A line of + signs marks the beginning of a new slide. Material below almost identical to what was presented in the PPTX-based lecture.

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- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$
- Expression of the force:
$$-\ddot{\mathbf{r}}_i^P \; dm_i(P)$$
- Virtual work produced:
$$[\delta \mathbf{r}_i^P]^T \cdot [-\ddot{\mathbf{r}}_i^P \; dm_i(P)]$$
- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:
$$\int\limits_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P)$$

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- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. A force per unit mass, $\mathbf{f}_i(P)$, is assumed to act at point P.
- Expression of the force:
$$\mathbf{f}_i(P) \; dm_i(P)$$
- Virtual work produced:
$$[\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P) \; dm_i(P)$$
- Comments:
  - The total virtual work produced by this type of force is obtained by summing over all points of body i:
$$\int\limits_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P) \; dm_i(P)$$
  - This type of force is rarely seen in classical multibody dynamics. Exception: the force due to the gravitational field, which leads to the weight of the body. In this case $\mathbf{f}_i(P) = \mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration of magnitude $g \approx 9.81 \frac{m}{s^2}$ (in Madison, WI).

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- Framework: we are considering a point P of body i. This point is associated with an infinitesimal mass element $dm_i(P)$. We also consider an *arbitrary* point R on body i. The focus is on the **internal force** acting between the mass elements $dm_i(P)$ and $dm_i(R)$.

- The expression of this type of force acting at point P is obtained by considering the contribution of each point R of the body:

$$\int_{m_i} \mathbf{f}_i(P,R)\; dm_i(R)$$

- Virtual work produced:

$$[\delta \mathbf{r}_i^P]^T \cdot \int_{m_i} \mathbf{f}_i(P,R)\; dm_i(R)$$

- Comments:
  - The total virtual work produced by this type of force when acting at all points of body i:

$$\int_{m_i} [\delta \mathbf{r}_i^P]^T \left[\int_{m_i} \mathbf{f}_i(P,R)\; dm_i(R) \right] \; dm_i(P) = \int_{m_i} \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P,R)\; dm_i(R)\; dm_i(P)$$

  - The assumption that we make is that the force $\mathbf{f}_i(P,R)$ acts along the line connecting points P and R. In other words, $\mathbf{f}_i(P,R) dm_i(R) = k(\mathbf{r}_i^P - \mathbf{r}_i^R)$, where k is a scalar that might depend on the points P and R.

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- Framework: We assume that a set of constraints acts on body i. These constraints most often lead to the presence of reaction forces. We will assume that the constraints on body i are producing reaction (constraint) forces acting at a collection of points generically denoted by $\mathcal{Q}_i$.

- Expression of this type of force acting at point $Q \in \mathcal{Q}_i$:

$$\mathbf{F}_Q^r$$

- Virtual work produced by this set of forces:

$$\sum_{Q \in \mathcal{Q}_i} [\delta \mathbf{r}_i^Q]^T \cdot \mathbf{F}_Q^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction forces $\mathbf{F}_Q^r$ for $Q \in \mathcal{Q}_i$
  - A separate discussion will follow on the meaning of the points $\mathcal{Q}_i$

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- Framework: We assume that a set of constraints acts on body i. These constraints most often lead to the presence of reaction torques. We will assume that the constraints on body i are producing reaction (constraint) torques acting at a collection of points generically denoted by $\mathcal{Z}_i$.
- Expression of this type of torque acting at point $Z \in \mathcal{Z}_i$ when represented in the L-RF:

$$\bar{\mathbf{n}}_Z^r$$

- Virtual work produced by these reaction torques:

$$\sum_{Z \in \mathcal{Z}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_Z^r$$

- Comments:
  - One of the outcomes of solving the EOM will be to compute the value of the reaction torques $\bar{\mathbf{n}}_Z^r$ for $Z \in \mathcal{Z}_i$
  - Note that since we are talking about *rigid* bodies, we have the same virtual rotation $\delta \bar{\pi}_i$ no matter what point $Z \in \mathcal{Z}_i$ of the rigid body we are dealing with

