\mathbf{r}} + \tilde{\omega} \times \ddot{\mathbf{s}}^P + \tilde{\omega} \times \mathbf{s}^P
\]

- Using the Algebraic Vector representation:

\[
\mathbf{a}^P \equiv \dddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \mathbf{s}^P = \ddot{\mathbf{r}} + \tilde{\omega} \tilde{\omega} \mathbf{s}^P + \mathbf{\tilde{\omega}} \mathbf{s}^P = \ddot{\mathbf{r}} + \tilde{\omega} \tilde{\omega} \mathbf{s}^P + \tilde{\omega} \mathbf{s}^P
\]
3D Translation and Rotation OK. Now What?

- Given an arbitrary point \( P \) on a rigid body, we know how to:
  - Capture its position with respect to both a G-RF and a L-RF
  - Compute its velocity
  - Compute its acceleration

- This will become important when we discuss how to express in mathematical terms the fact that the motion of a body is constrained by the presence of joints that limit its relative motion to ground or to other bodies in a mechanical system.

- We start with a geometric perspective on the relative motion between two bodies and then formulate a set of equations in terms of algebraic vectors that enforce the kinematics; that is, capture the effect of the joint connecting the two bodies.
End: Kinematics of a Rigid Body in 3D

Begin: Kinematics Analysis of Mech. System
Kinematics Analysis: Definition

- Kinematics Analysis – the process of computing the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanical system independent of the forces that produce its motion
Motivating Example: Motion of Simple Pendulum

- A revolute (hinge) joint present at point \( P \)
- A motion \( \theta(t) = 4t^2 \) is applied to the pendulum
- Find the time evolution of this pendulum
Kinematic Analysis Stages

- **Position Analysis Stage**
  - Challenging

- **Velocity Analysis Stage**
  - Simple

- **Acceleration Analysis Stage**
  - OK

- To take care of all these stages, ONE step is critical:
  - Write down the constraint equations associated with the joints present in your mechanism

- Once you have the constraints, the rest is boilerplate
Why is Kinematics Important?

- It can be an end in itself…
  - Kinematic Analysis - Interested how components of a certain mechanism move when motion[s] are applied
  - Kinematic Synthesis – Interested in finding how to design a mechanism to perform a certain operation in a certain way
    - NOTE: ME751 only covers Kinematic Analysis

- It is important to understand Kinematics since the building blocks of the infrastructure here will be recycled when assembling the infrastructure for the Kinetic problem (“Dynamics Analysis”, discussed in Chapter 11)

- In general, the Dynamic Analysis sees more mileage compared to the Kinematic Analysis of mechanism
Nomenclature & Conventions

[1st out of 2]

- We are dealing with rigid bodies only

- Recall this: L-RF – is a **body fixed Reference Frame** used to describe the position and orientation of a rigid body in the 3D space

- Body Cartesian generalized coordinates (GCs)
  - Used to define the position and orientation of the L-RF mentioned above
  - We’ll use Euler Parameters (less headaches than Euler Angles…)
  - For body “i” the GCs that we’ll work with are

\[
q_i = \begin{bmatrix}
  r_i \\
  p_i
\end{bmatrix} = \begin{bmatrix}
  x_i \\
  y_i \\
  z_i \\
  e_{0,i} \\
  e_{1,i} \\
  e_{2,i} \\
  e_{3,i}
\end{bmatrix}
\]

- Tell us where the body is located
- Tell us how the body is oriented

Note: \( q_i \in \mathbb{R}^7 \)
The system GSs – the array of all bodies GCs

\[ q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{nb} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ e_{2, nb} \\ e_{3, nb} \end{bmatrix} \equiv \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{nc} \end{bmatrix} \in \mathbb{R}^{nc} \]

NOTE: for a mechanism with \( nb \) bodies, the number \( nc \) of Cartesian generalized coordinates is

\[ nc = 7 \cdot nb \]

“nc” stands for “number of coordinates”

Recall we have a number of \( nb \) “Euler Parameter normalization constraints”:

\[ \mathbf{p}_i^T \cdot \mathbf{p}_i = 1, \quad i = 1, 2, \ldots, nb \]
Putting Things in Perspective

