Chapter 9 Supplement
11/20/01

Contents

9.3 Euler Parameters

9.3.1 Invariants of Orientation Transformation and Euler Parameters

10-25-01

Invariants Of Orientation Transformation

Using Eq. 9.2.15 and the notation $\|\mathbf{a}\| = \sqrt{\left(\mathbf{a}^{\mathrm{T}}\mathbf{a}\right)}$ for length of a vector,

$$\|\mathbf{a}\|^2 = \mathbf{a}^{\mathrm{T}}\mathbf{a} = \mathbf{a}'^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{a}' = \mathbf{a}'^{\mathrm{T}}\mathbf{a}' = \|\mathbf{a}'\|^2 \tag{9.3.1.1}$$

Thus, the orientation transformation preserves length of a vector; i.e.,

$$\|\mathbf{a}\| = \|\mathbf{a}'\| \tag{9.3.1.2}$$

Using this result and the definition of scalar product,

$$\begin{aligned} \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta(\mathbf{a},\mathbf{b}) &= \mathbf{a}^{\mathrm{T}}\mathbf{b} = \mathbf{a}'^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{b}' = \mathbf{a}'^{\mathrm{T}}\mathbf{b}' \\ &= \|\mathbf{a}'\|\|\mathbf{b}'\|\cos\theta(\mathbf{a}',\mathbf{b}') = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta(\mathbf{a}',\mathbf{b}') \end{aligned} \tag{9.3.1.3}$$

Thus, scalar product is preserved and

$$\cos\theta(\mathbf{a},\mathbf{b}) = \cos\theta(\mathbf{a}',\mathbf{b}') \tag{9.3.1.4}$$

Recall from the Eq. 9.2.13 that $\mathbf{A} = [\mathbf{f}, \mathbf{g}, \mathbf{h}]$, where **f**, **g**, and **h** are unit vectors along the x'-y'-z' axes, so they are orthogonal. As is well-known from three-dimensional vector analysis [***reference***] and can be verified by direct calculation using the characterization of unit vectors **f**, **g**, and **h**,

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} f_x & g_x & h_x \\ f_y & g_y & h_y \\ f_z & g_z & h_z \end{vmatrix} = f_x\left(g_y h_z - g_z h_y\right) + f_y\left(g_z h_x - g_x h_z\right) + f_z\left(g_x h_y - g_y h_x\right) \\ &= \mathbf{f}^{\mathrm{T}} \begin{bmatrix} 0 & -g_z & g_y \\ g_z & 0 & -g_x \\ -g_y & g_x & 0 \end{bmatrix} \mathbf{h} = \mathbf{f}^{\mathrm{T}}\tilde{\mathbf{g}}\mathbf{h} \end{aligned}$$

Since the vectors **f**, **g**, and **h** form a right-hand Cartesian reference frame, $\mathbf{f} = \tilde{\mathbf{g}}\mathbf{h}$ and

$$\left|\mathbf{A}\right| = \mathbf{f}^{\mathrm{T}}\tilde{\mathbf{g}}\mathbf{h} = \mathbf{f}^{\mathrm{T}}\mathbf{f} = 1 \tag{9.3.1.5}$$

If the Cartesian reference frame had been left-handed, then $\mathbf{f} = -\tilde{\mathbf{g}}\mathbf{h}$ and $\left|\mathbf{A}\right| = \mathbf{f}^{\mathrm{T}}\tilde{\mathbf{g}}\mathbf{h} = \mathbf{f}^{\mathrm{T}}\left(-\mathbf{f}\right) = -1$

Direct manipulation yields the following relationship:

$$\left(\mathbf{A}-\mathbf{I}\right)\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} - \mathbf{A}^{\mathrm{T}} = \mathbf{I} - \mathbf{A}^{\mathrm{T}} = \left(\mathbf{I}-\mathbf{A}\right)^{\mathrm{T}} = -\left(\mathbf{A}-\mathbf{I}\right)^{\mathrm{T}} \tag{9.3.1.6}$$

Using the facts [***reference***] that the determinant of a product of matrices is the product of its determinant, the determinant of the transpose of a matrix is its determinant, and the determinant of the negative of a 3×3 matrix is $\left(-1\right)^{3} = -1$,

$$\left|\mathbf{A}-\mathbf{I}\right| = \left|\left(\mathbf{A}-\mathbf{I}\right)\mathbf{A}^{\mathrm{T}}\right| = \left|-\left(\mathbf{A}-\mathbf{I}\right)\right| = -\left|\mathbf{A}-\mathbf{I}\right| \tag{9.3.1.7}$$

Thus, $\left|\mathbf{A}-\mathbf{I}\right| = 0$, and $\lambda = 1$ is an eigenvalue of **A**. Thus, there exists a unit eigenvector **u** corresponding to this eigenvalue such that

$$\mathbf{A}\mathbf{u} = \mathbf{u} \tag{9.3.1.8}$$

Since $\mathbf{A}\mathbf{u}' = \mathbf{u}$, Eq. (9.3.1.8) shows that

$$\mathbf{u} = \mathbf{u}' \tag{9.3.1.9}$$

That is, the transformation leaves the eigenvector **u** unchanged.

Let $\mathbf{c}'$ and $\mathbf{d}'$ be vectors orthogonal to **u**. Since the transformation **A** preserves scalar product, $\mathbf{c}$ and $\mathbf{d}$ are also orthogonal to **u**. From the definition of vector product,

$$\begin{aligned}\left\|\mathbf{c}\right\|\left\|\mathbf{d}\right\|\sin\theta\left(\mathbf{c},\mathbf{d}\right)\mathbf{u} = \tilde{\mathbf{c}}\mathbf{d} = \mathbf{A}\tilde{\mathbf{c}}'\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{d}' &= \mathbf{A}\tilde{\mathbf{c}}'\mathbf{d}' \\ &= \mathbf{A}\left\|\mathbf{c}'\right\|\left\|\mathbf{d}'\right\|\sin\theta\left(\mathbf{c}',\mathbf{d}'\right)\mathbf{u}' \\ &= \mathbf{A}\left\|\mathbf{c}\right\|\left\|\mathbf{d}\right\|\sin\theta\left(\mathbf{c}',\mathbf{d}'\right)\mathbf{u}\end{aligned}$$

$$\sin\theta\left(\mathbf{c},\mathbf{d}\right) = \sin\theta\left(\mathbf{c}',\mathbf{d}'\right) \tag{9.3.1.10}$$

With Eq. (9.3.1.4), this implies $\theta\left(\mathbf{c},\mathbf{d}\right) = \theta\left(\mathbf{c}',\mathbf{d}'\right)$.

Next, let the angle between $\mathbf{c}'$ and $\mathbf{c}$ be χ and the angle between $\mathbf{c}'$ and $\mathbf{d}'$, hence between $\mathbf{c}$ and $\mathbf{d}$, be α, as shown by drawing both pairs of vectors in the same plane in Fig. 0.1 . Then,

$$\theta(\mathbf{d}',\mathbf{d}) = \theta(\mathbf{c}',\mathbf{d}) - \theta(\mathbf{c}',\mathbf{d}') = \chi + \alpha - \alpha = \chi \tag{9.3.1.11}$$

Thus, the transformation **A** may be interpreted as rotating every vector in the plane perpendicular to **u** by the same angle χ.

