ME751
Advanced Computational Multibody Dynamics

Review Calculus

Starting Chapter 9 of Haug book

January 26, 2010

"Motivation is what gets you started. Habit is what keeps you going." - James Ronald Ryun, Olympic athlete

Before we get started…

- Last Time:
  - Finished the Linear Algebra review
  - Started review of Calculus
    - Emphasis placed on partial derivatives and the chain rule

- Today:
  - Finish review of Calculus
  - Discuss the concept of Geometric Vector (starting Chapter 9 of the book…)
  - Introduce the concept of Reference Frame
  - Establish the connection between Geometric Vector and Algebraic Vector

- HW due on Jan. 28.

- Trip to John Deere and NADS confirmed: May 4^{th}
  - Contemplating one more trip to Oshkosh Truck and/or P&H Mining

Scenario 3: Function of Two Vectors

- **F** is a vector function of 2 vector variables **q** and **p** :

$$\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

- Both **q** and **p** in turn depend on a set of "k" other variables **x**=$[x_1, \ldots, x_k]^T$:

$$\mathbf{q} = \mathbf{q}(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_1}$$

$$\mathbf{p} = \mathbf{p}(x_1, \ldots, x_k) : \mathbb{R}^k \rightarrow \mathbb{R}^{n_2}$$

$$n = n_1 + n_2$$

- A new function $\mathbf{\Phi}$**(x)** is defined as:

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{F}(\mathbf{q}(\mathbf{x}), \mathbf{p}(\mathbf{x})) : \mathbb{R}^k \rightarrow \mathbb{R}^m$$

The Chain Rule

- How do you compute the partial derivative of $\mathbf{\Phi}$ with respect to **x** ?

$$\mathbf{\Phi} : \mathbb{R}^k \longrightarrow \mathbb{R}^m$$

$$\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{x}) \qquad \Rightarrow \qquad \mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \ ??$$

- Theorem: Chain rule for function of two vectors

$$\mathbf{\Phi}_{\mathbf{x}} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{x}} = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial \mathbf{x}} = \mathbf{F}_{\mathbf{q}} \cdot \mathbf{q}_{\mathbf{x}} + \mathbf{F}_{\mathbf{p}} \cdot \mathbf{p}_{\mathbf{x}}$$

(This theorem is proved in your elementary calculus class)

Example:

Assume that $\mathbf{q} = \mathbf{q}(\mathbf{x}) \in \mathbb{R}^3$, and $\mathbf{p} = \mathbf{p}(\mathbf{x}) \in \mathbb{R}^3$. Show that:

$$\frac{\partial(\mathbf{q}^T\mathbf{p})}{\partial\mathbf{x}} = \mathbf{q}^T\mathbf{p}_{\mathbf{x}} + \mathbf{p}^T\mathbf{q}_{\mathbf{x}}$$

Scenario 4: Time Derivatives

- On the previous slides we talked about functions f of y, while y in turn depended on yet another variable x

- The relevant case is when the variable x is actually time, t
  - This scenario is super common in 751:
    - You have a function that depends on the generalized coordinates **q**, and in turn the generalized coordinates are functions of time (they change in time, since we are talking about kinematics/dynamics here…)

  - Case 1: scalar function that depends on an array of m generalized coordinates that in turn depend on time

$$\Phi = \Phi(\mathbf{q}(t)) \in \mathbb{R}$$

  - Case 2: vector function (of dimension n) that depends on an array of m generalized coordinates that in turn depend on time

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(\mathbf{q}(t)) \in \mathbb{R}^n$$

A Special Case: Time Derivatives (Cntd)

- Of interest if finding the time derivative of Φ and $\mathbf{\Phi}$

- Apply the chain rule, the scalar function Φ case first:

$$\dot{\Phi} = \frac{d\Phi}{dt} = \frac{d\Phi(\mathbf{q}(t))}{dt} = \frac{\partial \Phi}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \Phi_{\mathbf{q}}\dot{\mathbf{q}} \in \mathbb{R}$$

