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- Starting point – the virtual work expression:

$$\begin{aligned} \delta W &= \sum_{i=1}^{nb} [\, \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r\right) \\ &+ \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r\right) \,] = 0 \end{aligned}$$

- Recall that this is valid for any set of arbitrary displacements $(\delta \mathbf{r}_1, \delta \bar{\pi}_1)$, $(\delta \mathbf{r}_2, \delta \bar{\pi}_2)$,. . ., $(\delta \mathbf{r}_{nb}, \delta \bar{\pi}_{nb})$
- We saw that

$$\mathbf{F}_i^r = -\mathbf{\Phi}_{\mathbf{r}_i}^T \lambda \qquad \bar{\mathbf{n}}_i^r = -\bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda$$

- Rewrite the expression of the virtual work as

$$\begin{aligned} \delta W &= \sum_{i=1}^{nb} [\, \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a - \bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda\right) \,] = 0 \end{aligned}$$

+ +

- Recall this basic thing: the expression of the virtual work on the previous slide represents the work done by a collection of forces when the bodies were nudged a bit and their orientation was changed a bit (as reflected by a change $\delta \mathbf{A}_i$ in the orientation matrix)
- Before, we chose to capture this virtual work using the $\delta \mathbf{r}$–$\delta \bar{\pi}$ representation of the virtual displacements. In what follows we'll express this underlying set of virtual displacements in terms of $\delta \mathbf{r}$–$\delta \mathbf{p}$
- Question: So what is then the set of perturbations $\delta \mathbf{p}_i, i = 1, \ldots, nb$ that we need to apply so that we get the same change in orientations as was the case when we used $\delta \bar{\pi}$?
  - The reason this question is relevant is that we don't want to change anything in the expression of the virtual work. We have the same force and the same virtual displacements applied. Therefore, if the virtual work was zero before, it continues to be zero now in spite of using different generalized coordinates

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Slide with some useful identities:

- Recall that we proved when dealing with the Kinematics component of the course that for each body i

$$\bar{\omega}_i = 2\mathbf{G}_i\dot{\mathbf{p}}_i$$

- From identity above, for each body i, by taking the time derivative of both sides of the equal sign leads to (part of your HW)

$$\dot{\bar{\omega}}_i = 2\mathbf{G}_i\ddot{\mathbf{p}}_i$$

- We showed before that given a virtual rotation characterized by $\delta\bar{\pi}_i$, the $\delta\mathbf{p}_i$ that leads to the same change in orientation; i.e., the *equivalent* $\delta\mathbf{p}_i$, is

$$\delta\mathbf{p}_i = \frac{1}{2}\mathbf{G}_i^T\delta\bar{\pi}_i$$

- Recall that we also have the reciprocal of this: if a virtual change of orientation is characterized by $\delta\mathbf{p}_i$ (recall that we must have $\mathbf{p}_i^T\delta\mathbf{p}_i = 0$ for a healthy $\delta\mathbf{p}_i$), then the *equivalent* $\delta\bar{\pi}$ is

$$\delta\bar{\pi}_i = 2\mathbf{G}_i\delta\mathbf{p}_i$$

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- Question: How does the fact that $\delta\bar{\pi}_i$ and $\delta\mathbf{p}_i$ are equivalent manifest itself?

- Answer: If they are equivalent, it means that for any function $\mathbf{u}$ (assumed here for no particular reason to be a vector) that depends on position and orientation $\mathbf{u}(\mathbf{r}_i, \mathbf{A}_i)$, its variation in response to the virtual displacement $\delta\mathbf{r}_i$–$\delta\bar{\pi}_i$, it's going to be identical to its variation in response to the virtual displacement $\delta\mathbf{r}_i$–$\delta\mathbf{p}_i$.