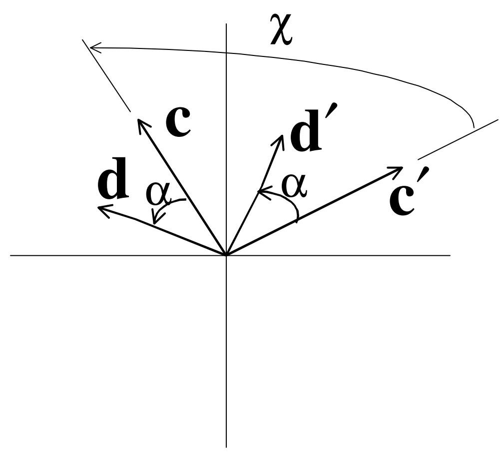

Figure 0.1 Angles Between Vectors In Plane Orthogonal To **u**

Rotation of Reference Frames

Since $\cos\theta(\mathbf{u},\mathbf{a}) = \cos\theta(\mathbf{u}',\mathbf{a}') = \cos\theta(\mathbf{u},\mathbf{a}')$, when drawn in the same reference frame in Fig. 0.2, both $\mathbf{a}'$ and $\mathbf{a}$ have the same projection on **u**, given by

$$\mathbf{v} \equiv \|\mathbf{a}\|\cos\theta(\mathbf{u},\mathbf{a})\mathbf{u} = \|\mathbf{a}'\|\cos\theta(\mathbf{u},\mathbf{a}')\mathbf{u} = (\mathbf{a}^{\mathrm{T}}\mathbf{u})\mathbf{u} \tag{9.3.1.12}$$

so the length of **v** is $\|\mathbf{v}\| = \|\mathbf{a}\|\cos\theta(\mathbf{a},\mathbf{u})$. Also, $\mathbf{v}' = \mathbf{A}^{\mathrm{T}}\mathbf{v} = (\mathbf{a}^{\mathrm{T}}\mathbf{u})\mathbf{A}^{\mathrm{T}}\mathbf{u} = (\mathbf{a}^{\mathrm{T}}\mathbf{u})\mathbf{u} = \mathbf{v}$.
Since norm is preserved by the transformation **A**, the vectors $\mathbf{a} - \mathbf{v}$ and $\mathbf{a}' - \mathbf{v} = \mathbf{a}' - \mathbf{v}'$ have the same length,

$$\|\mathbf{a} - \mathbf{v}\| = \|\mathbf{a}' - \mathbf{v}\| = \|\mathbf{a}\|\sin\theta(\mathbf{a},\mathbf{u}) \equiv \|\tilde{\mathbf{a}}\mathbf{u}\| \tag{9.3.1.13}$$

so they form radii of a circle in the same plane orthogonal to **u**, as shown in Fig. 0.2 . Note also that $\sin\theta(\mathbf{a},\mathbf{u}) = \|\mathbf{a} - \mathbf{v}\| / \|\mathbf{a}\| = \|\mathbf{a}' - \mathbf{v}\| / \|\mathbf{a}'\| = \sin\theta(\mathbf{a},\mathbf{u}')$.

Since $\mathbf{v} = \mathbf{A}\mathbf{v}$,

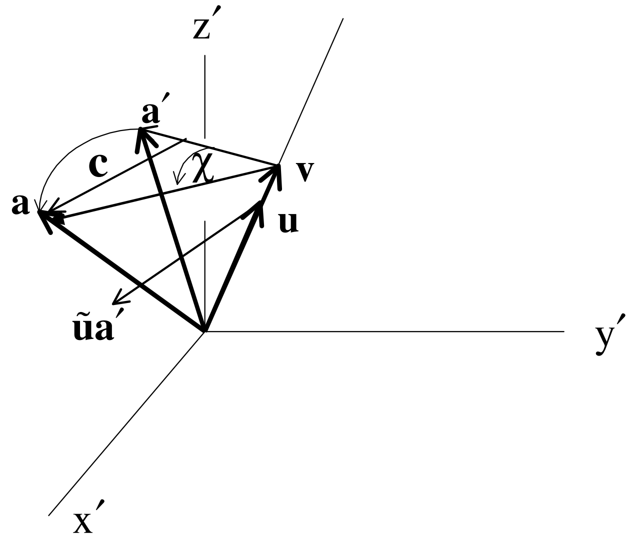

Figure 0.2 Relation Between Transformed Vectors

$$\mathbf{a} - \mathbf{v} = \mathbf{A}\mathbf{a}' - \mathbf{A}\mathbf{v} = \mathbf{A}\left(\mathbf{a}' - \mathbf{v}\right) \tag{9.3.1.14}$$

and, since both $\mathbf{a} - \mathbf{v}$ and $\mathbf{a}' - \mathbf{v}$ lie in the same plane orthogonal to **u**, the angle between these vectors is χ, as shown in Fig. 0.2. Since every vector passing through the origin of the reference frame transforms by a rotation χ of the plane formed by the intersection of it and the vector **u** about the vector **u**, this is true of each of the three coordinate axes. This result may be interpreted geometrically in the form of a classical theorem.

Euler's Theorem: If the origins of two right-hand Cartesian reference frames coincide, they may be brought into coincidence by a single rotation about some axis.

The vector **c** orthogonal to $\mathbf{a}' - \mathbf{v}$ and passing through $\mathbf{a}$ is perpendicular to the plane formed by **u** and $\mathbf{a}'$, so it is parallel to $\tilde{\mathbf{u}}\mathbf{a}'$. Equation (9.3.1.13) shows that the length of $\tilde{\mathbf{u}}\mathbf{a}'$ is the radius of the above noted circle, so **c** is given by

$$\mathbf{c} = \sin\chi\tilde{\mathbf{u}}\mathbf{a}' \tag{9.3.1.15}$$

Finally,

$$\begin{aligned}\mathbf{a} &= \mathbf{v} + \cos\chi\left(\mathbf{a}' - \mathbf{v}\right) + \mathbf{c} \\ &= \left(\mathbf{a}'^{\mathrm{T}}\mathbf{u}\right)\mathbf{u} + \cos\chi\left\{\mathbf{a}' - \left(\mathbf{a}'^{\mathrm{T}}\mathbf{u}\right)\mathbf{u}\right\} + \sin\chi\tilde{\mathbf{u}}\mathbf{a}' \\ &= \mathbf{u}\left(\mathbf{u}^{\mathrm{T}}\mathbf{a}'\right) + \cos\chi\left\{\mathbf{a}' - \mathbf{u}\left(\mathbf{u}^{\mathrm{T}}\mathbf{a}'\right)\right\} + \sin\chi\tilde{\mathbf{u}}\mathbf{a}' \\ &= \left[\left(1 - \cos\chi\right)\mathbf{u}\mathbf{u}^{\mathrm{T}} + \cos\chi\mathbf{I} + \sin\chi\tilde{\mathbf{u}}\right]\mathbf{a}'\end{aligned} \tag{9.3.1.16}$$

Since $\mathbf{a} = \mathbf{A}\mathbf{a}'$, Eq. (9.3.1.16) yields

$$\mathbf{A} = (1-\cos\chi)\mathbf{u}\mathbf{u}^{\mathrm{T}} + \cos\chi\mathbf{I} + \sin\chi\tilde{\mathbf{u}} \tag{9.3.1.17}$$

Using the trigonometric identities

$$\begin{aligned}
1-\cos\chi &= 2\sin^2\frac{\chi}{2} \\
\sin\chi &= 2\sin\frac{\chi}{2}\cos\frac{\chi}{2} \\
\cos\chi &= 2\cos^2\frac{\chi}{2} - 1 \\
\cos^2\frac{\chi}{2} + \sin^2\frac{\chi}{2} &= 1
\end{aligned} \tag{9.3.1.18}$$

the orientation transformation matrix of Eq. (9.3.1.17) is

$$\begin{aligned}
\mathbf{A} &= 2\sin^2\frac{\chi}{2}\mathbf{u}\mathbf{u}^{\mathrm{T}} + \left(2\cos^2\frac{\chi}{2} - 1\right)\mathbf{I} + 2\sin\frac{\chi}{2}\cos\frac{\chi}{2}\tilde{\mathbf{u}} \\
&= 2\sin^2\frac{\chi}{2}\mathbf{u}\mathbf{u}^{\mathrm{T}} + \left(\cos^2\frac{\chi}{2} - \sin^2\frac{\chi}{2}\right)\mathbf{I} + 2\sin\frac{\chi}{2}\cos\frac{\chi}{2}\tilde{\mathbf{u}}
\end{aligned} \tag{9.3.1.19}$$

