Kinematics Key Formulas

Algebraic Vectors

$$\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$$

$$\tilde{\mathbf{a}} \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}, \tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a}, \tilde{\mathbf{a}}\mathbf{a} = \mathbf{0}$$

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I} \quad \tilde{\mathbf{a}}\tilde{\mathbf{b}} + \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{b}}\tilde{\mathbf{a}} + \mathbf{b}\mathbf{a}^T$$

$$\widetilde{(\tilde{\mathbf{a}}\mathbf{b})} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}}$$

Euler Parameters

$$\mathbf{p} = \begin{bmatrix} e_0 & \mathbf{e}^T \end{bmatrix}^T = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix}^T$$

$$e_0 = \cos\tfrac{\chi}{2} \quad \mathbf{e} = \sin\tfrac{\chi}{2}\mathbf{u} \quad \mathbf{p}^T\mathbf{p} = 1$$

$$\mathbf{A}(\mathbf{p}) = \mathbf{A} = \left(e_0^2 - \mathbf{e}^T\mathbf{e}\right)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0\tilde{\mathbf{e}}$$

$$= \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}$$

$$\mathbf{a} = \mathbf{A}\mathbf{a}' \quad \mathbf{a}' = \mathbf{A}^T\mathbf{a} \quad \tilde{\mathbf{a}} = \mathbf{A}\tilde{\mathbf{a}}'\mathbf{A}^T \quad \tilde{\mathbf{a}}' = \mathbf{A}^T\tilde{\mathbf{a}}\mathbf{A}$$

Euler Parameter Identities

$\mathbf{a} \in R^3 \quad \mathbf{p} \in R^4$, *not normalized*

(*results hold only if $\mathbf{p}^T\mathbf{p} = 1$)

$$\mathbf{E}(\mathbf{p}) = \mathbf{E} \equiv \begin{bmatrix} -\mathbf{e} & \tilde{\mathbf{e}} + e_0\mathbf{I} \end{bmatrix} \quad \mathbf{E}\mathbf{p} = \mathbf{0}$$

$$\mathbf{G}(\mathbf{p}) = \mathbf{G} \equiv \begin{bmatrix} -\mathbf{e} & -\tilde{\mathbf{e}} + e_0\mathbf{I} \end{bmatrix} \quad \mathbf{G}\mathbf{p} = \mathbf{0}$$

$$\mathbf{A} = \mathbf{E}\mathbf{G}^T \quad \dot{\mathbf{A}} = 2\mathbf{E}\dot{\mathbf{G}}^T = 2\dot{\mathbf{E}}\mathbf{G}^T \quad \mathbf{p}^T\dot{\mathbf{p}} = 0^*$$

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \left(\mathbf{p}^T\mathbf{p}\right)^2\mathbf{I} = \mathbf{I}^*$$

$$\mathbf{E}\mathbf{E}^T = \mathbf{G}\mathbf{G}^T = \mathbf{p}^T\mathbf{p}\mathbf{I} = \mathbf{I}^*$$

$$\mathbf{E}^T\mathbf{E} = \mathbf{G}^T\mathbf{G} = \mathbf{p}^T\mathbf{p}\mathbf{I} - \mathbf{p}\mathbf{p}^T = \mathbf{I}^* - \mathbf{p}\mathbf{p}^T$$

$$\mathbf{G}(\mathbf{p}_i)\mathbf{p}_j = -\mathbf{G}(\mathbf{p}_j)\mathbf{p}_i \quad \mathbf{E}(\mathbf{p}_i)\mathbf{p}_j = -\mathbf{E}(\mathbf{p}_j)\mathbf{p}_i$$

$$\mathbf{E}(\mathbf{p}_i)\mathbf{G}^T(\mathbf{p}_j) = \mathbf{E}(\mathbf{p}_j)\mathbf{G}^T(\mathbf{p}_i)$$

$$\mathbf{G}(\mathbf{p}_i)\mathbf{E}^T(\mathbf{p}_j) = \mathbf{G}(\mathbf{p}_j)\mathbf{E}^T(\mathbf{p}_i)$$