- Translation, of the above statement in math terms:

$$\left.\begin{array}{rcl} \delta\mathbf{u}(\mathbf{r}_i, \mathbf{A}_i) & = & \mathbf{u}_{\mathbf{r}_i}\delta\mathbf{r}_i + \bar{\mathbf{\Pi}}_i(\mathbf{u})\delta\bar{\pi}_i \\ \delta\mathbf{u}(\mathbf{r}_i, \mathbf{A}_i) & = & \mathbf{u}_{\mathbf{r}_i}\delta\mathbf{r}_i + \mathbf{u}_{\mathbf{p}_i}\delta\mathbf{p}_i \end{array}\right\} \quad \Rightarrow \quad \bar{\mathbf{\Pi}}_i(\mathbf{u})\delta\bar{\pi}_i = \mathbf{u}_{\mathbf{p}_i}\delta\mathbf{p}_i$$

- To conclude, if $\delta\mathbf{r}_i$–$\delta\bar{\pi}_i$ are *equivalent*, then for any function $\mathbf{u}$ that depends on the position and orientation of body i one has that

$$\bar{\mathbf{\Pi}}_i(\mathbf{u})\delta\bar{\pi}_i = \mathbf{u}_{\mathbf{p}_i}\delta\mathbf{p}_i$$

- Remark: Assume that two sets of virtual displacements, one expressed in the $\delta\mathbf{r}_i$–$\delta\bar{\pi}_i$ form, while the other expressed in the $\delta\mathbf{r}_i$–$\delta\mathbf{p}_i$ form, are equivalent. Then, if one of them is consistent with a set of constraints $\mathbf{\Phi}$ it follows that the other one is consistent as well (since the variation $\delta\mathbf{\Phi}$ is the same in both cases)

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- Restate the expression of the virtual work, single out though the virtual work done by the reaction torques:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} [\ \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a\right) - \delta \bar{\pi}_i^T \bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda\ ] = 0\end{aligned}$$

- Note that since we now we account for the expression of the reaction force/torque, the condition above holds for arbitrary $\delta \mathbf{r}_i$–$\delta \bar{\pi}_i$
- The basic idea is simple: wherever you see $\delta \bar{\pi}_i$ in the expression of the virtual work of few slides ago, replace it with $2\mathbf{G}_i \delta \mathbf{p}_i$; wherever you see $\dot{\bar{\omega}}_i$, replace it with $2\mathbf{G}_i \ddot{\mathbf{p}}_i$. Additionally, when dealing with the virtual work of the reaction torques, recall the identity in red on the previous slide. We end up with the following:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} [\ \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \delta \mathbf{p}_i^T 2\mathbf{G}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i 2\mathbf{G}_i \ddot{\mathbf{p}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a\right) - \delta \mathbf{p}_i^T \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda\ ] = 0\end{aligned}$$

- In the expression above, the variations in Euler Parameters are arbitrary as long as they are healthy; i.e.,

$$\mathbf{p}_i^T \delta \mathbf{p}_i = 0 \qquad i = 1, \ldots, nb \qquad \text{i.e.,} \qquad \mathbf{\Phi}_{\mathbf{p}}^{\mathbf{p}} \delta \mathbf{p} = \mathbf{0}_{nb}$$

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- Concentrate on the $\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i$ term
- Recall that (Haug's book, pp. 345 Eq. 9.3.33 and Eq. 9.3.24 and 9.3.29):

$$\tilde{\bar{\omega}}_i = 2\mathbf{G}_i \dot{\mathbf{G}}_i^T \qquad \mathbf{G}_i^T \mathbf{G}_i = \mathbf{I}_4 - \mathbf{p}_i \mathbf{p}_i^T \qquad \mathbf{G}_i^T \dot{\mathbf{p}}_i = -\dot{\mathbf{G}}_i^T \mathbf{p}_i$$

- Then,

$$\begin{aligned}&\delta \mathbf{p}_i^T 2\mathbf{G}_i^T \left(\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i\right) = 8\delta \mathbf{p}_i^T\ \mathbf{G}_i^T \mathbf{G}_i \dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i^T \dot{\mathbf{p}}_i = 8\delta \mathbf{p}_i^T\ (\mathbf{I}_4 - \mathbf{p}_i \mathbf{p}_i^T) \dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i^T \dot{\mathbf{p}}_i \\ = \ &8\delta \mathbf{p}_i^T \dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i^T \dot{\mathbf{p}}_i - 8\delta \mathbf{p}_i^T \mathbf{p}_i \mathbf{p}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i^T \dot{\mathbf{p}}_i = -8\delta \mathbf{p}_i^T\ \dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \dot{\mathbf{G}}_i^T \mathbf{p}_i\end{aligned}$$