Euler Parameter Orientation Coordinates

Defining the four *Euler parameters* as

$$\begin{aligned}
\mathrm{e}_0 &\equiv \cos\frac{\chi}{2} \\
\mathbf{e} &\equiv \sin\frac{\chi}{2}\mathbf{u}
\end{aligned} \tag{9.3.1.20}$$

the orientation transformation matrix on the second line of Eq. (9.3.1.19) can be written in the form

$$\mathbf{A} = \left(\mathrm{e}_0^2 - \mathbf{e}^{\mathrm{T}}\mathbf{e}\right)\mathbf{I} + 2\mathbf{e}\mathbf{e}^{\mathrm{T}} + 2\mathrm{e}_0\tilde{\mathbf{e}} \tag{9.3.1.21}$$

Note that the first and second terms in Eq. (9.3.1.21) are symmetric matrices and the third term is skew-symmetric. In component form the orientation transformation matrix of Eq. (9.3.1.21) is

$$\mathbf{A} = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix} \tag{9.3.1.22}$$

If the form of the orientation transformation matrix on the first line of Eq. (9.3.1.19) had been used, the orientation transformation matrix could be written in terms of Euler parameters in the equivalent form

$$\bar{\mathbf{A}} = \left(2e_0{}^2 - 1\right)\mathbf{I} + 2\left(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}\right) \tag{9.3.1.23}$$

This form of the orientation transformation matrix has been used in most recent textbooks [Wittenburg, 1977; Nikravesh, 1988; Haug, 1989]. It suffers, however, from requiring satisfaction of the Euler parameter normalization condition, in order that a key factorization of $\bar{\mathbf{A}}$ as a product of two matrices that are linear in Euler parameters holds. This condition is not satisfied during interactive numerical solution procedures, hence precluding use of the factorization in important computational applications. It is interesting to note that the form of Eq. (9.3.1.21) is in fact used in classical developments such as [Goldstein, 1980; Roberson and Schwertassek, 1988].

The 4-vector of Euler parameters is defined as

$$\mathbf{p} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix}^T = \begin{bmatrix} e_0 & \mathbf{e}^T \end{bmatrix}^T \tag{9.3.1.24}$$

A direct calculation, using Eq. (9.3.1.20), yields

$$\mathbf{p}^T\mathbf{p} = e_0^2 + \mathbf{e}^T\mathbf{e} = \cos^2\frac{\chi}{2} + \sin^2\frac{\chi}{2}\mathbf{u}^T\mathbf{u} = \cos^2\frac{\chi}{2} + \sin^2\frac{\chi}{2}$$

where the condition that **u** is a unit vector has been used. This yields the identity

$$\mathbf{p}^T\mathbf{p} = 1 \tag{9.3.1.25}$$

which is called the Euler parameter normalization condition.

Mapping from Transformation Matrix to Euler Parameters

The trace of a given orthogonal transformation matrix **A**, denoted tr**A**, is defined as the sum of the diagonal elements of the matrix; i.e.,

$$\text{tr}\mathbf{A} \equiv a_{11} + a_{22} + a_{33} \tag{9.3.1.26}$$

Equating the sum of diagonal terms in the orientation transformation matrix of Eq. (9.3.1.22) to the trace of the given orthogonal transformation matrix **A** and using the identity of Eq. (9.3.1.25),

$$\mathrm{tr}\mathbf{A} = 3e_0^2 - e_1 - e_2^2 - e_3^2 = 4e_0^2 - 1 \tag{9.3.1.27}$$

Thus,

$$e_0^2 = \frac{\mathrm{tr}\mathbf{A} + 1}{4} \tag{9.3.1.28}$$

Equating diagonal terms of the given orthogonal transformation matrix **A** and corresponding terms in Eq. (9.3.1.22), manipulating, and using the identity of Eq. (9.3.1.25) yields

$$\begin{aligned}
a_{11} &= e_0^2 + e_1^2 - e_2^2 - e_3^2 = e_0^2 + 2e_1^2 - e_1^2 - e_2^2 - e_3^2 = 2e_0^2 + 2e_1^2 - 1 \\
a_{22} &= e_0^2 - e_1^2 + e_2^2 - e_3^2 = e_0^2 - e_1^2 + 2e_2^2 - e_2^2 - e_3^2 = 2e_0^2 + 2e_2^2 - 1 \\
a_{33} &= e_0^2 - e_1^2 - e_2^2 + e_3^2 = e_0^2 - e_1^2 - e_2^2 + 2e_3^2 - e_3^2 = 2e_0^2 + 2e_3^2 - 1
\end{aligned} \tag{9.3.1.29}$$

Using these relations and Eq. (9.3.1.28),

$$\begin{aligned}
e_1^2 &= \frac{a_{11} - 2e_0^2 + 1}{2} = \frac{2a_{11} - \mathrm{tr}\mathbf{A} + 1}{4} \\
e_2^2 &= \frac{a_{22} - 2e_0^2 + 1}{2} = \frac{2a_{22} - \mathrm{tr}\mathbf{A} + 1}{4} \\
e_3^2 &= \frac{a_{33} - 2e_0^2 + 1}{2} = \frac{2a_{33} - \mathrm{tr}\mathbf{A} + 1}{4}
\end{aligned} \tag{9.3.1.30}$$

Thus, the magnitudes of each of the four Euler parameters are uniquely defined by diagonal elements of the given orthogonal transformation matrix **A**. Summing the left sides of Eqs. (9.3.1.28) and (9.3.1.30) yields

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = \frac{2\left(a_{11} + a_{22} + a_{33}\right) - 2\mathrm{tr}\mathbf{A} + 4}{4} = 1 \tag{9.3.1.31}$$

so the Euler parameter normalization condition is satisfied.

In order to determine the algebraic signs of the Euler parameters, off-diagonal terms of the given matrix **A** must be employed. Adding symmetrically placed off-diagonal terms in the given matrix **A** and the matrix of Eq. (9.3.1.22) yields

$$\begin{aligned} a_{12} + a_{21} &= 4e_1e_2 \\ a_{13} + a_{31} &= 4e_1e_3 \\ a_{32} + a_{23} &= 4e_2e_3 \end{aligned} \tag{9.3.1.32}$$

Similarly, subtracting symmetrically placed off-diagonal terms yields

$$\begin{aligned} a_{12} - a_{21} &= 4e_0e_3 \\ a_{13} - a_{31} &= 4e_0e_2 \\ a_{32} - a_{23} &= 4e_0e_1 \end{aligned} \tag{9.3.1.33}$$

Selecting the largest Euler parameter in magnitude from Eqs. (9.3.1.28) and (9.3.1.30), its algebraic sign may be selected as positive or negative. For the sake of argument, select the positive sign. If e_0 is largest in magnitude, Eqs. (9.3.1.33) then uniquely determine the algebraic signs of the remaining Euler parameters. If one of the other Euler parameters is largest in magnitude, say e_1 for argument, then the first two of Eqs. (9.3.1.32)uniquely determines the algebraic sign of e_2 and e_3 and the third of Eqs. (9.3.1.33)uniquely determines the algebraic sign of e_0.

The Euler parameter vector determined above by assigning a positive value to the parameter with largest magnitude yields a unique Euler parameter vector denoted $\mathbf{p}^+$. If a negative sign were assigned to the Euler parameter of largest magnitude, the above procedure would be followed to obtain a unique Euler parameter denoted $\mathbf{p}^-$. Since $\mathbf{p}^+ = -\mathbf{p}^-$, $\left\|\mathbf{p}^+ - \mathbf{p}^-\right\|^2 = \left\|2\mathbf{p}^+\right\|^2 = 4\mathbf{p}^{+T}\mathbf{p}^+ = 4$, so the alternative values of Euler parameter determined by the given orthogonal orientation transformation matrix are far apart.