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- Framework: We assume that a set of active forces acts on body i. These active forces are acting at a collection of points generically denoted by $\mathcal{U}_i$.
- Expression of this type of force acting at point $U \in \mathcal{U}_i$:

$$\mathbf{F}_U^a$$

- Virtual work produced by this set of forces:

$$\sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}_U^a$$

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- Framework: We assume that a set of active torques acts on body i. We will assume that these active torques are acting at a collection of points generically denoted by $\mathcal{V}_i$.
- Expression of this type of torque acting at point $V \in \mathcal{V}_i$, expressed in the L-RF$_i$:

$$\bar{\mathbf{n}}_V^a$$

- Virtual work produced by this set of torques:

$$\sum_{V \in \mathcal{V}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_V^a$$

- Comments: Note that since we are talking about *rigid* bodies, we have the same virtual rotation $\delta \bar{\pi}_i$ no matter which of the torques acting on the rigid body we are dealing with

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- We will choose the L-RF of each body so that it is a centroidal reference frame. In other words, the L-RF$_i$ is located at the center of mass of body i, for $i \in \{1, \ldots, nb\}$
- For a centroidal reference frame, by definition,

$$\int_{m_i} \bar{\mathbf{s}}^P \, dm_i(P) = \mathbf{0}_3$$

- The definition of the mass moment of inertia tensor:

$$\bar{\mathbf{J}} = \int_{m_i} -\tilde{\bar{\mathbf{s}}}^P \, \tilde{\bar{\mathbf{s}}}^P \, dm_i(P) = \begin{bmatrix} \bar{J}_{xx} & \bar{J}_{xy} & \bar{J}_{xz} \\ \bar{J}_{yx} & \bar{J}_{yy} & \bar{J}_{yz} \\ \bar{J}_{zx} & \bar{J}_{zy} & \bar{J}_{zz} \end{bmatrix}$$

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- The constant matrix $\bar{\mathbf{J}}$ represents an attribute that depends on the shape (geometry) of the body and the distribution of mass within that geometry
- Recall that a careful choice of the orientation of L-RF leads to this matrix $\bar{\mathbf{J}}$ being diagonal. When L-RF is chosen like that, it becomes a principal reference frame
- To conclude, to keep things simple yet without any loss of generality in terms of formulating the EOM, we will assume that for each body i we selected the L-RF$_i$ so that it is a centroidal and principal RF

- NOTE: When can't you assume to have a centroidal and principal RF? When (a) the body if flexible, or (b) when you are solving an optimization problem and the geometry (shape) of the body changes in response to the very optimization process

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- Principle of Virtual Work, applied for a collection of rigid bodies interconnected through an arbitrary collection of constraints:

$$\delta W = \sum_{i=1}^{nb} \Bigg[\int_{m_i} -[\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P) + \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P) \; dm_i(P)$$

$$+ \int_{m_i} \int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P, R) \; dm_i(R) \; dm_i(P) + \sum_{Q \in \mathcal{Q}_i} [\delta \mathbf{r}_i^Q]^T \cdot \mathbf{F}_Q^r + \sum_{Z \in \mathcal{Z}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_Z^r$$

$$+ \sum_{U \in \mathcal{U}_i} [\delta \mathbf{r}_i^U]^T \cdot \mathbf{F}_U^a + \sum_{V \in \mathcal{V}_i} [\delta \bar{\pi}_i]^T \cdot \bar{\mathbf{n}}_V^a \Bigg] = 0$$

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- Recall the expression of a virtual translation of a point P on body i as a result of a virtual displacement $\delta \mathbf{r}_i$ and $\delta \bar{\pi}_i$ of the L-RF$_i$

$$\delta \mathbf{r}_i^P = \delta \mathbf{r}_i - \mathbf{A}_i \tilde{\bar{\mathbf{s}}}_i^P \delta \bar{\pi}_i$$

$$\Downarrow$$

$$\left[\delta \mathbf{r}_i^P\right]^T = \delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T$$

- We'll need the acceleration of an arbitrary point P, obtained as:

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i \bar{\mathbf{s}}_i^P$$

$$\Downarrow$$

$$\dot{\mathbf{r}}_i^P = \dot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P$$

$$\Downarrow$$

$$\ddot{\mathbf{r}}_i^P = \ddot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P + \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \bar{\mathbf{s}}_i^P$$