- For the vector function case, applying the chain rule leads to the same formula, only the size of the result is different…

$$\dot{\mathbf{\Phi}} = \frac{d\mathbf{\Phi}}{dt} = \frac{d\mathbf{\Phi}(\mathbf{q}(t))}{dt} = \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} \in \mathbb{R}^n$$

Example, Scalar Function Φ

- Assume a set of generalized coordinates is defined through array **q**. Also, a scalar function Φ of **q** is provided:

$$\mathbf{q}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix}$$

$$\Phi(\mathbf{q}) = 3x(t) + 2L\sin\theta(t)$$

- Find time derivative of Φ

$$\dot{\Phi} = ?$$

Example, Vector Function Φ

- Assume a set of generalized coordinates is defined through array **q**. Also, a vector function Φ of **q** is provided:

$$\mathbf{q}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} \qquad \boldsymbol{\Phi}(\mathbf{q}) = \begin{bmatrix} 3x(t) & + & 2L\sin\theta(t) \\ y(t) & - & 2L\cos\theta(t) \end{bmatrix}$$

- Find time derivative of $\boldsymbol{\Phi}$

$$\dot{\boldsymbol{\Phi}} = ?$$

Useful Formulas

- A couple of useful formulas, some of them you had to derive as part of the HW

$$\frac{\partial(\mathbf{g}^T\mathbf{p})}{\partial \mathbf{q}} = \mathbf{g}^T\mathbf{p}_{\mathbf{q}} + \mathbf{p}^T\mathbf{g}_{\mathbf{q}}$$

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{C}\mathbf{q}) = \mathbf{C}$$

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^T\mathbf{C}\mathbf{y}) = \mathbf{y}^T\mathbf{C}^T$$

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{C}\mathbf{q}) = \dot{\mathbf{p}}^T\mathbf{C}\mathbf{q} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$

Assumptions:
$\mathbf{g} = \mathbf{g}(\mathbf{q})$
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix
$\mathbf{y}$ doesn't depend on $\mathbf{x}$

The dimensions of the vectors and matrix above such that all the operations listed can be carried out.

Example

- Derive the last equality on previous slide
- Can you expand that equation further?

Assumptions:
$\mathbf{p} = \mathbf{p}(\mathbf{q})$
$\mathbf{C}$ - constant matrix

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{C}\mathbf{q}) = \dot{\mathbf{p}}^T\mathbf{C}\mathbf{q} + \mathbf{p}^T\mathbf{C}\dot{\mathbf{q}}$$

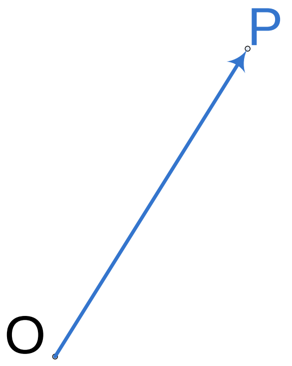
End: Review of Calculus

Begin: 3D Kinematics of a Rigid Body

Geometric Entities: Their Relevance

- Kinematics (and later Dynamics) of systems of rigid bodies:
  - Requires the ability to describe as function of time the position, velocity, and acceleration of each rigid body in the system
  - In the Euclidian 3D space, geometric vectors and second order tensors are extensively used to this end
    - Geometric vectors - used to locate points on a body or the center of mass of a rigid body
    - Second order tensors - used to describe the orientation of a body

Geometric Vectors



- What is a "Geometric Vector"?
  - A quantity that has three attributes:
    - A direction (given by the blue line)
    - A sense (from O to P)
    - A magnitude, ||OP||
  - Note that all geometric vectors are defined in relation to an origin **O**

- IMPORTANT:
  - Geometric vectors are entities that are independent of any reference frame