Euler Parameter Identities

Define the 3×4 matrices

$$\mathbf{E}(\mathbf{p}) = \mathbf{E} \equiv \left[-\mathbf{e} \quad \tilde{\mathbf{e}} + e_0\mathbf{I}\right] \tag{9.3.1.34}$$

$$\mathbf{G}(\mathbf{p}) = \mathbf{G} \equiv \left[-\mathbf{e} \quad -\tilde{\mathbf{e}} + e_0\mathbf{I}\right] \tag{9.3.1.35}$$

Multiplication and manipulation with vector identities verifies that

$$\begin{aligned}\mathbf{E}\mathbf{G}^{\mathrm{T}} &= \begin{bmatrix}-\mathbf{e} & \tilde{\mathbf{e}}+\mathrm{e}_0\mathbf{I}\end{bmatrix}\begin{bmatrix}-\mathbf{e}^{\mathrm{T}} \\ \tilde{\mathbf{e}}+\mathrm{e}_0\mathbf{I}\end{bmatrix} \\ &= \mathbf{e}\mathbf{e}^{\mathrm{T}} + \tilde{\mathbf{e}}\tilde{\mathbf{e}} + 2\mathrm{e}_0\tilde{\mathbf{e}} + \mathrm{e}_0^2\mathbf{I} \\ &= \mathbf{e}\mathbf{e}^{\mathrm{T}} + \mathbf{e}\mathbf{e}^{\mathrm{T}} - \mathbf{e}^{\mathrm{T}}\mathbf{e}\mathbf{I} + 2\mathrm{e}_0\tilde{\mathbf{e}} + \mathrm{e}_0^2\mathbf{I} \\ &= \left(\mathrm{e}_0^2 - \mathbf{e}^{\mathrm{T}}\mathbf{e}\right)\mathbf{I} + 2\mathbf{e}\mathbf{e}^{\mathrm{T}} + 2\mathrm{e}_0\tilde{\mathbf{e}}\end{aligned} \tag{9.3.1.36}$$

Noting that the right sides of Eqs. (9.3.1.21) and (9.3.1.36) are identical,

$$\mathbf{A} = \mathbf{E}\mathbf{G}^{\mathrm{T}} \tag{9.3.1.37}$$

Two observations are important at this point. First, Eq. (9.3.1.37) was obtained without the requirement that the vector **p** be normalized. This is important in computation, since during iterative solution, the Euler parameter vector will only be normalized as convergence is achieved. Second, from Eqs. (9.3.1.22), (9.3.1.34), and (9.3.1.35) it is observed that all the entries in the matrix $\mathbf{A}(\mathbf{p})$ are quadratic in components of **p**, whereas each of the factors on the right of Eq. (9.3.1.37) are matrices whose entries are linear in **p**.

Some of the identities presented in Section 9.3 of CAKD require that the vector of Euler parameters satisfy the normalization constraint of Eq. 9.3.9. They should therefore, not be used in numerical computations when $\mathbf{p}^{\mathrm{T}}\mathbf{p} \neq 1$. A number of useful identities involving Euler parameters can be obtained without requiring satisfaction of the normalization constraint. These identities are particularly valuable in developing relationships for derivatives of kinematic expressions with respect to Euler parameters, which are to be used in iterative computation in which the Euler parameter normalization condition may not be satisfied . In particular, let $\mathbf{a} \in \mathrm{R}^3$, and $\mathbf{p} \in \mathrm{R}^4$, **p** *not necessarily normalized*; i.e., $\mathbf{p}^{\mathrm{T}}\mathbf{p}$ not necessarily unity.

Carrying out the manipulation indicated shows that

$$\mathbf{E}\mathbf{p} = \begin{bmatrix}-\mathbf{e} & \tilde{\mathbf{e}}+\mathrm{e}_0\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathrm{e}_0 \\ \mathbf{e}\end{bmatrix} = -\mathrm{e}_0\mathbf{e} + \tilde{\mathbf{e}}\mathbf{e} + \mathrm{e}_0\mathbf{e} = \mathbf{0} \tag{9.3.1.38}$$

where no Euler parameter normalization condition was used. Similarly,

$$\mathbf{G}\mathbf{p} = \mathbf{0} \tag{9.3.1.39}$$

Carrying out the multiplication indicated,

$$\begin{aligned}\mathbf{E}\mathbf{E}^{\mathrm{T}} &= \left[-\mathbf{e} \quad \tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I}\right]\begin{bmatrix} -\mathbf{e}^{\mathrm{T}} \\ -\tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I} \end{bmatrix} \\ &= \mathbf{e}\mathbf{e}^{\mathrm{T}} - \tilde{\mathbf{e}}\tilde{\mathbf{e}} + \mathrm{e}_0^2\mathbf{I} \\ &= \mathbf{e}\mathbf{e}^{\mathrm{T}} - \mathbf{e}\mathbf{e}^{\mathrm{T}} + \mathbf{e}^{\mathrm{T}}\mathbf{e}\mathbf{I} + \mathrm{e}_0^2\mathbf{I} \\ &= \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I}\end{aligned} \tag{9.3.1.40}$$

Only if the Euler parameter normalization constraint is satisfied is the conventional identity obtained; i.e.,

$$\mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{I} \tag{9.3.1.41}$$

Similarly, for any vector **p**,

$$\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I} \tag{9.3.1.42}$$

Only if the normalization constraint is satisfied is the conventional results obtained,

$$\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{I} \tag{9.3.1.43}$$

Carrying out the multiplication indicated,

$$\begin{aligned}\mathbf{E}^{\mathrm{T}}\mathbf{E} &= \left[-\mathbf{e} \quad \tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I}\right]^{\mathrm{T}}\left[-\mathbf{e} \quad \tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I}\right] \\ &= \begin{bmatrix} \mathbf{e}^{\mathrm{T}}\mathbf{e} & -\mathrm{e}_0\mathbf{e}^{\mathrm{T}} \\ -\mathrm{e}_0\mathbf{e} & -\tilde{\mathbf{e}}\tilde{\mathbf{e}} + \mathrm{e}_0^2\mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}^{\mathrm{T}}\mathbf{e} + \mathrm{e}_0^2 - \mathrm{e}_0^2 & -\mathrm{e}_0\mathbf{e}^{\mathrm{T}} \\ -\mathrm{e}_0\mathbf{e} & -\mathbf{e}\mathbf{e}^{\mathrm{T}} + \mathbf{e}^{\mathrm{T}}\mathbf{e}\mathbf{I} + \mathrm{e}_0^2\mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}^{\mathrm{T}}\mathbf{p} & \mathbf{0} \\ \mathbf{0} & \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathrm{e}_0^2 & \mathrm{e}_0\mathbf{e}^{\mathrm{T}} \\ \mathrm{e}_0\mathbf{e} & \mathbf{e}\mathbf{e}^{\mathrm{T}} \end{bmatrix} \\ &= \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I}_4 - \mathbf{p}\mathbf{p}^{\mathrm{T}}\end{aligned} \tag{9.3.1.44}$$

Only if the normalization constraint is satisfied is the conventional result obtained; i.e.,

$$\mathbf{E}^{\mathrm{T}}\mathbf{E} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^{\mathrm{T}} \tag{9.3.1.45}$$

Similarly,

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I}_4 - \mathbf{p}\mathbf{p}^{\mathrm{T}} \tag{9.3.1.46}$$

Only if the normalization constraint is satisfied is the conventional result obtained; i.e.,

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^{\mathrm{T}} \tag{9.3.1.47}$$

A direct manipulation, using Eqs.(9.3.1.37), (9.3.1.40), (9.3.1.42), (9.3.1.44), and (9.3.1.46), shows that

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{E}\mathbf{G}^{\mathrm{T}}\mathbf{G}\mathbf{E}^{\mathrm{T}} = \mathbf{E}\left(\mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right)\mathbf{E}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{E}\mathbf{E}^{\mathrm{T}} = \left(\mathbf{p}^{\mathrm{T}}\mathbf{p}\right)^2 \mathbf{I} \tag{9.3.1.48}$$

and

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{G}\mathbf{E}^{\mathrm{T}}\mathbf{E}\mathbf{G}^{\mathrm{T}} = \mathbf{G}\left(\mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{I} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right)\mathbf{G}^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{p}\mathbf{G}\mathbf{G}^{\mathrm{T}} = \left(\mathbf{p}^{\mathrm{T}}\mathbf{p}\right)^2 \mathbf{I} \tag{9.3.1.49}$$

Thus, only when $\mathbf{p}^{\mathrm{T}}\mathbf{p} = 1$ does $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$.