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- Virtual work produced by the reaction forces and torques is

$$\sum_{Q\in\mathcal{Q}_i}\left[\delta\mathbf{r}_i^T + \delta\bar{\pi}_i^T\tilde{\bar{\mathbf{s}}}_i^Q\mathbf{A}_i^T\right]\cdot\mathbf{F}_Q^r + \sum_{Z\in\mathcal{Z}_i}\delta\bar{\pi}_i^T\cdot\bar{\mathbf{n}}_Z^r$$

$$= \delta\mathbf{r}_i^T\cdot\sum_{Q\in\mathcal{Q}_i}\mathbf{F}_Q^r + \delta\bar{\pi}_i^T\cdot\left[\sum_{Q\in\mathcal{Q}_i}\tilde{\bar{\mathbf{s}}}_i^Q\mathbf{A}_i^T\mathbf{F}_Q^r + \sum_{Z\in\mathcal{Z}_i}\bar{\mathbf{n}}_Z^r\right]$$

$$= \delta\mathbf{r}_i^T\mathbf{F}_i^r + \delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^r$$

- Notation used:
  - Total reaction force acting on body i:

$$\mathbf{F}_i^r = \sum_{Q\in\mathcal{Q}_i}\mathbf{F}_Q^r$$

  - Total reaction torque acting on body i:

$$\bar{\mathbf{n}}_i^r = \sum_{Q\in\mathcal{Q}_i}\tilde{\bar{\mathbf{s}}}_i^Q\mathbf{A}_i^T\mathbf{F}_Q^r + \sum_{Z\in\mathcal{Z}_i}\bar{\mathbf{n}}_Z^r$$

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- Virtual work produced by the active forces and torques is

$$\sum_{U\in\mathcal{U}_i}\left[\delta\mathbf{r}_i^T + \delta\bar{\pi}_i^T\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\right]\cdot\mathbf{F}_U^a + \sum_{V\in\mathcal{V}_i}\delta\bar{\pi}_i^T\cdot\bar{\mathbf{n}}_V^a$$

$$= \delta\mathbf{r}_i^T\cdot\sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a + \delta\bar{\pi}_i^T\cdot\left[\sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a + \sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a\right]$$

$$= \delta\mathbf{r}_i^T\mathbf{F}_i^a + \delta\bar{\pi}_i^T\bar{\mathbf{n}}_i^a$$

- Notation used:
  - Total active force acting on body i:

$$\mathbf{F}_i^a = \sum_{U\in\mathcal{U}_i}\mathbf{F}_U^a$$

  - Total active torque acting on body i:

$$\bar{\mathbf{n}}_i^a = \sum_{U\in\mathcal{U}_i}\tilde{\bar{\mathbf{s}}}_i^U\mathbf{A}_i^T\mathbf{F}_U^a + \sum_{V\in\mathcal{V}_i}\bar{\mathbf{n}}_V^a$$

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- Based on discussion at pp. 418 of Haug's book (see Eqs. 11.1.4, 11.1.5), the virtual work of the internal forces in a rigid body is zero:

$$\int_{m_i}\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P,R)\; dm_i(R)\; dm_i(P) = 0$$

- This goes back to the fact that (a) $\mathbf{f}_i(P,R)dm_i(R) = k(\mathbf{r}_i^P - \mathbf{r}_i^R)$, where k is a scalar that might depend on the points P and R, an assumption made a couple of slides ago, and (b) the body is rigid, from where $(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = \text{const.}$

$$(\mathbf{r}^P - \mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = \text{const.} \qquad \Rightarrow \qquad (\delta\mathbf{r}^P - \delta\mathbf{r}^R)^T(\mathbf{r}^P - \mathbf{r}^R) = 0$$

- Part of next assignment

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- Move on to the mass distributed internal force:

$$\begin{aligned}
\int_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \mathbf{f}_i(P)\; dm_i(P) &= \int_{m_i} [\delta \mathbf{r}_i^T + \delta\bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T] \cdot \mathbf{f}_i(P)\; dm_i(P) \\
&= \delta \mathbf{r}_i^T \int_{m_i} \mathbf{f}_i(P)\; dm_i(P) + \delta\bar{\pi}_i^T \int_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T \mathbf{f}_i(P)\; dm_i(P) \\
&\equiv \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta\bar{\pi}_i^T \cdot \bar{\mathbf{n}}_i^m
\end{aligned}$$