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- Expression of virtual work becomes:

$$\begin{aligned}\delta W &= \sum_{i=1}^{nb} [\, \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda\right) \\ &+ \; \delta \mathbf{p}_i^T \left(8\dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \dot{\mathbf{G}}_i^T \mathbf{p}_i - 4\mathbf{G}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i \ddot{\mathbf{p}}_i + 2\mathbf{G}_i^T \bar{\mathbf{n}}_i^m + 2\mathbf{G}_i^T \bar{\mathbf{n}}_i^a - \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda\right) \,] = 0\end{aligned}$$

- Expression above should hold as long as $\delta \mathbf{p}_i^T \mathbf{p}_i = 0$, $i = 1, \ldots, nb$
- Using like we did before Lagrange's Multiplier theorem, it means that there is one additional set of nb Lagrange Multipliers, called $\lambda_1^{\mathbf{p}}, \ldots, \lambda_{nb}^{\mathbf{p}}$ so that

$$\left.\begin{aligned}-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a - \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda &= \mathbf{0}_{3\times 1} \\ 8\dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \dot{\mathbf{G}}_i^T \mathbf{p}_i - 4\mathbf{G}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i \ddot{\mathbf{p}}_i + 2\mathbf{G}_i^T \bar{\mathbf{n}}_i^m + 2\mathbf{G}_i^T \bar{\mathbf{n}}_i^a - \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda - \mathbf{p}_i \lambda_i^{\mathbf{p}} &= \mathbf{0}_{4\times 1}\end{aligned}\right\} \quad i = 1, \ldots, nb$$

+++

- Use the following notation to clean up the relations above:

$$\mathbf{F}_i \equiv \mathbf{F}_i^m + \mathbf{F}_i^a \qquad\qquad \hat{\tau}_i \equiv 2\mathbf{G}_i^T (\bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a) + 8\dot{\mathbf{G}}_i^T \bar{\mathbf{J}}_i \dot{\mathbf{G}}_i^T \mathbf{p}_i$$

- Then, the matrix-free form of the EOM in the $\mathbf{r}$–$\mathbf{p}$ formulation are

$$\left.\begin{aligned}m_i \ddot{\mathbf{r}}_i + \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda &= \mathbf{F}_i \\ 4\mathbf{G}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i \ddot{\mathbf{p}}_i + \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda + \mathbf{p}_i \lambda_i^{\mathbf{p}} &= \hat{\tau}_i\end{aligned}\right\} \quad \text{for} \quad i = 1, \ldots, nb$$

- Above, we have a set of $3nb$ second order Ordinary Differential Equations (ODEs) associated the translational degrees of freedom and $4nb$ second order Ordinary Differential Equations associated with the rotational degrees of freedom
- Note that $\lambda_i^{\mathbf{p}}$ is a scalar, while λ is a vector having nc components (the number of rows in **Phi**).

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- We will use the following matrix notation (same has already been introduced in conjunction with the $\mathbf{r}$–ω formulation of the EOM)

$$\mathbf{M} = \begin{bmatrix} m_1 \mathbf{I}_3 & \mathbf{0}_{3\times 3} & \ldots & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & m_2 \mathbf{I}_3 & \ldots & \mathbf{0}_{3\times 3} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \ldots & m_{nb} \mathbf{I}_3 \end{bmatrix}_{3nb\times 3nb} \qquad \mathbf{J}^{\mathbf{p}} = \begin{bmatrix} 4\mathbf{G}_1^T \bar{\mathbf{J}}_1 \mathbf{G}_1 & \mathbf{0}_{4\times 4} & \ldots & \mathbf{0}_{4\times 4} \\ \mathbf{0}_{4\times 4} & 4\mathbf{G}_1^T \bar{\mathbf{J}}_1 \mathbf{G}_1 & \ldots & \mathbf{0}_{4\times 4} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \ldots & 4\mathbf{G}_{nb}^T \bar{\mathbf{J}}_{nb} \mathbf{G}_{nb} \end{bmatrix}_{4}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^T & \mathbf{0}_{1\times 4} & \cdots & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{1\times 4} & \mathbf{p}_2^T & \cdots & \mathbf{0}_{1\times 4} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 4} & \mathbf{p}_{nb}^T \end{bmatrix}_{nb\times 4nb} \qquad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{nb} \end{bmatrix}_{3nb} \qquad \hat{\tau} = \begin{bmatrix} \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_{nb} \end{bmatrix}_{4nb}$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix} \qquad \lambda^{\mathbf{p}} = \begin{bmatrix} \lambda_1^{\mathbf{p}} \\ \vdots \\ \lambda_{nb}^{\mathbf{p}} \end{bmatrix} \qquad \gamma^{\mathbf{p}} = \begin{bmatrix} -2\dot{\mathbf{p}}_1^T \dot{\mathbf{p}}_1 \\ \cdots \\ -2\dot{\mathbf{p}}_{nb}^T \dot{\mathbf{p}}_{nb} \end{bmatrix}$$