Forming the product, manipulating the order of terms, and using Eq. 9.1.25,

$$\begin{aligned}
\mathbf{E}(\mathbf{p}_i)\mathbf{p}_j &= \begin{bmatrix} -\mathbf{e}_i & \tilde{\mathbf{e}}_i + e_{0_i}\mathbf{I} \end{bmatrix} \begin{bmatrix} e_{0_j} \\ \mathbf{e}_j \end{bmatrix} \\
&= -e_{0_j}\mathbf{e}_i + \tilde{\mathbf{e}}_i\mathbf{e}_j + e_{0_i}\mathbf{e}_j \\
&= -\left(-e_{0_i}\mathbf{e}_j + \tilde{\mathbf{e}}_j\mathbf{e}_i + e_{0_j}\mathbf{e}_i\right) \\
&= -\begin{bmatrix} -\mathbf{e}_j & \tilde{\mathbf{e}}_j + e_{0_j}\mathbf{I} \end{bmatrix} \begin{bmatrix} e_{0_i} \\ \mathbf{e}_i \end{bmatrix}
\end{aligned} \tag{9.3.1.50}$$

which is simply

$$\mathbf{E}(\mathbf{p}_i)\mathbf{p}_j = -\mathbf{E}(\mathbf{p}_j)\mathbf{p}_i \tag{9.3.1.51}$$

Note that the manipulation required in deriving this identity did not require that either of the vectors $\mathbf{p}_i$ or $\mathbf{p}_j$ satisfy a normalization constraint. A similar manipulation yields the identity

$$\mathbf{G}(\mathbf{p}_i)\mathbf{p}_j = -\mathbf{G}(\mathbf{p}_j)\mathbf{p}_i \tag{9.3.1.52}$$

Direct manipulation, using Eqs. 9.1.23 and 9.1.31, yields

$$\begin{aligned}
\mathbf{E}(\mathbf{p}_i)\mathbf{G}^{\mathrm{T}}(\mathbf{p}_j) &= \begin{bmatrix} -\mathbf{e}_i & \tilde{\mathbf{e}}_i + e_{0_i}\mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{e}_j & -\tilde{\mathbf{e}}_j + e_{0_j}\mathbf{I} \end{bmatrix}^{\mathrm{T}} \\
&= \mathbf{e}_i\mathbf{e}_j^{\mathrm{T}} + \tilde{\mathbf{e}}_i\tilde{\mathbf{e}}_j + e_{0_j}\tilde{\mathbf{e}}_i + e_{0_i}\tilde{\mathbf{e}}_j + e_{0_i}e_{0_j}\mathbf{I} \\
&= \mathbf{e}_j\mathbf{e}_i^{\mathrm{T}} + \tilde{\mathbf{e}}_j\tilde{\mathbf{e}}_i + e_{0_i}\tilde{\mathbf{e}}_j + e_{0_j}\tilde{\mathbf{e}}_i + e_{0_j}e_{0_i}\mathbf{I}
\end{aligned} \tag{9.3.1.53}$$

Since the last two lines are identical, with the exception that indices i and j are interchanged,

$$\mathbf{E}(\mathbf{p}_i)\mathbf{G}^{\mathrm{T}}(\mathbf{p}_j) = \mathbf{E}(\mathbf{p}_j)\mathbf{G}^{\mathrm{T}}(\mathbf{p}_i) \tag{9.3.1.54}$$

where no Euler parameter normalization condition was required. A virtually identical manipulation yields the identity

$$\mathbf{G}(\mathbf{p}_i)\mathbf{E}^{\mathrm{T}}(\mathbf{p}_j) = \mathbf{G}(\mathbf{p}_j)\mathbf{E}^{\mathrm{T}}(\mathbf{p}_i) \tag{9.3.1.55}$$

If the Euler parameter normalization condition $\mathbf{p}^{\mathrm{T}}\mathbf{p} = 1$ is satisfied for all time, it may be differentiated with respect to time to obtain $\mathbf{p}^{\mathrm{T}}\dot{\mathbf{p}} + \dot{\mathbf{p}}^{\mathrm{T}}\mathbf{p} = 2\mathbf{p}^{\mathrm{T}}\dot{\mathbf{p}} = 0$

$$\mathbf{p}^{\mathrm{T}}\dot{\mathbf{p}} = 0 \tag{9.3.1.56}$$

It is important to recall that this condition is only satisfied if the Euler parameter normalization constraint is imposed for all time.

Taking the time derivative of Eq. (9.3.1.37), using the fact that **E** and **G** are linear in **p**, and using Eqs. (9.3.1.50) and (9.3.1.51),

$$\begin{aligned}\dot{\mathbf{A}} &= \dot{\mathbf{E}}\mathbf{G}^{\mathrm{T}} + \mathbf{E}\dot{\mathbf{G}}^{\mathrm{T}} \\ &= \mathbf{E}(\dot{\mathbf{p}})\mathbf{G}^{\mathrm{T}}(\mathbf{p}) + \mathbf{E}(\mathbf{p})\mathbf{G}^{\mathrm{T}}(\dot{\mathbf{p}}) \\ &= 2\mathbf{E}(\mathbf{p})\mathbf{G}^{\mathrm{T}}(\dot{\mathbf{p}}) \\ &= 2\mathbf{E}\dot{\mathbf{G}}^{\mathrm{T}} \\ &= 2\dot{\mathbf{E}}\mathbf{G}^{\mathrm{T}}\end{aligned} \tag{9.3.1.57}$$

Note that this result holds, even if the Euler parameter normalization condition is not satisfied.

Defining the 4×4 matrices

$$\mathbf{R}(\mathbf{a}) \equiv \begin{bmatrix} \mathbf{0} & -\mathbf{a}^{\mathrm{T}} \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \tag{9.3.1.58}$$

$$\mathbf{S}(\mathbf{a}) \equiv \begin{bmatrix} \mathbf{0} & -\mathbf{a}^{\mathrm{T}} \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix} \tag{9.3.1.59}$$

direct manipulation yields

$$
\begin{aligned}
\mathbf{R}(\mathbf{a})\,\mathbf{p} &= \begin{bmatrix} \mathbf{0} & -\mathbf{a}^{\mathrm{T}} \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \begin{bmatrix} \mathrm{e}_0 \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} -\mathbf{a}^{\mathrm{T}}\mathbf{e} \\ \mathrm{e}_0\mathbf{a} + \tilde{\mathbf{a}}\mathbf{e} \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{e}^{\mathrm{T}}\mathbf{a} \\ \mathrm{e}_0\mathbf{a} - \tilde{\mathbf{e}}\mathbf{a} \end{bmatrix} = \begin{bmatrix} -\mathbf{e}^{\mathrm{T}} \\ \mathrm{e}_0 - \tilde{\mathbf{e}} \end{bmatrix} \mathbf{a} = \begin{bmatrix} -\mathbf{e} & \tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I} \end{bmatrix}^{\mathrm{T}} \mathbf{a}
\end{aligned} \qquad (9.3.1.60)
$$

where no use of a normalization condition was required. From the definition of Eq. 9.3.18, this is the identity

$$
\mathbf{R}(\mathbf{a})\,\mathbf{p} = \mathbf{E}^{\mathbf{T}}(\mathbf{p})\,\mathbf{a} \qquad (9.3.1.61)
$$

A similar manipulation yields the identity

$$
\mathbf{S}(\mathbf{a})\,\mathbf{p} = \mathbf{G}^{\mathbf{T}}(\mathbf{p})\,\mathbf{a} \qquad (9.3.1.62)
$$