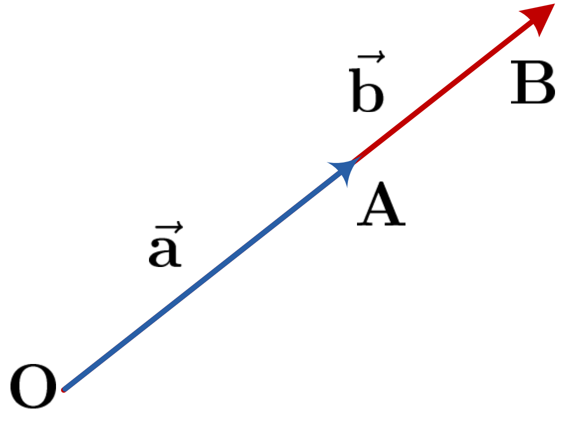
- ME751 deals spatial kinematics and dynamics
  - We assume that all the vectors are defined in the 3D Euclidian space
  - A basis for the Euclidian space is any collection of three independent vectors

Geometric Vectors: Operations

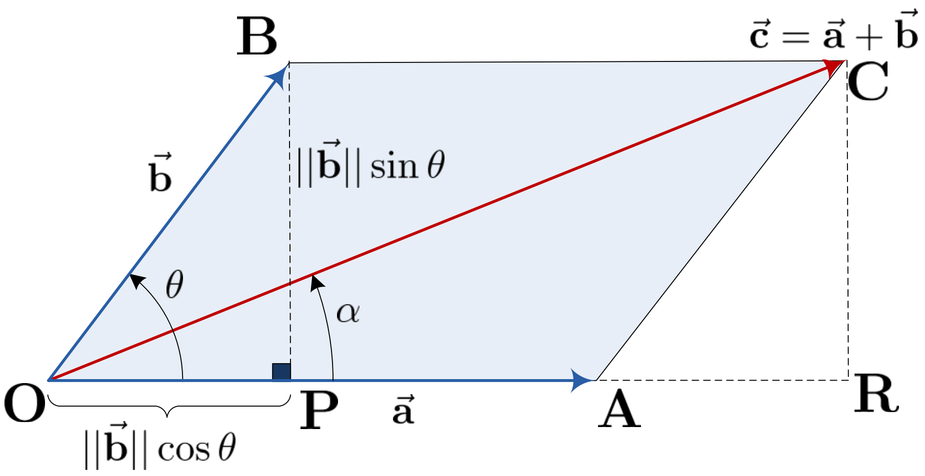
- What geometric vectors operations are <u>defined</u> out there?
  - Scaling by a scalar α
  - Addition of geometric vectors (the parallelogram rule)
  - Multiplication of two geometric vectors
    - The inner product rule (leads to a number)
    - The outer product rule (leads to a vector)
  - One can measure the angle θ between two geometric vectors
- A review these definitions follows over the next couple of slides

G. Vector Operation : Scaling by α

- By definition, scaling one geometric vector $\vec{\mathbf{a}}$ by a scalar $\alpha \neq 0$ leads to a new vector $\vec{\mathbf{b}} \equiv \alpha\vec{\mathbf{a}}$ that has the following three attributes:
  - $\vec{\mathbf{b}}$ has the same direction as the vector $\vec{\mathbf{a}}$
  - $\vec{\mathbf{b}}$ has the sense of $\vec{\mathbf{a}}$ if $\alpha > 0$, and opposite sense if $\alpha < 0$
  - The magnitude of $\vec{\mathbf{b}}$ is $b = |\alpha| a$
- Note that if $\alpha = 0$, then $\vec{\mathbf{b}}$ is the null vector.