- Notation used:

$$\mathbf{F}_i^m = \int_{m_i} \mathbf{f}_i(P)\; dm_i(P) \qquad\qquad \bar{\mathbf{n}}_i^m = \int_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T \mathbf{f}_i(P)\; dm_i(P)$$

- Superscript m indicates that this is mass distributed force (force per unit mass)
- If the $\mathbf{f}_i(P) = \mathbf{g}$; i.e., we only have the unit of mass subject to the gravitational force, then

$$\mathbf{F}_i^m = m_i \mathbf{g} \qquad\qquad \bar{\mathbf{n}}_i^m = \mathbf{0}_3$$

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- Virtual work of the inertia force turns out to be more challenging:

$$\int\limits_{m_i} [\delta \mathbf{r}_i^P]^T \cdot \ddot{\mathbf{r}}_i^P \; dm_i(P) = \int\limits_{m_i} [\delta \mathbf{r}_i^T + \delta \bar{\pi}_i^T \tilde{\bar{\mathbf{s}}}_i^P \mathbf{A}_i^T] \cdot \left[\ddot{\mathbf{r}}_i + \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P + \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \bar{\mathbf{s}}_i^P \right] \; dm_i(P)$$

$$= \delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i \int\limits_{m_i} dm_i(P) + \delta \mathbf{r}_i^T \mathbf{A}_i \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \; dm_i(P) + \delta \mathbf{r}_i^T \mathbf{A}_i \dot{\tilde{\bar{\omega}}}_i \int\limits_{m_i} \bar{\mathbf{s}}_i^P \; dm_i(P)$$

$$+ \delta \bar{\pi}_i^T \int\limits_{m_i} \bar{\mathbf{s}}_i^P \; dm_i(P) \mathbf{A}_i \ddot{\mathbf{r}}_i + \delta \bar{\pi}_i^T \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P \; dm_i(P) - \delta \bar{\pi}_i^T \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\mathbf{s}}}_i^P \; dm_i(P) \dot{\bar{\omega}}_i$$

$$= \delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i + \delta \bar{\pi}_i^T \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i + \delta \bar{\pi}_i^T \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i$$

- We used the fact that $\bar{\mathbf{J}} = \int\limits_{m_i} -\tilde{\bar{\mathbf{s}}}^P \; \tilde{\bar{\mathbf{s}}}^P \; dm_i(P)$ and the following identity (see Haug's book, bottom of pp.420)

$$\int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\omega}}_i \tilde{\bar{\omega}}_i \bar{\mathbf{s}}_i^P \; dm_i(P) = \tilde{\bar{\omega}}_i \left[- \int\limits_{m_i} \tilde{\bar{\mathbf{s}}}_i^P \tilde{\bar{\mathbf{s}}}_i^P \; dm_i(P) \right] \bar{\omega}_i = \tilde{\bar{\omega}}_i \bar{\mathbf{J}} \bar{\omega}_i$$

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- At this point, the expression of the virtual work assumes the form:

$$\begin{aligned} \delta W &= \sum_{i=1}^{nb} \left[-\delta \mathbf{r}_i^T \ddot{\mathbf{r}}_i m_i - \delta \bar{\pi}_i^T \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \delta \bar{\pi}_i^T \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \delta \mathbf{r}_i^T \cdot \mathbf{F}_i^m + \delta \bar{\pi}_i^T \cdot \bar{\mathbf{n}}_i^m \right. \\ &+ \left. \delta \mathbf{r}_i^T \mathbf{F}_i^a + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^a + \delta \mathbf{r}_i^T \mathbf{F}_i^r + \delta \bar{\pi}_i^T \bar{\mathbf{n}}_i^r \; \right] = 0 \end{aligned}$$

- Alternatively,

$$\begin{aligned} \delta W &= \sum_{i=1}^{nb} [\; \delta \mathbf{r}_i^T \left(-\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \right) \\ &+ \delta \bar{\pi}_i^T \left(-\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r \right) \;] = 0 \end{aligned} \tag{1}$$

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- Since Eq.(1) on previous slide should hold for *any* set of virtual displacements $(\delta \mathbf{r}_1, \delta \bar{\pi}_1)$, $(\delta \mathbf{r}_2, \delta \bar{\pi}_2)$,..., $(\delta \mathbf{r}_{nb}, \delta \bar{\pi}_{nb})$, then we necessarily have that for $i = 1, \ldots, nb$:

$$\begin{aligned} -\ddot{\mathbf{r}}_i m_i + \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r &= \mathbf{0}_3 \\ -\tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i - \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r &= \mathbf{0}_3 \end{aligned}$$