9.3.2 Euler Parameter Derivative Identities

10-25-01

As in Section 9.3A, let $\mathbf{a},\mathbf{b} \in \mathrm{R}^3$ and $\mathbf{p},\boldsymbol{\gamma} \in \mathrm{R}^4$, where $\mathbf{p},\boldsymbol{\gamma}$ are not necessarily normalized. Expanding the product indicated and taking the derivative with respect to **p**,

$$\begin{aligned}
\left(\mathbf{A}(\mathbf{p})\mathbf{a}'\right)_{\mathbf{p}} &= \left\{\left(e_0{}^2 - \mathbf{e}^{\mathrm{T}}\mathbf{e}\right)\mathbf{a}' + 2\left(\mathbf{e}\mathbf{e}^{\mathrm{T}} + e_0\tilde{\mathbf{e}}\right)\mathbf{a}'\right\}_{\mathbf{p}} \\
&= \left[2e_0\mathbf{a}' + 2\tilde{\mathbf{e}}\mathbf{a}' \quad -2\mathbf{a}'\mathbf{e}^{\mathrm{T}} + 2\mathbf{e}^{\mathrm{T}}\mathbf{a}'\mathbf{I} + 2\mathbf{e}\mathbf{a}'^{\mathrm{T}} - 2e_0\tilde{\mathbf{a}}'\right] \\
&= \left[2e_0\mathbf{a}' + 2\tilde{\mathbf{e}}\mathbf{a}' \quad -2\tilde{\mathbf{e}}\tilde{\mathbf{a}}' + 2\mathbf{e}\mathbf{a}'^{\mathrm{T}} - 2e_0\tilde{\mathbf{a}}'\right] \\
&= 2\left[\left(e_0\mathbf{I} + \tilde{\mathbf{e}}\right)\mathbf{a}' \quad \mathbf{e}\mathbf{a}'^{\mathrm{T}} - \left(e_0\mathbf{I} + \tilde{\mathbf{e}}\right)\tilde{\mathbf{a}}'\right]
\end{aligned} \tag{9.3.2.1}$$

where $\left(\mathbf{e}^{\mathrm{T}}\mathbf{e}\mathbf{a}'\right)_{\mathbf{e}} = \left(\mathbf{a}'\mathbf{e}^{\mathrm{T}}\mathbf{e}\right)_{\mathbf{e}} = \mathbf{a}'\left(\mathbf{e}^{\mathrm{T}}\mathbf{e}\right)_{\mathbf{e}} = 2\mathbf{a}'\mathbf{e}^{\mathrm{T}}$, $\left(\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{a}'\right)_{\mathbf{e}} = \left\{\mathbf{e}\left(\mathbf{a}'^{\mathrm{T}}\mathbf{e}\right)\right\}_{\mathbf{e}} = \mathbf{e}\mathbf{a}'^{\mathrm{T}} + \left(\mathbf{a}'^{\mathrm{T}}\mathbf{e}\right)\mathbf{I}$, Eq. 9.1.25, and Eq. 9.1.28 have been used. Defining

$$\mathbf{B}(\mathbf{p},\mathbf{a}') \equiv 2\left[\left(e_0\mathbf{I} + \tilde{\mathbf{e}}\right)\mathbf{a}' \quad \mathbf{e}\mathbf{a}'^{\mathrm{T}} - \left(e_0\mathbf{I} + \tilde{\mathbf{e}}\right)\tilde{\mathbf{a}}'\right] \tag{9.3.2.2}$$

the following derivative identity is obtained from Eq. (9.3.2.1):

$$\left(\mathbf{A}(\mathbf{p})\mathbf{a}'\right)_{\mathbf{p}} = \mathbf{B}(\mathbf{p},\mathbf{a}') \tag{9.3.2.3}$$

Expanding the product indicated, using Eqs. 9.1.25 and 9.1.31,

$$\begin{aligned}
\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j &= 2\left[\left(e_{0_i}\mathbf{I} + \tilde{\mathbf{e}}_i\right)\mathbf{a}' \quad \mathbf{e}_i\mathbf{a}'^{\mathrm{T}} - \left(e_{0_i}\mathbf{I} + \tilde{\mathbf{e}}_i\right)\tilde{\mathbf{a}}'\right]\mathbf{p}_j \\
&= 2\left[\left(e_{0_i}e_{0_j}\mathbf{I} + e_{0_j}\tilde{\mathbf{e}}_i\right)\mathbf{a}' + \mathbf{e}_i\mathbf{a}'^{\mathrm{T}}\mathbf{e}_j + e_{0_i}\tilde{\mathbf{e}}_j\mathbf{a}' + \tilde{\mathbf{e}}_i\tilde{\mathbf{e}}_j\mathbf{a}'\right] \\
&= 2\left[\left(e_{0_j}e_{0_i}\mathbf{I} + e_{0_j}\tilde{\mathbf{e}}_i\right)\mathbf{a}' + \mathbf{e}_i\mathbf{e}_j{}^{\mathrm{T}}\mathbf{a}' + e_{0_i}\tilde{\mathbf{e}}_j\mathbf{a}' + \tilde{\mathbf{e}}_i\tilde{\mathbf{e}}_j\mathbf{a}'\right] \\
&= 2\left[\left(e_{0_j}e_{0_i}\mathbf{I} + e_{0_i}\tilde{\mathbf{e}}_j\right)\mathbf{a}' + \mathbf{e}_j\mathbf{e}_i{}^{\mathrm{T}}\mathbf{a}' + e_{0_j}\tilde{\mathbf{e}}_i\mathbf{a}' + \tilde{\mathbf{e}}_j\tilde{\mathbf{e}}_i\mathbf{a}'\right] \\
&= 2\left[\left(e_{0_j}e_{0_i}\mathbf{I} + e_{0_i}\tilde{\mathbf{e}}_j\right)\mathbf{a}' + \mathbf{e}_j\mathbf{a}'^{\mathrm{T}}\mathbf{e}_i + e_{0_j}\tilde{\mathbf{e}}_i\mathbf{a}' + \tilde{\mathbf{e}}_j\tilde{\mathbf{e}}_i\mathbf{a}'\right]
\end{aligned} \tag{9.3.2.4}$$

Since the second and last expressions on the right are identical, with the exception that the roles of i and j have been reversed, this yields the identity

$$\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j = \mathbf{B}(\mathbf{p}_j,\mathbf{a}')\mathbf{p}_i \tag{9.3.2.5}$$

It is important to note that this result was obtained with no requirement that either $\mathbf{p}_i$ or $\mathbf{p}_j$ be normalized.

Using the same approach employed to obtain Eq. (9.3.2.3),

$$\begin{aligned}\left(\mathbf{A}^T(\mathbf{p})\mathbf{a}\right)_{\mathbf{p}} &= \left(\left(e_0^2 - \mathbf{e}^T\mathbf{e}\right)\mathbf{a} + 2\mathbf{e}\mathbf{e}^T\mathbf{a} - 2e_0\tilde{\mathbf{e}}\mathbf{a}\right)_{\mathbf{p}} \\ &= \left[2e_0\mathbf{a} - 2\tilde{\mathbf{e}}\mathbf{a} \quad -2\mathbf{a}\mathbf{e}^T + 2\mathbf{a}^T\mathbf{e}\mathbf{I} + 2\mathbf{e}\mathbf{a}^T + 2e_0\tilde{\mathbf{a}}\right] \\ &= 2\left[\left(e_0\mathbf{I} - \tilde{\mathbf{e}}\right)\mathbf{a} \quad +\mathbf{e}\mathbf{a}^T - \tilde{\mathbf{e}}\tilde{\mathbf{a}} + e_0\tilde{\mathbf{a}}\right] \\ &= 2\left[\left(e_0\mathbf{I} - \tilde{\mathbf{e}}\right)\mathbf{a} \quad +\mathbf{e}\mathbf{a}^T + \left(e_0\mathbf{I} - \tilde{\mathbf{e}}\right)\tilde{\mathbf{a}}\right]\end{aligned} \tag{9.3.2.6}$$

and a derivative relation is obtained, with no requirement for normalization of **p**,

$$\left(\mathbf{A}^T(\mathbf{p})\mathbf{a}\right)_{\mathbf{p}} = \mathbf{C}(\mathbf{p},\mathbf{a}) \tag{9.3.2.7}$$

where

$$\mathbf{C}(\mathbf{p},\mathbf{a}) \equiv 2\left[\left(e_0\mathbf{I} - \tilde{\mathbf{e}}\right)\mathbf{a} \quad \mathbf{e}\mathbf{a}^T + \left(e_0\mathbf{I} - \tilde{\mathbf{e}}\right)\tilde{\mathbf{a}}\right] \tag{9.3.2.8}$$

