ME751 – Advanced Computational Multibody Dynamics

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Generalized Coordinates Partitioning Algorithm

The starting point of this discussion is the set of Newton-Euler equations of motion augmented with the acceleration kinematic constraint equations:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 4nb} & \mathbf{0}_{3nb\times nb} & \boldsymbol{\Phi}_{\mathbf{r}}^T \\ \mathbf{0}_{4nb\times 3nb} & \mathbf{J}^{\mathbf{p}} & \mathbf{P}^T & \boldsymbol{\Phi}_{\mathbf{p}}^T \\ \mathbf{0}_{nb\times 3nb} & \mathbf{P} & \mathbf{0}_{nb\times nb} & \mathbf{0}_{nb\times nc} \\ \boldsymbol{\Phi}_{\mathbf{r}} & \boldsymbol{\Phi}_{\mathbf{p}} & \mathbf{0}_{nc\times nb} & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \lambda^{\mathbf{p}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\boldsymbol{\tau}} \\ \boldsymbol{\gamma}^{\mathbf{p}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} \tag{1}$$

The following notation will be used in order to simplify the presentation:

$$\mathbf{M}^F(\mathbf{q}) \equiv \begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 4nb} \\ \mathbf{0}_{4nb\times 3nb} & \mathbf{J}^{\mathbf{p}} \end{bmatrix} \quad \boldsymbol{\Phi}_{\mathbf{q}}^F \equiv \begin{bmatrix} \mathbf{0}_{nb\times 3nb} & \mathbf{P} \\ \boldsymbol{\Phi}_{\mathbf{r}} & \boldsymbol{\Phi}_{\mathbf{p}} \end{bmatrix} \quad \mathbf{Q}^A \equiv \begin{bmatrix} \mathbf{F} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} \quad \lambda^F \equiv \begin{bmatrix} \lambda^{\mathbf{p}} \\ \lambda \end{bmatrix} \quad \boldsymbol{\gamma}^F \equiv \begin{bmatrix} \boldsymbol{\gamma}^{\mathbf{p}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} \tag{2}$$

The linear system in Eq. (1) is equivalently expressed as

$$\begin{bmatrix} \mathbf{M}^F(\mathbf{q}) & \left[\boldsymbol{\Phi}_{\mathbf{q}}^F\right]^T \\ \boldsymbol{\Phi}_{\mathbf{q}}^F & \mathbf{0}_{(nb+nc)\times(nb+nc)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \lambda^F \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^A \\ \boldsymbol{\gamma}^F \end{bmatrix} \tag{3}$$

In the discussion below, the notation will become pretty cumbersome. In order to try to keep it simple, the superscript *F* is going to be dropped. Thus, all references made below to $\mathbf{M}$ should be understood to be to $\mathbf{M}^F(\mathbf{q})$, references to $\boldsymbol{\Phi}_{\mathbf{q}}$ should be understood to be to $\boldsymbol{\Phi}_{\mathbf{q}}^F$, references to $\boldsymbol{\lambda}$ should be understood to be to $\boldsymbol{\lambda}^F$, and references to $\boldsymbol{\gamma}$ should be understood to be to $\boldsymbol{\gamma}^F$.

Central to the coordinate partitioning based DAE-to-ODE reduction is the notion of partitioning the vector of generalized coordinates $\mathbf{q}$ into dependent and independent vectors $\mathbf{u}$ and $\mathbf{v}$, respectively. Let $\mathbf{q}_0$ be a consistent configuration of the mechanical

system; i. e., $\mathbf{\Phi}(\mathbf{q}_0) = \mathbf{0}$. In this configuration, the constraint Jacobian $\mathbf{\Phi}_\mathbf{q}$ is evaluated and numerically factored using the Gaussian elimination with full pivoting [1], yielding

$$\mathbf{\Phi}_\mathbf{q}(\mathbf{q}_0) \xrightarrow{Gauss} [\mathbf{\Phi}_\mathbf{u}^* | \mathbf{\Phi}_\mathbf{v}^*] \tag{4}$$

where, provided the Jacobian is full row rank, the matrix $\mathbf{\Phi}_\mathbf{u}^*$ is upper triangular and

$$\det(\mathbf{\Phi}_\mathbf{u}^*) \neq 0 \tag{5}$$

Let $\sigma(i), i = 1,\ldots,nc$ be the set of column permutations done during the Gaussian elimination algorithm. By formally rearranging the generalized coordinates in the vector $\mathbf{q}$ such that

$$\mathbf{q}_{new}(i) = \mathbf{q}(\sigma(i)), \qquad i = 1,\ldots,7nb \tag{6}$$

the first nc components of $\mathbf{q}_{new}$ comprise the vector of dependent coordinates $\mathbf{u}$, while the remaining $ndof \equiv 7nb - nc$ components form the vector of independent coordinates $\mathbf{v}$.

