ME751
Advanced Computational Multibody Dynamics

Review Calculus
Starting Chapter 9 of Haug book
January 26, 2010

"Motivation is what gets you started. Habit is what keeps you going." - James Ronald Ryun, Olympic athlete
Before we get started…

- **Last Time:**
  - Finished the Linear Algebra review
  - Started review of Calculus
    - Emphasis placed on partial derivatives and the chain rule

- **Today:**
  - Finish review of Calculus
  - Discuss the concept of Geometric Vector (starting Chapter 9 of the book…)
  - Introduce the concept of Reference Frame
  - Establish the connection between Geometric Vector and Algebraic Vector

- HW due on Jan. 28.

- Trip to John Deere and NADS confirmed: May 4\textsuperscript{th}
  - Contemplating one more trip to Oshkosh Truck and/or P&H Mining
Scenario 3: Function of Two Vectors

- **F** is a vector function of 2 vector variables **q** and **p**:
  
  \[
  F : \mathbb{R}^n \rightarrow \mathbb{R}^m
  \]

- Both **q** and **p** in turn depend on a set of “k” other variables \( x = [x_1, \ldots, x_k]^T \):
  
  \[
  q = q(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_1}
  \]
  
  \[
  p = p(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_2}
  \]
  
  \[
  n = n_1 + n_2
  \]

- A new function \( \Phi(x) \) is defined as:
  
  \[
  \Phi(x) = F(q(x), p(x)) : \mathbb{R}^k \rightarrow \mathbb{R}^m
  \]
The Chain Rule

- How do you compute the partial derivative of $\Phi$ with respect to $x$?

  $$\Phi : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

  $$\Phi = \Phi(x) \quad \Rightarrow \quad \Phi_x = \frac{\partial \Phi}{\partial x} = ??$$

- **Theorem**: Chain rule for function of two vectors

  $$\Phi_x = \frac{\partial \Phi}{\partial x} = \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} = F_q \cdot q_x + F_p \cdot p_x$$

  (This theorem is proved in your elementary calculus class)
Example:

Assume that $\mathbf{q} = \mathbf{q}(\mathbf{x}) \in \mathbb{R}^3$, and $\mathbf{p} = \mathbf{p}(\mathbf{x}) \in \mathbb{R}^3$. Show that:

$$\frac{\partial (\mathbf{q}^T \mathbf{p})}{\partial \mathbf{x}} = \mathbf{q}^T \mathbf{p}_x + \mathbf{p}^T \mathbf{q}_x$$
Scenario 4: Time Derivatives

- On the previous slides we talked about functions $f$ of $y$, while $y$ in turn depended on yet another variable $x$

- The relevant case is when the variable $x$ is actually time, $t$
  - This scenario is super common in 751:
    - You have a function that depends on the generalized coordinates $\mathbf{q}$, and in turn the generalized coordinates are functions of time (they change in time, since we are talking about kinematics/dynamics here…)

- Case 1: scalar function that depends on an array of $m$ generalized coordinates that in turn depend on time
  \[ \Phi = \Phi(\mathbf{q}(t)) \in \mathbb{R} \]

- Case 2: vector function (of dimension $n$) that depends on an array of $m$ generalized coordinates that in turn depend on time
  \[ \Phi = \Phi(\mathbf{q}(t)) \in \mathbb{R}^n \]
A Special Case: Time Derivatives (Cntd)

- Of interest if finding the time derivative of $\Phi$ and $\Phi$

- Apply the chain rule, the scalar function $\Phi$ case first:

$$\dot{\Phi} = \frac{d\Phi}{dt} = \frac{d\Phi(q(t))}{dt} = \frac{\partial \Phi}{\partial q} \cdot \frac{dq}{dt} = \Phi_q \dot{q} \in \mathbb{R}$$

- For the vector function case, applying the chain rule leads to the same formula, only the size of the result is different…

$$\dot{\Phi} = \frac{d\Phi}{dt} = \frac{d\Phi(q(t))}{dt} = \frac{\partial \Phi}{\partial q} \cdot \frac{dq}{dt} = \Phi_q \dot{q} \in \mathbb{R}^n$$
Example, Scalar Function $\Phi$