Using the same approach employed to derive Eq. (9.3.2.5), the following identity is obtained:

$$\mathbf{C}(\mathbf{p}_i,\mathbf{a})\mathbf{p}_j = \mathbf{C}(\mathbf{p}_j,\mathbf{a})\mathbf{p}_i \tag{9.3.2.9}$$

Equation (9.3.2.5) provides an easy result for the derivative

$$\left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j\right)_{\mathbf{p}_i} = \left(\mathbf{B}(\mathbf{p}_j,\mathbf{a}')\mathbf{p}_i\right)_{\mathbf{p}_i}$$

namely,

$$\left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}')\mathbf{p}_j\right)_{\mathbf{p}_i} = \mathbf{B}(\mathbf{p}_j,\mathbf{a}') \tag{9.3.2.10}$$

Similarly, Eq. (9.3.2.9) yields

$$\left(\mathbf{C}(\mathbf{p}_i,\mathbf{a})\mathbf{p}_j\right)_{\mathbf{p}_i} = \mathbf{C}(\mathbf{p}_j,\mathbf{a}) \tag{9.3.2.11}$$

Expanding the product indicated, using Eqs. 9.1.23, 25, 31, and 28,

$$
\begin{aligned}
\mathbf{B}^{\mathrm{T}}(\mathbf{p},\mathbf{a}')\,\mathbf{b} &= 2\left[(\mathrm{e}_0\mathbf{I}+\tilde{\mathbf{e}})\mathbf{a}' \quad \mathbf{e}\mathbf{a}'^{\mathrm{T}} - (\mathrm{e}_0\mathbf{I}+\tilde{\mathbf{e}})\tilde{\mathbf{a}}'\right]^{\mathrm{T}}\mathbf{b} \\
&= 2\begin{bmatrix} \mathrm{e}_0\mathbf{a}'^{\mathrm{T}}\mathbf{b} - \mathbf{a}'^{\mathrm{T}}\tilde{\mathbf{e}}\mathbf{b} \\ \mathbf{a}'\mathbf{e}^{\mathrm{T}}\mathbf{b} + \mathrm{e}_0\tilde{\mathbf{a}}'\mathbf{b} - \tilde{\mathbf{a}}'\tilde{\mathbf{e}}\mathbf{b} \end{bmatrix} = 2\begin{bmatrix} \mathrm{e}_0\mathbf{a}'^{\mathrm{T}}\mathbf{b} + \mathbf{a}'^{\mathrm{T}}\tilde{\mathbf{b}}\mathbf{e} \\ \mathrm{e}_0\tilde{\mathbf{a}}'\mathbf{b} + \mathbf{a}'\mathbf{b}^{\mathrm{T}}\mathbf{e} + \tilde{\mathbf{a}}'\tilde{\mathbf{b}}\mathbf{e} \end{bmatrix} \\
&= 2\begin{bmatrix} \mathbf{a}'^{\mathrm{T}}\mathbf{b} & \mathbf{a}'^{\mathrm{T}}\tilde{\mathbf{b}} \\ \tilde{\mathbf{a}}'\mathbf{b} & \mathbf{a}'\mathbf{b}^{\mathrm{T}} + \tilde{\mathbf{a}}'\tilde{\mathbf{b}} \end{bmatrix}\mathbf{p} = 2\begin{bmatrix} \mathbf{a}'^{\mathrm{T}}\mathbf{b} & \mathbf{a}'^{\mathrm{T}}\tilde{\mathbf{b}} \\ \tilde{\mathbf{a}}'\mathbf{b} & \mathbf{a}'\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}'^{\mathrm{T}} - \mathbf{a}'^{\mathrm{T}}\mathbf{b}\mathbf{I} \end{bmatrix}\mathbf{p}
\end{aligned} \tag{9.3.2.12}
$$

where no normalization condition on the vector **p** is required. This may be written as the identity

$$
\mathbf{B}^{\mathrm{T}}(\mathbf{p},\mathbf{a}')\,\mathbf{b} = \mathbf{K}(\mathbf{a}',\mathbf{b})\,\mathbf{p} \tag{9.3.2.13}
$$

where

$$
\mathbf{K}(\mathbf{a}',\mathbf{b}) \equiv 2\begin{bmatrix} \mathbf{a}'^{\mathrm{T}}\mathbf{b} & \mathbf{a}'^{\mathrm{T}}\tilde{\mathbf{b}} \\ \tilde{\mathbf{a}}'\mathbf{b} & \mathbf{a}'\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}'^{\mathrm{T}} - \mathbf{a}'^{\mathrm{T}}\mathbf{b}\mathbf{I} \end{bmatrix} \tag{9.3.2.14}
$$

A direct manipulation shows that $\mathbf{K}(\mathbf{a}',\mathbf{b})$ is symmetric.

Direct use of Eq. (9.3.2.13) yields the derivative relation

$$
\left(\mathbf{B}^{\mathrm{T}}(\mathbf{p},\mathbf{a}')\,\boldsymbol{\gamma}\right)_{\mathbf{p}} = \mathbf{K}(\mathbf{a}',\boldsymbol{\gamma}) \tag{9.3.2.15}
$$

Manipulations identical to those used in obtaining Eq. (9.3.2.12) yield

$$
\begin{aligned}
\mathbf{C}(\mathbf{p},\mathbf{a})^{\mathrm{T}}\,\mathbf{b} &= 2\left[(\mathrm{e}_0\mathbf{I}-\tilde{\mathbf{e}})\mathbf{a} \quad \mathbf{e}\mathbf{a}^{\mathrm{T}} + (\mathrm{e}_0\mathbf{I}-\tilde{\mathbf{e}})\tilde{\mathbf{a}}\right]^{\mathrm{T}}\mathbf{b} \\
&= 2\begin{bmatrix} \mathrm{e}_0\mathbf{a}^{\mathrm{T}}\mathbf{b} + \mathbf{a}^{\mathrm{T}}\tilde{\mathbf{e}}\mathbf{b} \\ \mathbf{a}\mathbf{e}^{\mathrm{T}}\mathbf{b} - \mathrm{e}_0\tilde{\mathbf{a}}\mathbf{b} - \tilde{\mathbf{a}}\tilde{\mathbf{e}}\mathbf{b} \end{bmatrix} = 2\begin{bmatrix} \mathrm{e}_0\mathbf{a}^{\mathrm{T}}\mathbf{b} - \mathbf{a}^{\mathrm{T}}\tilde{\mathbf{b}}\mathbf{e} \\ \mathbf{a}\mathbf{b}^{\mathrm{T}}\mathbf{e} - \mathrm{e}_0\tilde{\mathbf{a}}\mathbf{b} + \tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{e} \end{bmatrix} \\
&= 2\begin{bmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} & -\mathbf{a}^{\mathrm{T}}\tilde{\mathbf{b}} \\ -\tilde{\mathbf{a}}\mathbf{b} & \mathbf{a}\mathbf{b}^{\mathrm{T}} + \tilde{\mathbf{a}}\tilde{\mathbf{b}} \end{bmatrix}\mathbf{p} = 2\begin{bmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} & -\mathbf{a}^{\mathrm{T}}\tilde{\mathbf{b}} \\ -\tilde{\mathbf{a}}\mathbf{b} & \mathbf{a}\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}^{\mathrm{T}} - \mathbf{a}^{\mathrm{T}}\mathbf{b}\mathbf{I} \end{bmatrix}\mathbf{p}
\end{aligned} \tag{9.3.2.16}
$$

where no normalization condition on the vector **p** is required. This may be written as the identity

$$
\mathbf{C}(\mathbf{p},\mathbf{a})^{\mathrm{T}}\,\mathbf{b} = \mathbf{L}(\mathbf{a},\mathbf{b})\,\mathbf{p} \tag{9.3.2.17}
$$

where

$$
\mathbf{L}(\mathbf{a},\mathbf{b}) \equiv 2\begin{bmatrix} \mathbf{a}^{\mathrm{T}}\mathbf{b} & -\mathbf{a}^{\mathrm{T}}\tilde{\mathbf{b}} \\ -\tilde{\mathbf{a}}\mathbf{b} & \mathbf{a}\mathbf{b}^{\mathrm{T}} + \mathbf{b}\mathbf{a}^{\mathrm{T}} - \mathbf{a}^{\mathrm{T}}\mathbf{b}\mathbf{I} \end{bmatrix} \tag{9.3.2.18}
$$