- Before getting lost in the details of the Kinematics Analysis:
  - Recall that we presented a collection of terms that will help understand the “language” of Kinematics
  - We are about to give a 30,000 feet perspective of things to come to justify the need for the material presented over the next two lectures
  - Among the concepts introduced today, here are the more important ones:
    - Constraint equations (as a means to specifying the geometry associated with the motion of a mechanism)
    - Jacobian matrix (or simply, the Jacobian)
Joints (Physical System) vs. Constraint Equations (Virtual System)

- **Physical Mechanical System:**
  - A mechanical system (mechanism) uses joints to connect bodies
  - Moreover, some of its components are driven in a predefined fashion

- **Virtual System:**
  - A set of constraint equations needs to be specified to capture the effect of the joints present in the physical model
  - Some of these equations will be time dependent to capture motions

- **Constraint Equations, taxonomy**
  - Holonomic vs. Nonholonomic constraint
    - Holonomic: only depends on generalized coordinates, not on their time derivative
  - Scleronomic (“Kinematic”) vs. Rheonomic (“Driving”) constraints
    \[ \Phi^K(q) = 0 \]  \[ \Phi^D(q, t) = 0 \]
Kinematic Constraints, $\Phi^K(q) = 0$

- What are they, and what role do they play?
  - A collection of equations that, if satisfied, coerce the bodies in the model to move like the bodies of the mechanism
    - They enforce the geometry of the motion

- Most important thing in relation to constraints:
  - For each joint in the model, the equations of constraint that you use must imply the relative motion allowed by the joint
    - This is where we’ll spend a lecture

- Keep in mind: the way you model should resemble the physical system (the geometry of the motion)

- Notation: We’ll use $m_K$ to denote the number of kinematic (or scleronomic) constraints present in the model:
  $\Phi^K(q) \in \mathbb{R}^{m_K}$
Driving Constraints, $\Phi^D(q, t) = 0$

- What are they, and what role do they play?
  - An equation, such as $\theta(t) = \pi t + \pi/2$, that specifies how a generalized coordinate or a relation that depends on model GCs changes in time
    - They define (prescribe) the motion

- Notation:
  - We’ll use $m_D$ to denote the number of driving (or rheonomic) constraints present in the model
    
    
    $\Phi^D(q, t) \in \mathbb{R}^{m_D}$
  
  - We’ll use $m$ to denote the total number of constraints (kinematic and driving) present in the model:
    
    $m = m_K + m_D$
Example: Handling a Spherical Joint

- Define a set of Kinematic Constraints that reflect the existence of a spherical joint between points $P$ on body $i$ and $Q$ on body $j$

\begin{align*}
\mathbf{r}^P &= \mathbf{r}_i + \mathbf{s}_i^P = \mathbf{r}_i + \mathbf{A}_i(\mathbf{p}_i)\mathbf{s}_i^P \\
\mathbf{r}^Q &= \mathbf{r}_j + \mathbf{s}_j^Q = \mathbf{r}_j + \mathbf{A}_j(\mathbf{p}_j)\mathbf{s}_j^Q \\
\mathbf{r}^P &= \mathbf{r}^Q \implies \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i^P = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j^Q \\
\Phi(q_i, q_j) &= \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i^P - \mathbf{r}_j - \mathbf{A}_j\mathbf{s}_j^Q = 0_3
\end{align*}

- Note that the spherical joint condition is enforced by requiring that the points $P$ and $Q$ coincide at all times
Example: Specifying Motions

- Wrecker boom with two motions prescribed (ME451 example)

- Prescribed motions:

  \[ ||P_1 P_2|| = 3 + 0.1t \]

  \[ \phi_1(t) = 0.1t \]
Degrees of Freedom

- Number of degrees of freedom (NDOF, ndof) is equal to total number of generalized coordinates minus the number of constraints that these coordinates must satisfy
  - Sometimes also called “Gruebler Count”

\[ NDOF = nc - m_K - m_D \]

- Quick Remarks:
  - NDOF is an attribute of the model, and it is independent of the set of generalized coordinates used to represent the motion of the mechanism
  - When using Euler Parameters for body orientation, \( m_K \) should also include the set of nb normalization constraints

- In general, for carrying out Kinematic Analysis, NDOF=0
  - For Dynamics Analysis, NDOF \( \geq 0 \)
Motion: Causes

- How can one set a mechanical system in motion?