- Equivalently,

$$\begin{aligned} m_i \ddot{\mathbf{r}}_i &= \mathbf{F}_i^m + \mathbf{F}_i^a + \mathbf{F}_i^r \\ \bar{\mathbf{J}}_i \dot{\bar{\omega}}_i &= \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^r - \tilde{\bar{\omega}}_i \bar{\mathbf{J}}_i \bar{\omega}_i \end{aligned}$$

- The set of equations above represent the EOM for the system of nb bodies.

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- Unfortunately, we have the EOM but in the current form we can't use them since they contain the reaction forces and torques $\mathbf{F}_i^r$ and $\bar{\mathbf{n}}_i^r$, $i = 1, \ldots, nb$, that we don't know.
  - It turns out that in order to get $\ddot{\mathbf{r}}_i$ and $\dot{\bar{\omega}}_i$, which are quantities of primary interest, we have to find a sensible way to handle the reaction torques and moments, $\mathbf{F}_i^r$ and $\bar{\mathbf{n}}_i^r$, respectively.
- The key observation, and the reason for using the principle of Virtual Work:
  - If we are careful about choosing the virtual displacements $(\delta\mathbf{r}_1, \delta\bar{\pi}_1)$, $(\delta\mathbf{r}_2, \delta\bar{\pi}_2)$,. . ., $(\delta\mathbf{r}_{nb}, \delta\bar{\pi}_{nb})$, we can get eliminate the contribution of the reaction forces
  - Specifically, if the virtual displacements $(\delta\mathbf{r}_1, \delta\bar{\pi}_1)$, $(\delta\mathbf{r}_2, \delta\bar{\pi}_2)$,. . ., $(\delta\mathbf{r}_{nb}, \delta\bar{\pi}_{nb})$, are chosen so that they are consistent with the set of constraints that produce the reaction torques and moments, $\mathbf{F}_i^r$ and $\bar{\mathbf{n}}_i^r$, respectively, then their total virtual work is zero

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- Recall that the following condition should hold for a set of virtual displacements to be consistent (see two lectures ago):

$$\begin{bmatrix} \mathbf{\Phi_r} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \bar{\mathbf{R}}(\mathbf{\Phi}) \cdot \begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix} = \mathbf{0}$$

- The matrix vector notation that was used above, along with some more that will be used shortly is as follows:

$$\begin{bmatrix} \delta\mathbf{r} \\ \delta\bar{\pi} \end{bmatrix}_{6nb} = \begin{bmatrix} \delta\mathbf{r}_1 \\ \vdots \\ \delta\mathbf{r}_{nb} \\ \delta\bar{\pi}_1 \\ \vdots \\ \delta\bar{\pi}_{nb} \end{bmatrix}_{6nb} \qquad \mathbf{F}^r = \begin{bmatrix} \mathbf{F}_1^r \\ \vdots \\ \mathbf{F}_{nb}^r \end{bmatrix}_{3nb} \qquad \bar{\mathbf{n}}^r = \begin{bmatrix} \bar{\mathbf{n}}_1^r \\ \vdots \\ \bar{\mathbf{n}}_{nb}^r \end{bmatrix}_{3nb}$$

+ +

- Going back to the observation that the *total* virtual work of the reaction forces is zero when the virtual displacements are consistent, we have that

$$\delta\mathbf{r}_1^T\mathbf{F}_1^r + \ldots + \delta\mathbf{r}_{nb}^T\mathbf{F}_{nb}^r + \delta\bar{\pi}_1^T\bar{\mathbf{n}}_1^r + \ldots + \delta\bar{\pi}_{nb}^T\bar{\mathbf{n}}_{nb}^r = 0$$

provided

$$\boldsymbol{\Phi}_{\mathbf{r}_1}\delta\mathbf{r}_1 + \ldots + \boldsymbol{\Phi}_{\mathbf{r}_{nb}}\delta\mathbf{r}_{nb} + \bar{\boldsymbol{\Pi}}_1(\boldsymbol{\Phi})\delta\bar{\pi}_1 + \ldots + \bar{\boldsymbol{\Pi}}_{nb}(\boldsymbol{\Phi})\delta\bar{\pi}_{nb} = \mathbf{0}_{nc}$$

- In matrix form,

$$\delta\mathbf{r}^T\mathbf{F}^r + \delta\bar{\pi}^T\bar{\mathbf{n}}^r = 0$$

provided

$$\boldsymbol{\Phi}_{\mathbf{r}}\delta\mathbf{r} + \bar{\boldsymbol{\Pi}}(\boldsymbol{\Phi})\delta\bar{\pi} = \mathbf{0}_{nc}$$

- NOTE: I used nc for the number of constraints, rather than m, which is what I used in the past. This was in order to avoid confusion with the superscript m.

