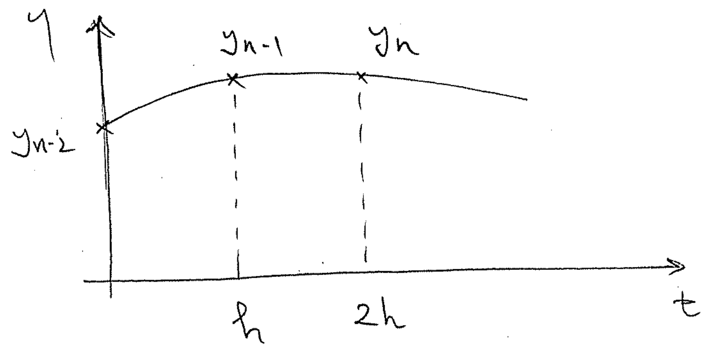
Deriving the BDF formula of order 2

step 1 : Find the polynomial that passes through

y_{n-2}, y_{n-1}, y_n

Step 2 : Take a time derivative of this polynomial at t_n and equate it to $f(t_n, y_n)$.

Step. 1 :



* Integration step-size : h.
* $p(t)$: poly that passes through y_{n-2}, y_{n-1}, y_n..

I have three points $(y_{n-2}, y_{n-1}, y_n) \Rightarrow p(t)$ is going to have order 2:

$$p(t) = a_0 t^2 + a_1 t + a_2$$

$$\left.\begin{array}{l} p(0) = y_{n-2} \\ p(h) = y_{n-1} \\ p(2h) = y_n \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_2 = y_{n-2} \\ a_0 \cdot h^2 + a_1 h + a_2 = y_{n-1} \\ a_0 (4h^2) + a_1 \cdot (2h) + a_2 = y_n \end{array}\right.$$

Solve above 3x3 linear system to get:

$$a_0 = \frac{y_{n-2} - 2y_{n-1} + y_n}{2h^2} \qquad a_1 = \frac{-3y_{n-2} + 4y_{n-1} - y_n}{2h} \qquad a_2 = y_{n-2}$$

Then the poly assumes the form:

$$p(t) = \frac{y_{n-2} - 2y_{n-1} + y_n}{2h^2} t^2 - \frac{3y_{n-2} - 4y_{n-1} + y_n}{2h} t + y_{n-2}$$

Step 2:

$$\dot{p}(t) = \frac{y_{n-2} - 2y_{n-1} + y_n}{h^2} \cdot t - \frac{3y_{n-2} - 4y_{n-1} + y_n}{2h}$$

set the condition $\dot{p}(2h) = f(t_n, y_n)$ to get

$$\frac{y_{n-2} - 2y_{n-1} + y_n}{h^2} \cdot 2h - \frac{3y_{n-2} - 4y_{n-1} + y_n}{2h} = f(t_n, y_n)$$

$$\Rightarrow \quad y_{n-2} - 4y_{n-1} + 3y_n = 2h f(t_n, y_n)$$

Equivalently,

$$\boxed{y_n = \frac{4}{3} y_{n-1} - \frac{1}{3} y_{n-2} + \frac{2h}{3} f(t_n, y_n)}$$