- For the matrices defined above (not the vectors), notice the very large number of zeros. This matrices are sparse, typically less then 3% are nonzero entries.
- Moreover, they have a certain sparsity pattern, which is taken advantage of in commercial software in order to reduce their footprint and speed up computation

+++

- The set of matrix-free representation of the EOM of a few slides ago assumes the following matrix-vector form:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T \lambda &= \mathbf{F} \\ \mathbf{J}^{\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{\Phi}_{\mathbf{p}}^T \lambda + \mathbf{P}^T \lambda^{\mathbf{p}} &= \hat{\tau} \end{aligned}$$

- Counting the equations and unknowns:
  - The number of equations: $7nb$
  - The number of unknowns in those equations $3nb$ for $\ddot{\mathbf{r}}$, $4nb$ for $\ddot{\mathbf{p}}$, nc for λ, nb for $\lambda^{\mathbf{p}} \quad \Rightarrow \quad 8nb + nc$ unknowns
  - We have more unknowns than equation...
  - Recall that in the $\mathbf{r}$–$\bar{\omega}$ formulation we had $6nb + nc$ unknowns.

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- Recall that what we did in the $\mathbf{r}$–$\bar{\omega}$ formulation of the EOM to address the imbalance between the number of equations and unknowns
  - We added to the above EOM the set of acceleration *kinematic* constraint equations

$$\mathbf{\Phi_r}\ddot{\mathbf{r}} + \mathbf{\Phi_p}\ddot{\mathbf{p}} = \hat{\gamma} \qquad \Rightarrow \qquad nc \text{ more equations.}$$

- Moreover, in the **r**–**p** formulation, beyond the set of acceleration *kinematic* constraint equations we have a set of Euler Parameter normalization acceleration constraint equations. In matrix form,

$$\mathbf{P}\ddot{\mathbf{p}} = \gamma^{\mathbf{P}} \qquad \Rightarrow \qquad nb \text{ more equations.}$$

- At this point we have as many unknowns as equations. The unknowns: $\ddot{\mathbf{r}}$, $\ddot{\mathbf{p}}$, λ, and $\lambda^{\mathbf{P}}$ are obtained as the solution of a *linear* system of equations:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 4nb} & \mathbf{0}_{3nb\times nb} & \mathbf{\Phi_r}^T \\ \mathbf{0}_{4nb\times 3nb} & \mathbf{J^P} & \mathbf{P}^T & \mathbf{\Phi_p}^T \\ \mathbf{0}_{nb\times 3nb} & \mathbf{P} & \mathbf{0}_{nb\times nb} & \mathbf{0}_{nb\times nc} \\ \mathbf{\Phi_r} & \mathbf{\Phi_p} & \mathbf{0}_{nc\times nb} & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \lambda^{\mathbf{P}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\tau} \\ \gamma^{\mathbf{P}} \\ \hat{\gamma} \end{bmatrix}$$

- Color code: BLUE for known quantities, RED for unknown quantities

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IMPORTANT OBSERVATION:

- If you follow the derivation of the EOM, you'll notice that the Lagrange Multiplier λ in the equation on the previous slide and the Lagrange Multiplier λ in the **r**–$\bar{\omega}$ formulation assume the same value.