It will be assumed henceforth that reordering has been done, so the permutation array is equal to the identity; i. e.,

$$\boldsymbol{\sigma}(i) = i, \qquad i = 1,\ldots,n \tag{7}$$

Thus, no column permutation is necessary during the factorization process. This is not a limiting assumption, and it is introduced here to simplify the notation and enhance the clarity of the presentation. Under this assumption, the kinematic constrained equations at position, velocity, and acceleration levels, along with the Newton-Euler form of the equations of motion can be expressed in partitioned form as

$$\mathbf{M}^{\mathbf{vv}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{v}} + \mathbf{M}^{\mathbf{vu}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{u}} + \mathbf{\Phi}_\mathbf{v}^\mathbf{T}(\mathbf{u},\mathbf{v})\boldsymbol{\lambda} = \mathbf{Q}^\mathbf{v}(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \tag{8}$$

$$\mathbf{M}^{\mathbf{uv}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{v}} + \mathbf{M}^{\mathbf{uu}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{u}} + \mathbf{\Phi}_\mathbf{u}^\mathbf{T}(\mathbf{u},\mathbf{v})\boldsymbol{\lambda} = \mathbf{Q}^\mathbf{u}(\mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}) \tag{9}$$

$$\mathbf{\Phi}(\mathbf{u},\mathbf{v}) = \mathbf{0} \tag{10}$$

$$\boldsymbol{\Phi}_{\mathbf{u}}(\mathbf{u},\mathbf{v})\dot{\mathbf{u}} + \boldsymbol{\Phi}_{\mathbf{v}}(\mathbf{u},\mathbf{v})\dot{\mathbf{v}} = \mathbf{0} \tag{11}$$

$$\boldsymbol{\Phi}_{\mathbf{u}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{u}} + \boldsymbol{\Phi}_{\mathbf{v}}(\mathbf{u},\mathbf{v})\ddot{\mathbf{v}} = \boldsymbol{\gamma}\left(\ \mathbf{u},\mathbf{v},\dot{\mathbf{u}},\dot{\mathbf{v}}\right) \tag{12}$$

The condition of Eq. (5) implies that

$$\det(\boldsymbol{\Phi}_{\mathbf{u}}) \neq 0 \tag{13}$$

over some time interval. The implicit function theorem [2] guarantees that Eq. (10) can be locally solved in a neighborhood of the consistent configuration $\mathbf{q}_0$ for $\mathbf{u}$ as a function of $\mathbf{v}$. Thus,

$$\mathbf{u} = \mathbf{g}(\mathbf{v}) \tag{14}$$

where the function $\mathbf{g}(\mathbf{v})$ has as many continuous derivatives as does the constraint function $\boldsymbol{\Phi}(\mathbf{q})$. In other words, at an admissible configuration $\mathbf{q}_0$, there exist neighborhoods $U_1(\mathbf{v}_0)$ and $U_2(\mathbf{u}_0)$, and a function $\mathbf{g}:U_1 \to U_2$, such that for any $\mathbf{v} \in U_1$, Eq. (10) is identically satisfied when $\mathbf{u}$ is as given by Eq. (14). Note that the dependency of $\mathbf{u}$ as a function of $\mathbf{v}$ in Eq. (14) it is not explicitly determined, but is a theoretical result that enables DAE-to-ODE reduction.

Since the coefficient matrix $\boldsymbol{\Phi}_{\mathbf{u}}$ in Eq. (11) is nonsingular, $\dot{\mathbf{u}}$ can be expressed in terms of $\mathbf{v}$ and $\dot{\mathbf{v}}$, as

$$\dot{\mathbf{u}} = -\boldsymbol{\Phi}_{\mathbf{u}}^{-1}\boldsymbol{\Phi}_{\mathbf{v}}\dot{\mathbf{v}} \equiv \mathbf{H}\dot{\mathbf{v}} \tag{15}$$

The dependency of the quantities in Eq. (15) and the following equations on $\mathbf{v}$ and $\dot{\mathbf{v}}$ is suppressed for notational simplicity. Following the same argument, the dependent accelerations are expressed as function of independent positions, velocities, and accelerations using Eqs. (12), (14), and (15),

$$\ddot{\mathbf{u}} = \mathbf{H}\ddot{\mathbf{v}} + \boldsymbol{\Phi}_{\mathbf{u}}^{-1}\boldsymbol{\gamma} \tag{16}$$

Finally, Lagrange multipliers are formally expressed as function of independent positions, velocities, and accelerations, by using Eq. (9) to obtain

$$\lambda = \Phi_{\mathbf{u}}^{-\mathrm{T}}[\mathbf{Q}^{\mathbf{u}} - \mathbf{M}^{\mathbf{uv}}\ddot{\mathbf{v}} - \mathbf{M}^{\mathbf{uu}}(\mathbf{H}\ddot{\mathbf{v}} + \Phi_{\mathbf{u}}^{-1}\gamma)] \tag{17}$$