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- Going back to the expression of the virtual work, we'll ignore the effect of the reaction forces/torques at the price of not having quite any arbitrary set of virtual displacements. Instead, we will have to ensure that they are consistent.

- Then,

$$\delta W = \sum_{i=1}^{nb}\left[\delta\mathbf{r}_i^T\left(-m_i\ddot{\mathbf{r}}_i + \mathbf{F}_i^m + \mathbf{F}_i^a\right) + \delta\bar{\pi}_i^T\left(-\tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i - \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a\right)\right] = 0$$

provided

$$\boldsymbol{\Phi}_{\mathbf{r}_1}\delta\mathbf{r}_1 + \ldots + \boldsymbol{\Phi}_{\mathbf{r}_{nb}}\delta\mathbf{r}_{nb} + \bar{\boldsymbol{\Pi}}_1(\boldsymbol{\Phi})\delta\bar{\pi}_1 + \ldots + \bar{\boldsymbol{\Pi}}_{nb}(\boldsymbol{\Phi})\delta\bar{\pi}_{nb} = \mathbf{0}_{nc}$$

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- We will introduce some more matrix–vector notation in order to set the stage for applying the Lagrange Multiplier theorem

- We introduce the following notation ($\mathbf{I}_3$ is the identity matrix of dimension 3):

$$\mathbf{M} = \begin{bmatrix} m_1\mathbf{I}_3 & \mathbf{0}_{3\times3} & \ldots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & m_2\mathbf{I}_3 & \ldots & \mathbf{0}_{3\times3} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \ldots & m_{nb}\mathbf{I}_3 \end{bmatrix} \qquad \bar{\mathbf{J}} = \begin{bmatrix} \bar{\mathbf{J}}_1 & \mathbf{0}_{3\times3} & \ldots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \bar{\mathbf{J}}_2 & \ldots & \mathbf{0}_{3\times3} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \ldots & \bar{\mathbf{J}}_{nb} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1^a + \mathbf{F}_1^m \\ \vdots \\ \mathbf{F}_{nb}^a + \mathbf{F}_{nb}^m \end{bmatrix}_{3nb} \qquad \bar{\mathbf{n}} = \begin{bmatrix} \bar{\mathbf{n}}_1^a + \bar{\mathbf{n}}_1^m - \tilde{\bar{\omega}}_1 \bar{\mathbf{J}}_1 \bar{\omega}_1 \\ \vdots \\ \bar{\mathbf{n}}_{nb}^a + \bar{\mathbf{n}}_{nb}^m - \tilde{\bar{\omega}}_{nb} \bar{\mathbf{J}}_{nb} \bar{\omega}_{nb} \end{bmatrix}_{3nb}$$

++

- The virtual work then assumes the expression (see previous slide):

$$\delta W = \delta \mathbf{r}^T (\mathbf{M}\ddot{\mathbf{r}} - \mathbf{F}) + \delta \bar{\pi}^T \left(\bar{\mathbf{J}} \dot{\bar{\omega}} - \bar{\mathbf{n}} \right) = 0$$

provided

$$\mathbf{\Phi_r} \delta \mathbf{r} + \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \delta \bar{\pi} = \mathbf{0}_{nc}$$

- Other way of posing this is as follows:

If the virtual displacements are consistent; i.e., $\begin{bmatrix} \mathbf{\Phi_r} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) \end{bmatrix} \cdot \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix} = \mathbf{0}_{nc}$

then it follows that

$$\delta W = \begin{bmatrix} \delta \mathbf{r} \\ \delta \bar{\pi} \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\ \bar{\mathbf{J}} \dot{\bar{\omega}} - \bar{\mathbf{n}} \end{bmatrix} = 0$$