  - Approach leading to Kinematic Analysis
    - Prescribe motions for various components of the mechanical system until NDOF = 0
    - For a well posed problem, you’ll be able to uniquely determine $\mathbf{q}(t)$ as the solution of an algebraic problem

  - Approach leading to Dynamics Analysis
    - Apply a set of forces upon the mechanism and specify a number of motions, but when doing the latter make sure you end up with NDOF $\geq 0$
    - For a well posed problem, $\mathbf{q}(t)$ found as the solution of a differential problem

(Forget this for now…)
Position Analysis

- How do you get the position configuration of the mechanism?
  - Kinematic Analysis key observation: The number of constraints (kinematic and driving) is equal to the number of generalized coordinates: \( m = nc \)
  - This is a prerequisite for Kinematic Analysis

The solution of the nonlinear system is found by using the so called “Newton-Raphson” algorithm

- We’ll elaborate on this later, for now just assume that you have a way to solve the above nonlinear system to find the solution \( q(t) \)
Exercise: Kinematic Analysis

- A particle moves on a circle of radius 1
- The generalized coordinates used are \( \mathbf{q} = [x, y]^T \)
- The y coordinate has a prescribed motion: \( y(t) = 0.1 \sin(50\pi t) \)
- Carry out Position Analysis for the given one particle system
Velocity Analysis

- Take one time derivative of constraints $\Phi(q,t)$ to obtain the **velocity equation**:

$$\frac{d}{dt} \Phi(q, t) = 0 \quad \Rightarrow \quad \Phi_q \dot{q} = -\Phi_t$$

- The Jacobian has as many rows ($m$) as it has columns ($nc$) since for Kinematics Analysis, NDOF=$nc-m =0$

- Therefore, you have a linear system that you need to solve to recover $\dot{q}$

$$\Phi_q \dot{q} = \nu$$
Acceleration Analysis

- Take yet one more time derivative to obtain the **acceleration equation**:

\[ \ddot{\Phi} = \frac{d^2}{dt^2} \Phi(q, t) = 0 \quad \Rightarrow \quad \Phi_q \ddot{q} = - (\Phi_q \dot{q})_q \dot{q} - 2 \Phi_t \dot{q} \dot{q} - \Phi_{tt} \]

- **NOTE**: Getting right-hand side of acceleration equation is tedious
  - One observation that simplifies the computation: note that the right side of the above equation is made up of everything in the expression of \( \ddot{\Phi} \) that does *not* depend on second time derivatives (accelerations)

- Just like we pointed out for the velocity analysis, you also have to solve a linear system of retrieve the acceleration \( \ddot{q} \)

\[ \Phi_q \ddot{q} = \gamma \]
Kinematics Analysis: Comments on the Three Stages

- The three stages of Kinematics Analysis: position analysis, velocity analysis, and acceleration analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity, and acceleration, respectively.

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\Phi_q$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\Phi_q \ x = b$$

Formulating Kinematic Constraints

- What are we after?

- Derive kinematic constraints that specify that the location and/or attitude of a body wrt the global (or absolute) RF is constrained in a certain way
  - Sometimes called *absolute* constraints

- Derive kinematic constraints that couple the relative motion of two bodies
  - Sometimes called *relative* constraints
The Drill…

- Step 1: Identify a joint or joint primitive (revolute, translational, relative distance, etc.; i.e., the physical thing) acting between two components of a mechanism

- Step 2: Formulate the algebraic equations $\Phi(q)=0$, that capture the effect of the joint
  - This is called “modeling”

- Step 3: Compute the Jacobian (or the sensitivity matrix) $\Phi_q$

- Step 4: Compute $v$, the right side of the velocity equation

- Step 5: Compute $\gamma$, the right side of the acceleration equation (tedious…)
Example

- The location of point O’ in the OXY global RF is \([x,y]^T\). The orientation of the bar is described by the angle \(\phi_1\). Find the location of C and D expressed in the global reference frame as functions of \(x\), \(y\), and \(\phi_1\).
Example 2.4.3: Slider Crank

- Based on information provided in figure (b), derive the position vector associated with point P (that is, find position of point P in the global reference frame OXY)

![Slider Crank Diagram](image)

Figure 2.4.6  Slider–crank mechanism. (a) Physical system. (b) Kinematic model.