- Assume a set of generalized coordinates is defined through array $q$. Also, a scalar function $\Phi$ of $q$ is provided:

  $$q(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} \quad \Phi(q) = 3x(t) + 2L \sin \theta(t)$$

- Find time derivative of $\Phi$

  $$\dot{\Phi} = ?$$
Example, Vector Function $\Phi$

- Assume a set of generalized coordinates is defined through array $\mathbf{q}$. Also, a vector function $\Phi$ of $\mathbf{q}$ is provided:

\[
\mathbf{q}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} \quad \Phi(\mathbf{q}) = \begin{bmatrix} 3x(t) + 2L \sin \theta(t) \\ y(t) - 2L \cos \theta(t) \end{bmatrix}
\]

- Find time derivative of $\Phi$

\[
\dot{\Phi} = ?
\]
Useful Formulas

- A couple of useful formulas, some of them you had to derive as part of the HW

\[
\frac{\partial (g^T p)}{\partial q} = g^T p_q + p^T g_q
\]

\[
\frac{\partial}{\partial q} (Cq) = C
\]

\[
\frac{\partial}{\partial x} (x^T Cy) = y^T C^T
\]

\[
\frac{d}{dt} (p^T Cq) = \dot{p}^T Cq + p^T C\dot{q}
\]

Assumptions:
- \(g = g(q)\)
- \(p = p(q)\)
- \(C\) - constant matrix
- \(y\) doesn’t depend on \(x\)

The dimensions of the vectors and matrix above such that all the operations listed can be carried out.
Example

- Derive the last equality on previous slide
- Can you expand that equation further?

\[
\frac{d}{dt}(p^T C q) = \dot{p}^T C q + p^T C \dot{q}
\]

Assumptions:
- \( p = p(q) \)
- \( C \) - constant matrix
End: Review of Calculus
Begin: 3D Kinematics of a Rigid Body
Geometric Entities: Their Relevance

- Kinematics (and later Dynamics) of systems of rigid bodies:
  - Requires the ability to describe as function of time the position, velocity, and acceleration of each rigid body in the system

- In the Euclidian 3D space, geometric vectors and second order tensors are extensively used to this end
  - Geometric vectors - used to locate points on a body or the center of mass of a rigid body
  - Second order tensors - used to describe the orientation of a body
Geometric Vectors

What is a “Geometric Vector”? 
- A quantity that has three attributes:
  - A direction (given by the blue line)
  - A sense (from O to P)
  - A magnitude, $||OP||$
- Note that all geometric vectors are defined in relation to an origin O

IMPORTANT:
- Geometric vectors are entities that are independent of any reference frame

ME751 deals spatial kinematics and dynamics
- We assume that all the vectors are defined in the 3D Euclidian space
- A basis for the Euclidian space is any collection of three independent vectors
Geometric Vectors: Operations

- What geometric vectors operations are defined out there?
  - Scaling by a scalar $\alpha$
  - Addition of geometric vectors (the parallelogram rule)
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
  - One can measure the angle $\theta$ between two geometric vectors
  - A review these definitions follows over the next couple of slides
G. Vector Operation: Scaling by $\alpha$

- By definition, scaling one geometric vector $\vec{a}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{b} \equiv \alpha \vec{a}$ that has the following three attributes:
  - $\vec{b}$ has the same direction as the vector $\vec{a}$
  - $\vec{b}$ has the sense of $\vec{a}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{b}$ is $b = |\alpha|a$

- Note that if $\alpha = 0$, then $\vec{b}$ is the null vector.
G. Vector Operation: Addition of Two G. Vectors

- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to see visualize, pretty messy to summarize in an analytical fashion:

\[
c = \sqrt{||\text{OR}||^2 + ||\text{RC}||^2} = \sqrt{(a + b \cos \theta)^2 + b^2 \sin^2 \theta}
\]

\[
\alpha = \tan^{-1} \frac{b \sin \theta}{a + b \cos \theta}
\]
G. Vector Operation: Inner Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

\[ \vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\vec{a}, \vec{b}) \]

- Note that operation is commutative

- Don’t call this the “dot product” of the two vectors
  - This name is saved for algebraic vectors
G. Vector Operation: Outer Product of Two G. Vectors