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- According to the Lagrange Multiplier theorem, there exists a vector of nc Lagrange Multipliers, $\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix}$, so that

$$\begin{bmatrix} \mathbf{M}\ddot{\mathbf{r}} - \mathbf{F} \\ \bar{\mathbf{J}} \dot{\bar{\omega}} - \bar{\mathbf{n}} \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi_r}^T \\ \bar{\mathbf{\Pi}}^T(\mathbf{\Phi}) \end{bmatrix} \lambda = \mathbf{0}_{6nb}$$

- Expression above is the most important equation in ME751: the Newton-Euler form of the EOM. Equivalently expressed as:

$$\begin{cases} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi_r}^T \lambda = \mathbf{F} \\ \bar{\mathbf{J}} \dot{\bar{\omega}} + \bar{\mathbf{\Pi}}^T(\mathbf{\Phi}) \lambda = \bar{\mathbf{n}} \end{cases}$$

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- Here is what we know:
  - $\mathbf{M}$ and $\bar{\mathbf{J}}$, attributes of the bodies making up the multibody system
  - $\mathbf{\Phi_r}$ and $\bar{\mathbf{\Pi}}(\mathbf{\Phi})$, two quantities that we obtain based on the constraints present in the multibody system
  - $\mathbf{F}$ and $\bar{\mathbf{n}}$, force/torque quantities that we obtain based on the forces and torques acting on each body in the system

- Here is what we do not know:
  - $\ddot{\mathbf{r}}$ and $\dot{\bar{\omega}}$, the translational and angular acceleration
  - λ, the set of Lagrange Multipliers associated with the constraints present in the system

- To summarize,
  - The number of equations on previous slide: $6nb$
  - The number of unknowns in those equations $3nb$ for $\ddot{\mathbf{r}}$, $3nb$ for $\dot{\bar{\omega}}$, and nc for $\lambda \quad \Rightarrow \quad 6nb + nc$ unknowns
  - We have more unknowns than equation...

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- What saves the day is the fact that we can also use the acceleration kinematic constraint equations:

$$\mathbf{\Phi_r}\ddot{\mathbf{r}} + \bar{\mathbf{\Pi}}(\mathbf{\Phi})\dot{\bar{\omega}} = \gamma \qquad \Rightarrow \qquad nc \text{ more equations.}$$

- We can assemble everything in matrix form and produce the following *system of linear equations* of dimension $6nb + nc$, whose solution will provide the unknowns $\ddot{\mathbf{r}}$, $\dot{\bar{\omega}}$, and λ:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 3nb} & \mathbf{\Phi_r}^T \\ \mathbf{0}_{3nb\times 3nb} & \bar{\mathbf{J}} & \bar{\mathbf{\Pi}}^T(\mathbf{\Phi}) \\ \mathbf{\Phi_r} & \bar{\mathbf{\Pi}}(\mathbf{\Phi}) & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\bar{\omega}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \bar{\mathbf{n}} \\ \gamma \end{bmatrix}$$

- Remark: the derivation was kind of tedious, but the problem we have to solve to get the quantities of interest looks simple since it is formulated in terms of quantities that we are already familiar with

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- In order to understand the role played by the Lagrange Multipliers λ, we express the Newton-Euler form of the EOM at the body level; i.e., for each body separately:

$$\left.\begin{array}{rcl} m_1\ddot{\mathbf{r}}_1 + \mathbf{\Phi}_{\mathbf{r}_1}^T\lambda & = & \mathbf{F}_1^a + \mathbf{F}_1^m \\ \dots & & \dots \\ m_i\ddot{\mathbf{r}}_i + \mathbf{\Phi}_{\mathbf{r}_i}^T\lambda & = & \mathbf{F}_i^a + \mathbf{F}_i^m \\ \dots & & \dots \\ m_{nb}\ddot{\mathbf{r}}_{nb} + \mathbf{\Phi}_{\mathbf{r}_{nb}}^T\lambda & = & \mathbf{F}_{nb}^a + \mathbf{F}_{nb}^m \end{array}\right\} \longrightarrow \quad m_i\ddot{\mathbf{r}}_i + \mathbf{\Phi}_{\mathbf{r}_i}^T\lambda = \mathbf{F}_i^a + \mathbf{F}_i^m \quad (i = 1, \dots, nb)$$