G. Vector Operation: Addition of Two G. Vectors



- Sum of two vectors (definition)
  - Obtained by the parallelogram rule
- Operation is commutative
- Easy to see visualize, pretty messy to summarize in an analytical fashion:

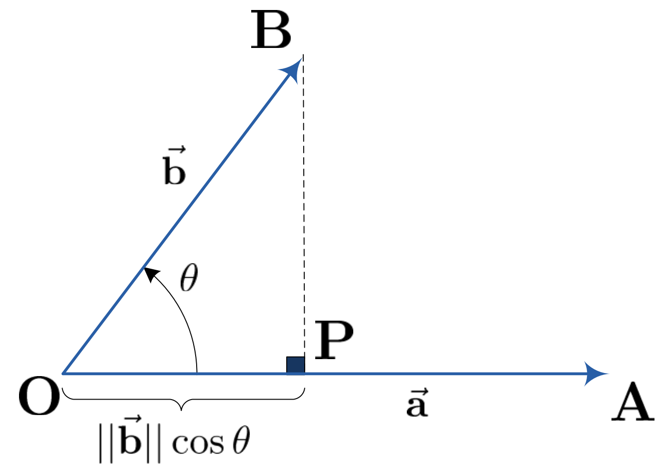
$$c = \sqrt{||\mathbf{OR}||^2 + ||\mathbf{RC}||^2} = \sqrt{(a + b\cos\theta)^2 + b^2\sin^2\theta}$$

$$\alpha = \tan^{-1}\frac{b\sin\theta}{a + b\cos\theta}$$

G. Vector Operation: Inner Product of Two G. Vectors

- The product between the magnitude of the first geometric vector and the projection of the second vector onto the first vector

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \cos(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$



- Note that operation is commutative

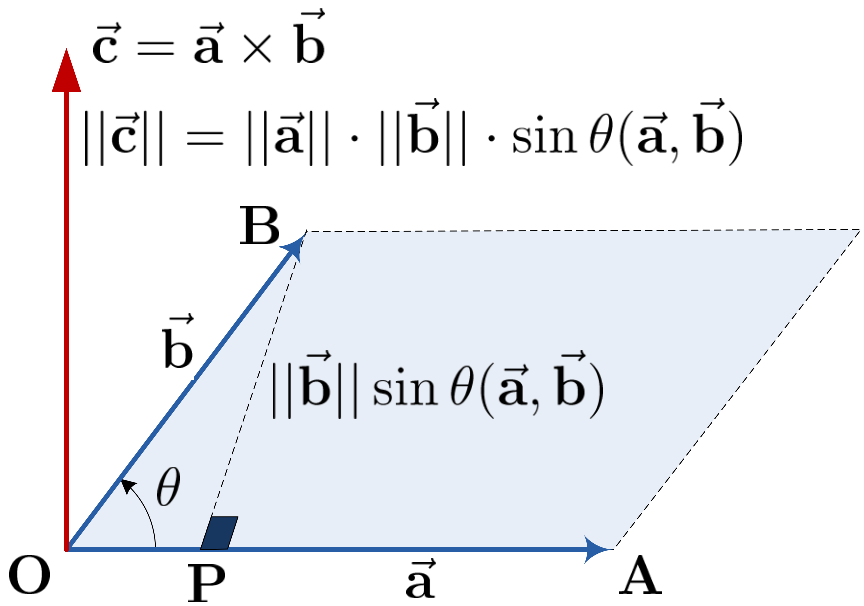
- Don’t call this the “dot product” of the two vectors
  - This name is saved for algebraic vectors

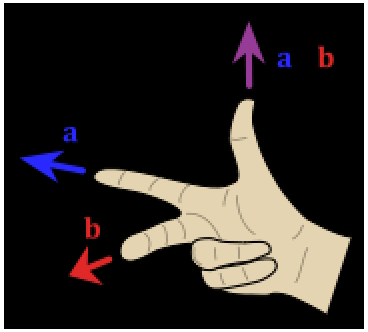
G. Vector Operation: Outer Product of Two G. Vectors