Once $\mathbf{u}$, $\dot{\mathbf{u}}$, $\ddot{\mathbf{u}}$, and $\boldsymbol{\lambda}$ are formally expressed as functions of independent variables, the DAE is reduced to a second order ODE called the state-space ODE (SSODE), which is obtained by substituting the dependent variables in Eq. (8) to yield

$$\hat{\mathbf{M}}\ddot{\mathbf{v}} = \hat{\mathbf{Q}} \tag{18}$$

where

$$\hat{\mathbf{M}} = \mathbf{M}^{\mathbf{vv}} + \mathbf{M}^{\mathbf{vu}}\mathbf{H} + \mathbf{H}^{\mathrm{T}}(\mathbf{M}^{\mathbf{uv}} + \mathbf{M}^{\mathbf{uu}}\mathbf{H}) \tag{19}$$

$$\hat{\mathbf{Q}} = \mathbf{Q}^{\mathbf{v}} + \mathbf{H}^{\mathrm{T}}\mathbf{Q}^{\mathbf{u}} - (\mathbf{M}^{\mathbf{vv}} + \mathbf{H}^{\mathrm{T}}\mathbf{M}^{\mathbf{uu}})\Phi_{\mathbf{u}}^{-1}\gamma \tag{20}$$

An argument based on the positive definiteness of the quadratic form associated with kinetic energy of any mechanical system is at the foundation of a result [3] that states that the coefficient matrix $\hat{\mathbf{M}}$ in Eq. (18) is positive definite. Therefore, the system in Eq. (18) has a unique solution at each time step, which is used to numerically advance the integration to the next time step.

From a practical standpoint, the independent accelerations are not computed by first evaluating the matrices $\hat{\mathbf{M}}$ and $\hat{\mathbf{Q}}$, and then solving the system in Eq. (18). This strategy for solving for independent accelerations would be inefficient, because of the costly matrix-matrix multiplications involved. Rather, an approach based on first solving the linear system of Eq. (1), and then extracting the independent accelerations $\ddot{\mathbf{v}}$ from the set of generalized accelerations $\ddot{\mathbf{q}}$, based on the permutation array $\boldsymbol{\sigma}$, is more efficient.

In this context, it is worth pointing out that efficient state-space based explicit integration of the DAE of Multibody Dynamics requires in the first place a fast method to

obtain the independent accelerations $\ddot{\mathbf{v}}$. Using an explicit integration formula, the independent positions $\mathbf{v}$, and velocities $\dot{\mathbf{v}}$ are obtained at the next time step by integrating the initial value problem (IVP) $\ddot{\mathbf{v}} = \mathbf{f}(t, \mathbf{v}, \dot{\mathbf{v}})$ (with $\mathbf{f} = \hat{\mathbf{M}}^{-1}\hat{\mathbf{Q}}$) for a given set of initial conditions $\mathbf{v}_0$ and $\dot{\mathbf{v}}_0$; then using Eqs. (10) and (11) $\mathbf{u}$ and $\dot{\mathbf{u}}$ at the new time step can be obtained. These two last stages amount to the solution of a set of non-linear equations and a set of linear equations, respectively. Thus, for explicit integration the linear algebra stage is central. It appears during each of the three steps of the process (computation of $\ddot{\mathbf{v}}$, recovery of $\mathbf{u}$, recovery of $\dot{\mathbf{u}}$) and the extent to which topology information of the mechanical system model is taken into account will determine the overall performance of the method.

Finally, note that one does not necessarily have to compute the accelerations by solving the linear system in Eq. (1). Instead, first solve the linear system

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times 3nb} & \boldsymbol{\Phi}_{\mathbf{r}}^{T} \\ \mathbf{0}_{3nb\times 3nb} & \mathbf{J} & \boldsymbol{\Pi}^{T}(\boldsymbol{\Phi}) \\ \boldsymbol{\Phi}_{\mathbf{r}} & \overline{\Pi}(\boldsymbol{\Phi}) & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\bar{\boldsymbol{\omega}}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \boldsymbol{\tau} \\ \gamma \end{bmatrix}_{6nb+nc}, \tag{21}$$

which is of lower dimension and has several constant block matrices, and then compute the accelerations as $\ddot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^{T}\dot{\bar{\boldsymbol{\omega}}} - \frac{1}{4}\bar{\boldsymbol{\omega}}^{T}\bar{\boldsymbol{\omega}}\mathbf{p}$. See [4] for more details.

References:

1. Atkinson, K.E., *An Introduction to Numerical Analysis*. Second ed. 1989, New York, NY: Wiley.
2. Corwin, L.J. and R.H. Szczarba, *Multivariable Calculus*. 1982, New York
3. Haug, E.J., *Computer-Aided Kinematics and Dynamics of Mechanical Systems. Volume I:Basic Methods*. 1989, Boston, MA: Allyn and Bacon.
4. Negrut, D. *ME751:Advanced Computational Multibody Dynamics:* http://sbel.wisc.edu/Courses/ME751/2010/index.htm. 2010 [cited 2010 March 7].