$$\left.\begin{array}{rcl} \bar{\mathbf{J}}_1\dot{\bar{\omega}}_1 + \bar{\mathbf{\Pi}}_1^T(\mathbf{\Phi})\lambda & = & \bar{\mathbf{n}}_1^a + \bar{\mathbf{n}}_1^m - \tilde{\bar{\omega}}_1\bar{\mathbf{J}}_1\bar{\omega}_1 \\ \dots & & \dots \\ \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda & = & \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^m - \tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i \\ \dots & & \dots \\ \bar{\mathbf{J}}_{nb}\dot{\bar{\omega}}_{nb} + \bar{\mathbf{\Pi}}_{nb}^T(\mathbf{\Phi})\lambda & = & \bar{\mathbf{n}}_{nb}^a + \bar{\mathbf{n}}_{nb}^m - \tilde{\bar{\omega}}_{nb}\bar{\mathbf{J}}_{nb}\bar{\omega}_{nb} \end{array}\right\} \longrightarrow \bar{\mathbf{J}}_i\dot{\bar{\omega}}_i + \bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda = \bar{\mathbf{n}}_i^a + \bar{\mathbf{n}}_i^m - \tilde{\bar{\omega}}_i\bar{\mathbf{J}}_i\bar{\omega}_i \quad (i = 1, \dots, nb$$

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- If we compare the expression of the EOM on previous slide with the one we obtained back when we still had $\mathbf{F}_i^r$ and $\bar{\mathbf{n}}_i^r$ around, it is easy to see that

$$\mathbf{F}_i^r = -\mathbf{\Phi}_{\mathbf{r}_i}^T\lambda \qquad \bar{\mathbf{n}}_i^r = -\bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda$$

- In other words, the Lagrange multipliers, obtained along with $\ddot{\mathbf{r}}$ and $\dot{\bar{\omega}}$ as the solution of a linear system, are the key ingredients needed to compute the reaction forces/torques produced by the constraints $\mathbf{\Phi}(\mathbf{q}, t)$ present in the system

- The way the reaction forces and torques should be interpreted is like this: the quantities $-\mathbf{\Phi}_{\mathbf{r}_i}^T\lambda$ and $-\bar{\mathbf{\Pi}}_i^T(\mathbf{\Phi})\lambda$ represent the net reaction force applied at the center of the L-RF$_i$ and the net reaction torque applied to the rigid body, respectively, them being a consequence of the presence of a set of constraints $\mathbf{\Phi}(\mathbf{q}, t)$

- Moreover, one can go to a finer level of granularity and investigate the contribution of a certain constraint Φ^α (with Lagrange Multiplier λ_α) on body i. The reaction force is simply $-[\Phi_{\mathbf{r}_i}^\alpha]^T\lambda_\alpha$, while the reaction torque expressed in L-RF$_i$ is $-\bar{\mathbf{\Pi}}_i^T(\Phi^\alpha)\lambda_\alpha$

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- For every constraint there is a Lagrange Multiplier. Once you have the Lagrange Multiplier you can compute the reaction force and reaction torque produced by the said constraint

- Just to reinforce this, if your mechanical system has 12 constraints, you will have 12 Lagrange Multipliers, with them computing the reaction forces and torques produced by the 12 constraints

- We said that the reaction force and torque associated with a basic GCon Φ^α that has the Lagrange Multiplier λ_α are

$$\mathbf{F}_i^{r,\alpha} = -[\Phi^\alpha_{\mathbf{r}_i}]^T \lambda_\alpha \qquad \bar{\mathbf{n}}_i^{r,\alpha} = -\bar{\mathbf{\Pi}}_i^T(\Phi^\alpha)\lambda_\alpha$$

  - Note that if $\mathbf{r}_i$ does not enter the expression of Φ^α then $\mathbf{F}_i^{r,\alpha} = \mathbf{0}_{3\times 1}$ since $\Phi^\alpha_{\mathbf{r}_i} = \mathbf{0}_{1\times 3}$
  - Likewise, if Φ^α has no explicit dependence on orientation, then $\bar{\mathbf{\Pi}}_i(\Phi^\alpha) = \mathbf{0}_{1\times 3}$ and therefore $\bar{\mathbf{n}}_i^{r,\alpha} = \mathbf{0}_{3\times 1}$

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