- Direction: perpendicular to the plane determined by the two geometric vectors
- Sense: provided by the “right-hand rule”
- Magnitude:

$$||\vec{\mathbf{c}}|| = ||\vec{\mathbf{a}}|| \cdot ||\vec{\mathbf{b}}|| \cdot \sin\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$$

- Operation is not commutative since, think right-hand rule





G. Vector Operation: Angle Between Two G. Vectors

- Regarding the angle between two vectors, note that

$$\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) \neq \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}}) \qquad \theta(\vec{\mathbf{a}}, \vec{\mathbf{b}}) = 2\pi - \theta(\vec{\mathbf{b}}, \vec{\mathbf{a}})$$

  - Important: Angles are positive counterclockwise

Combining Basic G. Vector Operations

- P1 – The sum of geometric vectors is associative

$$\vec{\mathbf{a}}+(\vec{\mathbf{b}}+\vec{\mathbf{c}})=(\vec{\mathbf{a}}+\vec{\mathbf{b}})+\vec{\mathbf{c}}$$

- P2 – Multiplication with a scalar is distributive with respect to the sum:

$$k\cdot(\vec{\mathbf{a}}+\vec{\mathbf{b}})=k\cdot\vec{\mathbf{a}}+k\cdot\vec{\mathbf{b}}$$

- P3 – The inner and outer products are distributive with respect to sum:

$$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}+\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}$$

$$\vec{\mathbf{a}}\times(\vec{\mathbf{b}}+\vec{\mathbf{c}})=\vec{\mathbf{a}}\times\vec{\mathbf{b}}+\vec{\mathbf{a}}\times\vec{\mathbf{c}}$$

- P4:

$$\vec{\mathbf{b}}(\alpha+\beta)=\alpha\cdot\vec{\mathbf{b}}+\beta\cdot\vec{\mathbf{b}}$$

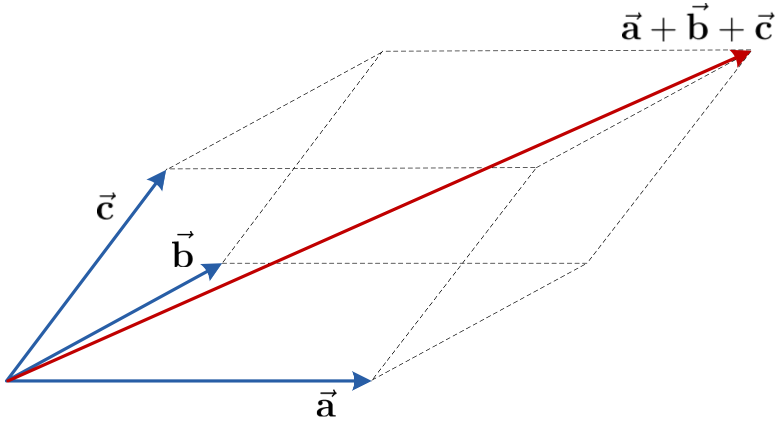
- Look innocent, but rather hard to prove true

[AO]

Exercise, P1:

- Prove that the sum of geometric vectors is associative:

$$\vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = (\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}}$$



[AO]
Exercise, P2:

- Prove that multiplication by a scalar is distributive with respect to the sum:

$$k\cdot(\vec{\mathbf{a}}+\vec{\mathbf{b}})=k\cdot\vec{\mathbf{a}}+k\cdot\vec{\mathbf{b}}$$