- In other words, whether you solve the linear system on the previous slide or the liner system that provides $\ddot{\mathbf{r}}$, $\dot{\bar{\omega}}$, and λ (discussed last lecture), the value of λ is going to be the same in both cases

- Meaning: the Lagrange Multiplier λ is an attribute of the system, more precisely of the set of constraints present in the system. The way the constraint makes its presence felt depends on the set of generalized coordinates used to capture the dynamics of the system.

  - The Lagrange Multiplier λ shows up in the **r**–$\bar{\omega}$ formulation of the EOM as $-\bar{\mathbf{\Pi}}^T(\mathbf{\Phi})\lambda$ to manifest the effect of the constraints upon the EOM governing the rotational motion
  - The Lagrange Multiplier λ shows up in the **r**–**p** formulation of the EOM as $-\mathbf{\Phi_p}^T\lambda$ to manifest the effect of the constraints upon the EOM governing the rotational motion

– Finally, in both the $\mathbf{r}$–$\bar{\omega}$ and $\mathbf{r}$–$\mathbf{p}$ formulations, λ shows up in the EOM as $-\mathbf{\Phi}_{\mathbf{r}}^T \lambda$ to manifest the effect of the constraints upon the EOM governing the translational generalized coordinates

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On the relevance of the Friction Forces

- ANY friction force shall be accounted for as being an external force
- If you have friction in a joint, you should NOT expect for the reaction force in that joint to capture the net effect of the friction
- This is because a joint is a union of GCons, which impose *geometric* constraints associated with the relative motion of bodies
  - Again, they impose geometric constraints, not the fact that there is friction between two bodies
  - Friction doesnt have to do anything with the geometry of the motion; in fact, its governed by different physics (all the way down to the atomic level)
  - When we used the fact that the virtual work of the reaction forces is zero we ignored any friction force that could appear in a joint, since their virtual work is *not* zero
- If you have friction in a system, you will have to come up with a friction model (such as Coulomb, or LuGre)
  - In about three weeks well talk about the Coulomb friction model

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Comment on the matrix-free and matrix-form formulation of the EOM

- Matrix-free: dealing with one body at a time, a 'microscopic' perspective

$$\left.\begin{array}{lcl} m_i \ddot{\mathbf{r}}_i + \mathbf{\Phi}_{\mathbf{r}_i}^T \lambda & = & \mathbf{F}_i \\ 4\mathbf{G}_i^T \bar{\mathbf{J}}_i \mathbf{G}_i \ddot{\mathbf{p}}_i + \mathbf{\Phi}_{\mathbf{p}_i}^T \lambda + \mathbf{p}_i \lambda_i^{\mathbf{p}} & = & \hat{\tau}_i \end{array}\right\} \quad \text{for} \quad i = 1, \ldots, nb$$

- The matrix-vector form of the EOM: aggregation of the collection of EOM associated with bodies $i = 1, \ldots, nb$:

$$\left\{\begin{array}{lcl} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T \lambda & = & \mathbf{F} \\ \mathbf{J}^{\mathbf{p}} \ddot{\mathbf{p}} + \mathbf{\Phi}_{\mathbf{p}}^T \lambda + \mathbf{P}^T \lambda^{\mathbf{p}} & = & \hat{\tau} \end{array}\right.$$

- The matrix-free and matrix-form formulations of the EOM capture the same thing (the second order ODE governing the time evolution of a collection of rigid bodies interconnected through geometric constraints). The matrix-form is useful when it's coupled with the Acceleration Kinematic Constraint Equations to provide the big *linear* system whose solution provides the accelerations and Lagrange multipliers:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 4nb} & \mathbf{0}_{3nb\times nb} & \mathbf{\Phi}_{\mathbf{r}}^T \\ \mathbf{0}_{4nb\times 3nb} & \mathbf{J^P} & \mathbf{P}^T & \mathbf{\Phi}_{\mathbf{p}}^T \\ \mathbf{0}_{nb\times 3nb} & \mathbf{P} & \mathbf{0}_{nb\times nb} & \mathbf{0}_{nb\times nc} \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_{\mathbf{p}} & \mathbf{0}_{nc\times nb} & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \lambda^{\mathbf{P}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\tau} \\ \gamma^{\mathbf{P}} \\ \hat{\gamma} \end{bmatrix}$$