$$\mathbf{R}(\mathbf{a}) \equiv \begin{bmatrix} \mathbf{0} & -\mathbf{a}^T \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \quad \mathbf{T}(\mathbf{a}) \equiv \begin{bmatrix} \mathbf{0} & -\mathbf{a}^T \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix}$$

$$\mathbf{R}(\mathbf{a})\mathbf{p} = \mathbf{E}^T(\mathbf{p})\mathbf{a} \quad \mathbf{T}(\mathbf{a})\mathbf{p} = \mathbf{G}^T(\mathbf{p})\mathbf{a}$$

Velocity and Acceleration

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \quad \dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{s}^P$$

$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\boldsymbol{\omega}}' \quad \tilde{\boldsymbol{\omega}}' = \mathbf{A}^T\dot{\mathbf{A}}$$

$$\boldsymbol{\omega}' = 2\mathbf{G}\dot{\mathbf{p}} \quad \dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\boldsymbol{\omega}' = \tfrac{1}{2}\mathbf{E}^T\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{G}\ddot{\mathbf{p}} \quad \ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\dot{\boldsymbol{\omega}}' - \tfrac{1}{4}\boldsymbol{\omega}'^T\boldsymbol{\omega}'\mathbf{p}$$

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{A}}\mathbf{s}'^P = \ddot{\mathbf{r}} + \left(\mathbf{A}\tilde{\dot{\boldsymbol{\omega}}}' + \mathbf{A}\tilde{\boldsymbol{\omega}}'\tilde{\boldsymbol{\omega}}'\right)\mathbf{s}'^P$$

Virtual Displacement and Rotation

$$\delta\tilde{\boldsymbol{\pi}}' = \mathbf{A}^T\delta\mathbf{A} \quad \delta\boldsymbol{\pi}' = \mathbf{A}^T\delta\boldsymbol{\pi} \quad \delta\mathbf{A} = \mathbf{A}\delta\tilde{\boldsymbol{\pi}}' = \delta\tilde{\boldsymbol{\pi}}\mathbf{A}$$

$$\delta\mathbf{r}^P = \delta\mathbf{r} + \delta\mathbf{A}\mathbf{s}'^P = \delta\mathbf{r} + \mathbf{A}\delta\tilde{\boldsymbol{\pi}}'\mathbf{s}'^P$$

$$\delta\boldsymbol{\pi}' = 2\mathbf{G}\delta\mathbf{p} \quad \delta\mathbf{p} = \tfrac{1}{2}\mathbf{G}^T\delta\boldsymbol{\pi}'$$

Derivative Identities

$\mathbf{a}, \mathbf{a}', \mathbf{b} \in R^3 \quad \mathbf{p}, \boldsymbol{\gamma} \in R^4$, *not normalized*

$$\left(\mathbf{E}(\mathbf{p})\boldsymbol{\gamma}\right)_{\mathbf{p}} = -\mathbf{E}(\boldsymbol{\gamma}) \quad \left(\mathbf{E}^T(\mathbf{p})\mathbf{a}\right)_{\mathbf{p}} = \mathbf{R}(\mathbf{a})$$

$$\left(\mathbf{A}(\mathbf{p})\mathbf{a}'\right)_{\mathbf{p}} = \mathbf{B}(\mathbf{p},\mathbf{a}')$$

$$\equiv 2\begin{bmatrix} (e_0\mathbf{I} + \tilde{\mathbf{e}})\mathbf{a}' & \mathbf{e}\mathbf{a}'^T - (e_0\mathbf{I} + \tilde{\mathbf{e}})\tilde{\mathbf{a}}' \end{bmatrix}$$

$$\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j = \mathbf{B}(\mathbf{p}_j,\mathbf{a}')\mathbf{p}_i \quad \left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j\right)_{\mathbf{p}_i} = \mathbf{B}(\mathbf{p}_j,\mathbf{a}')$$