Geometric Vectors: Making Things Simpler

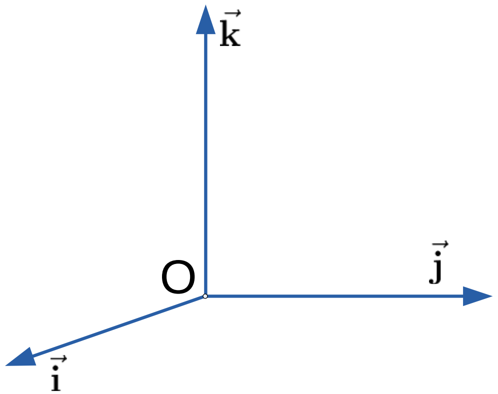
- Geometric vectors:
  - Easy to visualize but cumbersome to work with
  - The major drawback: hard to manipulate
    - Was very hard to carry out simple operations (recall proving the distributive property just discussed)
    - When it comes to computers, which are good at storing matrices and vectors, having to deal with a geometric entity is cumbersome
- We are about to address these drawbacks by first introducing a reference frame that we'll use to express all our vectors in
- First, the following observation will prove useful:
  - Three geometric vectors are *independent* if the third one does not belong to the plan defined by the first two ones

Using Reference Frames (RFs)

- Recall that three vectors that are independent are enough to represent all the other vectors in the 3D Euclidian space

- It's convenient to choose these three vectors to be mutually orthonormal
  - Length 1.0
  - Angle between them: $\pi/2$
  - Denoted by: $\vec{\mathbf{i}}, \quad \vec{\mathbf{j}}, \quad \vec{\mathbf{k}}$
  - Defined such that the following relations hold (right hand RF) :

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}$$
$$\vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}}$$
$$\vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}}$$



Representing a G. Vector in a RF; Cartesian Coordinates

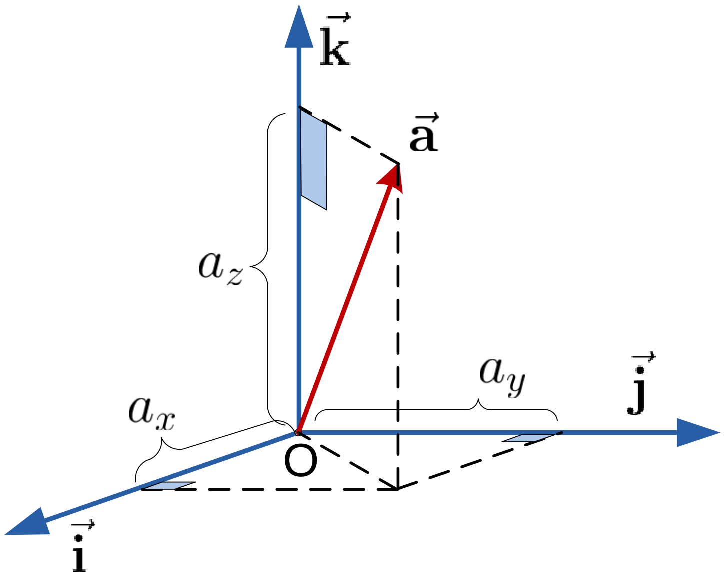
- Together, $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ define a right-hand **reference frame**
- The geometric vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ form a basis for the Euclidian space $\Rightarrow$ Any other vector can be expressed as a linear combination of the basis vectors:

$$\vec{\mathbf{a}} = a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}$$

- The scalars a_x, a_y, and a_z are called the Cartesian coordinates of $\vec{\mathbf{a}}$ in the reference frame defined by $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$
- Since $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ are mutually orthogonal we can compute easily the Cartesian coordinates:

$$a_x = \vec{\mathbf{i}} \cdot \vec{\mathbf{a}} \qquad a_y = \vec{\mathbf{j}} \cdot \vec{\mathbf{a}} \qquad a_z = \vec{\mathbf{k}} \cdot \vec{\mathbf{a}}$$

Representing a G. Vector in a RF



$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}}$$

- Inner product of two g. vectors, recall: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a\, b \cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$

- Since the angle between basis vectors is π/2:

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} = 1 \qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{i}} = 0$$

  - Therefore, the Cartesian coordinates are computed as

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}} \qquad a_z = \vec{\mathbf{a}} \cdot \vec{\mathbf{k}}$$