+++

- Recall that the claim two slides ago was that the quantities in BLUE were known, while the quantities in RED were unknown and supposed to be computed as the solution of a linear system
- Note that in order to evaluate the blue quantities and pose the problem whose solution provides the red quantities; i.e., the accelerations $\ddot{\mathbf{q}}$ and Lagrange Multipliers λ and $\lambda^{\mathbf{P}}$, you need to have *position* and *velocity* information, $\mathbf{q}$ and $\dot{\mathbf{q}}$, respectively.
- In other words, in order to compute $\ddot{\mathbf{q}}$, λ and $\lambda^{\mathbf{P}}$, you need to have $\mathbf{q}$ and $\dot{\mathbf{q}}$. There is no surprise here, since the EOM represent a set of second order ODEs. You will see next week that when you are dealing with a second order ODE, the initial conditions (ICs) provided are the value of $\mathbf{q}$ and $\dot{\mathbf{q}}$, which therefore allows one to compute $\ddot{\mathbf{q}}$, λ, and $\lambda^{\mathbf{P}}$

+++

IMPORTANT OBSERVATION:

- Please keep in mind the following: γ used to denote the RHS of the acceleration equation in the $\mathbf{r}$–$\bar{\omega}$ formulation.
- The quantity $\hat{\gamma}$ is used to denote the RHS of the acceleration equation in the $\mathbf{r}$–$\mathbf{p}$ formulation.
- Using the notation $\mathbf{q} \equiv \begin{bmatrix} \mathbf{r} \\ \mathbf{p} \end{bmatrix}$, we have

$$\hat{\gamma} = -(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}$$

- Note that we have not provided yet $\hat{\gamma}$ for the four basic GCons

– Basic approach to computing $\hat{\gamma}^\alpha$, where $\alpha \in \{DP1, DP2, D, CD\}$: take two time derivatives of Φ^α; set the result equal to zero; move to the right side all quantities that do not depend on $\ddot{\mathbf{r}}$ and $\ddot{\mathbf{p}}$. The quantity on the RHS is your $\hat{\gamma}^\alpha$, the right side of the kinematic acceleration constraint equation

++

- First, recall that we introduced a matrix $\mathbf{B}$ as follows:

$$\frac{\partial[\mathbf{A}(\mathbf{p}) \cdot \bar{\mathbf{s}}]}{\partial \mathbf{p}} \equiv \mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})$$

- Some helpful identities:

$$\dot{\mathbf{B}}(\mathbf{p}, \bar{\mathbf{s}}) = \mathbf{B}(\dot{\mathbf{p}}, \bar{\mathbf{s}}) \quad \longrightarrow \quad \text{due to the linearity of the } \mathbf{B}(\mathbf{p}, \bar{\mathbf{s}}) \text{ matrix in relation to the variable } \mathbf{p}$$

$$\frac{d[\mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})\dot{\mathbf{p}}]}{d\,t} = \mathbf{B}(\dot{\mathbf{p}}, \bar{\mathbf{s}})\dot{\mathbf{p}} + \mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})\ddot{\mathbf{p}}$$

$$\mathbf{a}_i = \mathbf{A}_i \bar{\mathbf{a}}_i \quad \Rightarrow \quad \dot{\mathbf{a}}_i = \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i \quad \Rightarrow \quad \ddot{\mathbf{a}}_i = \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i + \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i)\ddot{\mathbf{p}}_i$$

$$\dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \dot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

$$\ddot{\mathbf{d}}_{ij} = \ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\ddot{\mathbf{p}}_j + \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P)\ddot{\mathbf{p}}_i - \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

++
HOMEWORK:

- For the four basic GCons, prove that $\hat{\gamma}$ assumes the values indicated below:

$$\begin{aligned}
\Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) &= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{a}}_j - f(t) = 0 \\
\hat{\gamma}^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, f(t)) &= -\dot{\mathbf{a}}_i^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{a}}_j)\dot{\mathbf{p}}_j - \dot{\mathbf{a}}_j^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i - 2\dot{\mathbf{a}}_i^T \dot{\mathbf{a}}_j + \ddot{f}(t)
\end{aligned}$$

$$\begin{aligned}
\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T (\mathbf{r}_j + \mathbf{A}_j \bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i \bar{\mathbf{s}}_i^P) - f(t) = 0 \\
\hat{\gamma}^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) &= -\mathbf{a}_i^T \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + \mathbf{a}_i^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - \dot{\mathbf{d}}_{ij}^T \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i - 2\dot{\mathbf{a}}_i^T \dot{\mathbf{d}}_{ij} + \ddot{f}(t)
\end{aligned}$$

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{d}_{ij}^T\mathbf{d}_{ij} - f(t) = (\mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P)^T(\mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P) - f(t) = 0$$

$$\hat{\gamma}^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = -2\mathbf{d}_{ij}^T\mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + 2\mathbf{d}_{ij}^T\mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - 2\dot{\mathbf{d}}_{ij}^T\dot{\mathbf{d}}_{ij} + \ddot{f}(t)$$

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T\mathbf{d}_{ij} - f(t) = \mathbf{c}^T(\mathbf{r}_j + \mathbf{A}_j\bar{\mathbf{s}}_j^Q - \mathbf{r}_i - \mathbf{A}_i\bar{\mathbf{s}}_i^P) - f(t) = 0$$

$$\hat{\gamma}^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T\mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i - \mathbf{c}^T\mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j + \ddot{f}(t)$$

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BONUS HOMEWORK PROBLEM

- Follow the approach used to get the EOM in the $\mathbf{r}$–$\mathbf{p}$ to produce the EOM for the $\mathbf{r}$–ϵ formulation, where $\epsilon_i = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}_i$ denotes the set of three Euler Angles used to capture the orientation of body i in 3D space. Specifically, start from the expression of the virtual work to obtain the following second order ODEs:

$$\mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_{\mathbf{r}}^T\lambda = \mathbf{F}$$
$$\mathbf{J}^\epsilon\ddot{\epsilon} + \mathbf{\Phi}_{\epsilon}^T\lambda = \breve{\tau}$$

- Given that $\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{nb} \end{bmatrix}_{3nb}$, you will have to obtain the ODEs above and the expression of $\mathbf{J}^\epsilon$ and $\breve{\tau}$

- If you take upon this challenge, you'll have to recall that early on in the semester when we discussed about Euler Angles we showed that there is a [almost everywhere] nonsingular matrix $\mathbf{B}(\epsilon)$ so that (we'll denote by $\mathbf{D} = \mathbf{A}^T\mathbf{B}$)

$$\omega = \mathbf{B}\dot{\epsilon} \qquad \bar{\omega} = \mathbf{D}\dot{\epsilon}$$

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- Note that the EOMs on the previous slide is what ADAMS uses
- Just like before, the ODEs above are augmented with the acceleration kinematic constraint equation $\mathbf{\Phi}_{\mathbf{r}}\ddot{\mathbf{r}} + \mathbf{\Phi}_\epsilon\ddot{\epsilon} = \breve{\gamma}$ to obtain:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 3nb} & \mathbf{\Phi}_{\mathbf{r}}^T \\ \mathbf{0}_{3nb\times 3nb} & \mathbf{J}^\epsilon & \mathbf{\Phi}_\epsilon^T \\ \mathbf{\Phi}_{\mathbf{r}} & \mathbf{\Phi}_\epsilon & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\epsilon} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \breve{\tau} \\ \breve{\gamma} \end{bmatrix}$$

- You would have to determine now for the four basic GCons the expression of $\breve{\gamma}$
- Not going to pursue this since
  - Euler Angles have a nasty singularity that you can run into relatively easily
  - The ingredients that enter the EOMs and kinematic acceleration constraint equations, such as $\mathbf{J}^{\epsilon}$, $\mathbf{\Phi}_{\epsilon}$, etc., are very cumbersome and computationally expensive to evaluate
- Note: I uploaded on the class website a PDF that describes in detail the formulation of the EOM and their numerical solution
  - See link 'ADAMS/Solver Primer' (2004 vintage document)

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