- Direction: perpendicular to the plane determined by the two geometric vectors

- Sense: provided by the “right-hand rule”

- Magnitude:

\[
\|\vec{c}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin \theta(\vec{a}, \vec{b})
\]

- Operation is not commutative since, think right-hand rule
G. Vector Operation: Angle Between Two G. Vectors

- Regarding the angle between two vectors, note that

\[ \theta(\vec{a}, \vec{b}) \neq \theta(\vec{b}, \vec{a}) \quad \theta(\vec{a}, \vec{b}) = 2\pi - \theta(\vec{b}, \vec{a}) \]

- Important: Angles are positive counterclockwise
Combining Basic G. Vector Operations

- P1 – The sum of geometric vectors is associative
  \[ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \]

- P2 – Multiplication with a scalar is distributive with respect to the sum:
  \[ k \cdot (\vec{a} + \vec{b}) = k \cdot \vec{a} + k \cdot \vec{b} \]

- P3 – The inner and outer products are distributive with respect to sum:
  \[ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \]
  \[ \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \]

- P4:
  \[ \vec{b}(\alpha + \beta) = \alpha \cdot \vec{b} + \beta \cdot \vec{b} \]

- Look innocent, but rather hard to prove true
Exercise, P1:

- Prove that the sum of geometric vectors is associative:

\[ \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c} \]
Exercise, P2:

- Prove that multiplication by a scalar is distributive with respect to the sum:

\[ k \cdot (\vec{a} + \vec{b}) = k \cdot \vec{a} + k \cdot \vec{b} \]
Geometric Vectors: Making Things Simpler

- Geometric vectors:
  - Easy to visualize but cumbersome to work with

- The major drawback: hard to manipulate
  - Was very hard to carry out simple operations (recall proving the distributive property just discussed)
  - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entity is cumbersome

- We are about to address these drawbacks by first introducing a reference frame that we’ll use to express all our vectors in

- First, the following observation will prove useful:
  - Three geometric vectors are _independent_ if the third one does not belong to the plan defined by the first two ones
Using Reference Frames (RFs)

- Recall that three vectors that are independent are enough to represent all the other vectors in the 3D Euclidean space.

- It’s convenient to choose these three vectors to be mutually orthonormal:
  - Length 1.0
  - Angle between them: $\pi/2$
  - Denoted by: $\vec{i}, \vec{j}, \vec{k}$
  - Defined such that the following relations hold (right hand RF):

\[
\begin{align*}
\vec{i} \times \vec{j} &= \vec{k} \\
\vec{j} \times \vec{k} &= \vec{i} \\
\vec{k} \times \vec{i} &= \vec{j}
\end{align*}
\]
Representing a G. Vector in a RF; Cartesian Coordinates

- Together, \( \vec{i}, \vec{j}, \) and \( \vec{k} \) define a right-hand reference frame.

- The geometric vectors \( \vec{i}, \vec{j}, \) and \( \vec{k} \) form a basis for the Euclidian space \( \Rightarrow \)
  Any other vector can be expressed as a linear combination of the basis vectors:

\[
\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}
\]

- The scalars \( a_x, \ a_y, \) and \( a_z \) are called the Cartesian coordinates of \( \vec{a} \) in the reference frame defined by \( \vec{i}, \vec{j}, \) and \( \vec{k} \).

- Since \( \vec{i}, \vec{j}, \) and \( \vec{k} \) are mutually orthogonal we can compute easily the Cartesian coordinates:

\[
a_x = \vec{i} \cdot \vec{a} \quad a_y = \vec{j} \cdot \vec{a} \quad a_z = \vec{k} \cdot \vec{a}
\]
Representing a G. Vector in a RF

- Inner product of two g. vectors, recall: \( \mathbf{a} \cdot \mathbf{b} = a b \cos \theta(\mathbf{a}, \mathbf{b}) \)

- Since the angle between basis vectors is \( \pi/2 \):
  \[
  \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0
  \]
  Therefore, the Cartesian coordinates are computed as
  \[
  a_x = \mathbf{a} \cdot \mathbf{i} \\
  a_y = \mathbf{a} \cdot \mathbf{j} \\
  a_z = \mathbf{a} \cdot \mathbf{k}
  \]