$$\mathbf{B}^T(\mathbf{p},\mathbf{a}')\mathbf{b} = \mathbf{K}(\mathbf{a}',\mathbf{b})\mathbf{p} \quad \left(\mathbf{B}^T(\mathbf{p},\mathbf{a}')\mathbf{b}\right)_{\mathbf{p}} = \mathbf{K}(\mathbf{a}',\mathbf{b})$$

$$\mathbf{K}(\mathbf{a}',\mathbf{b}) \equiv 2\begin{bmatrix} \mathbf{a}'^T\mathbf{b} & \mathbf{a}'^T\tilde{\mathbf{b}} \\ \tilde{\mathbf{a}}'\mathbf{b} & \mathbf{a}'\mathbf{b}^T + \mathbf{b}\mathbf{a}'^T - \mathbf{a}'^T\mathbf{b}\mathbf{I} \end{bmatrix}$$

$$\left(\mathbf{G}(\mathbf{p})\boldsymbol{\gamma}\right)_{\mathbf{p}} = -\mathbf{G}(\boldsymbol{\gamma}) \quad \left(\mathbf{G}^T(\mathbf{p})\mathbf{a}\right)_{\mathbf{p}} = \mathbf{T}(\mathbf{a})$$

$$\left(\mathbf{A}^T(\mathbf{p})\mathbf{a}\right)_{\mathbf{p}} = \mathbf{C}(\mathbf{p},\mathbf{a})$$

$$\equiv 2\begin{bmatrix} (e_0\mathbf{I} - \tilde{\mathbf{e}})\mathbf{a} & \mathbf{e}\mathbf{a}^T + (e_0\mathbf{I} - \tilde{\mathbf{e}})\tilde{\mathbf{a}} \end{bmatrix}$$

$$\mathbf{C}(\mathbf{p}_i,\mathbf{a})\mathbf{p}_j = \mathbf{C}(\mathbf{p}_j,\mathbf{a})\mathbf{p}_i \quad \left(\mathbf{C}(\mathbf{p}_i,\mathbf{a})\mathbf{p}_j\right)_{\mathbf{p}_i} = \mathbf{C}(\mathbf{p}_j,\mathbf{a})$$

$$\mathbf{C}(\mathbf{p},\mathbf{a})^T\mathbf{b} = \mathbf{L}(\mathbf{a},\mathbf{b})\mathbf{p} \quad \left(\mathbf{C}(\mathbf{p},\mathbf{a})^T\mathbf{b}\right)_{\mathbf{p}} = \mathbf{L}(\mathbf{a},\mathbf{b})$$

$$\mathbf{L}(\mathbf{a},\mathbf{b}) \equiv 2\begin{bmatrix} \mathbf{a}^T\mathbf{b} & -\mathbf{a}^T\tilde{\mathbf{b}} \\ -\tilde{\mathbf{a}}\mathbf{b} & \mathbf{a}\mathbf{b}^T + \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I} \end{bmatrix}$$

$$\left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i')\mathbf{p}_j\right)_{\mathbf{a}_i'} \equiv \mathbf{N}(\mathbf{p}_i,\mathbf{p}_j) = 2\left\{\mathbf{E}(\mathbf{p}_i)\mathbf{G}^T(\mathbf{p}_j)\right\}$$

$$\left(\mathbf{C}(\mathbf{p}_i,\mathbf{a}_i)\mathbf{p}_j\right)_{\mathbf{a}_i} = \mathbf{N}^T(\mathbf{p}_i,\mathbf{p}_j) \quad \mathbf{N}(\mathbf{p}_i,\mathbf{p}_j) = \mathbf{N}(\mathbf{p}_j,\mathbf{p}_i)$$

$$\left(\mathbf{B}^T(\mathbf{p},\mathbf{a}')\mathbf{b}\right)_{\mathbf{a}'} = \mathbf{C}^T(\mathbf{p},\mathbf{b}) \quad \left(\mathbf{C}^T(\mathbf{p},\mathbf{b})\mathbf{a}'\right)_{\mathbf{b}} = \mathbf{B}^T(\mathbf{p},\mathbf{a}')$$