Equation (9.3.2.17) yields the derivative relation

$$\left(\mathbf{C}(\mathbf{p},\mathbf{a})^{\mathrm{T}}\,\mathbf{b}\right)_{\mathbf{p}} = \mathbf{L}(\mathbf{a},\mathbf{b}) \qquad (9.3.2.19)$$

Using the identities of Eqs. 9.3A.51 and 9.3A.512 of Section 9.3 A, the following derivatives are directly obtained:

$$\left(\mathbf{E}(\mathbf{p})\,\boldsymbol{\gamma}\right)_{\mathbf{p}} = -\mathbf{E}(\boldsymbol{\gamma}) \qquad (9.3.2.20)$$

$$\left(\mathbf{G}(\mathbf{p})\,\boldsymbol{\gamma}\right)_{\mathbf{p}} = -\mathbf{G}(\boldsymbol{\gamma}) \qquad (9.3.2.21)$$

Note that these results do not require that the vector **p** be normalized.

Expanding the product indicated, using Eqs. 9.1.23 and 25,

$$\mathbf{E}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a} = \left[-\mathbf{e} \quad \tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I}\right]^{\mathrm{T}}\mathbf{a} = \begin{bmatrix} -\mathbf{e}^{\mathrm{T}}\mathbf{a} \\ -\tilde{\mathbf{e}}\mathbf{a} + \mathrm{e}_0\mathbf{a} \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{a}^{\mathrm{T}} \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix}\mathbf{p} \qquad (9.3.2.22)$$

Using the definition of Eq. 0.59 of Section 9.3 A, this yields the identity

$$\mathbf{E}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a} = \mathbf{R}(\mathbf{a})\,\mathbf{p} \qquad (9.3.2.23)$$

which, upon differentiation, yields the derivative relationship

$$\left(\mathbf{E}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a}\right)_{\mathbf{p}} = \mathbf{R}(\mathbf{a}) \qquad (9.3.2.24)$$

Using the same manipulations above yields analogous identities and derivative relationships,

$$\mathbf{G}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a} = \left[-\mathbf{e} \quad -\tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I}\right]^{\mathrm{T}}\mathbf{a} = \begin{bmatrix} -\mathbf{e}^{\mathrm{T}}\mathbf{a} \\ \tilde{\mathbf{e}}\mathbf{a} + \mathrm{e}_0\mathbf{a} \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{a}^{\mathrm{T}} \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix}\mathbf{p} \qquad (9.3.2.25)$$

$$\mathbf{G}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a} = \mathbf{S}(\mathbf{a})\,\mathbf{p} \qquad (9.3.2.26)$$

$$\left(\mathbf{G}^{\mathrm{T}}(\mathbf{p})\,\mathbf{a}\right)_{\mathbf{p}} = \mathbf{S}(\mathbf{a}) \qquad (9.3.2.27)$$

where no normalization condition on the vector **p** is required.

9.3.3 Time Derivatives and Variations of Euler Parameters

10-31-01

Since orientation of the body and space in applications varies with time, the orientation transformation matrix is a function of time; i.e., $\mathbf{A} = \mathbf{A}(t)$. Thus, the Euler parameter vector $\mathbf{p}$ that defines $\mathbf{A}(t)$ must be a function of time; i.e.,

$$\mathbf{A}(t) = \mathbf{A}(\mathbf{p}(t)) \tag{9.3.3.1}$$

Since the Euler parameter normalization constraint must hold at all orientations; i.e. for all time,

$$\mathbf{p}^T(t)\mathbf{p}(t) = 1 \tag{9.3.3.2}$$

For all time.

Time Derivatives of Euler Parameters and Angular Velocity

Equation (9.3.3.2) may be differentiated with respect to time to obtain $\mathbf{p}^T\dot{\mathbf{p}} + \dot{\mathbf{p}}^T\mathbf{p} = 2\mathbf{p}^T\dot{\mathbf{p}} = 0$, or equivalently

$$\mathbf{p}^T\dot{\mathbf{p}} = 0 \tag{9.3.3.3}$$

It is important to recall that this condition is only satisfied if the Euler parameter normalization constraint is imposed for all time.

Taking the time derivative of $\mathbf{A}(t) = \mathbf{E}(t)\mathbf{G}^T(t)$ Eq., using the fact that $\mathbf{E}$ and $\mathbf{G}$ are linear in $\mathbf{p}$ and the identity $\mathbf{E}(\mathbf{p}_i)\mathbf{G}^T(\mathbf{p}_j) = \mathbf{E}(\mathbf{p}_j)\mathbf{G}^T(\mathbf{p}_i)$ with $\mathbf{p}_i = \dot{\mathbf{p}}$ and $\mathbf{p}_j = \mathbf{p}$,

$$\begin{aligned}\dot{\mathbf{A}} &= \dot{\mathbf{E}}\mathbf{G}^T + \mathbf{E}\dot{\mathbf{G}}^T \\ &= \mathbf{E}(\dot{\mathbf{p}})\mathbf{G}^T(\mathbf{p}) + \mathbf{E}(\mathbf{p})\mathbf{G}^T(\dot{\mathbf{p}}) \\ &= 2\mathbf{E}(\mathbf{p})\mathbf{G}^T(\dot{\mathbf{p}}) \\ &= 2\mathbf{E}\dot{\mathbf{G}}^T \\ &= 2\dot{\mathbf{E}}\mathbf{G}^T\end{aligned} \tag{9.3.3.4}$$

where the last result is obtained using the above identity, but with $\mathbf{p}_i = \mathbf{p}$ and $\mathbf{p}_j = \dot{\mathbf{p}}$ Note that this result holds, even if the Euler parameter normalization condition is not satisfied.

The product $\mathbf{G}\dot{\mathbf{p}}$ may be expanded as

$$\mathbf{G}\dot{\mathbf{p}} = \begin{bmatrix} -\mathbf{e} & -\tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathrm{e}}_0 \\ \dot{\mathbf{e}} \end{bmatrix} = -\dot{\mathrm{e}}_0\mathbf{e} - \tilde{\mathbf{e}}\dot{\mathbf{e}} + \mathrm{e}_0\dot{\mathbf{e}} \tag{9.3.3.5}$$

Applying the tilde operator to both sides of this equation and manipulating yields

$$\begin{aligned} \widetilde{\mathbf{G}\dot{\mathbf{p}}} &= \widetilde{-\dot{\mathrm{e}}_0\mathbf{e} - \tilde{\mathbf{e}}\dot{\mathbf{e}} + \mathrm{e}_0\dot{\mathbf{e}}} \\ &= -\dot{\mathrm{e}}_0\mathbf{e} - \widetilde{\tilde{\mathbf{e}}\dot{\mathbf{e}}} + \mathrm{e}_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{\mathrm{e}}_0\mathbf{e} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \tilde{\dot{\mathbf{e}}}\tilde{\mathbf{e}} + \mathrm{e}_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{\mathrm{e}}_0\mathbf{e} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \mathbf{e}\dot{\mathbf{e}}^{\mathrm{T}} - \dot{\mathbf{e}}^{\mathrm{T}}\mathbf{e}\mathbf{I} + \mathrm{e}_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{\mathrm{e}}_0\mathbf{e} - \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} + \mathbf{e}\dot{\mathbf{e}}^{\mathrm{T}} + \mathrm{e}_0\dot{\mathrm{e}}_0\mathbf{I} + \mathrm{e}_0\tilde{\dot{\mathbf{e}}} \\ &= \begin{bmatrix} -\mathbf{e} & -\tilde{\mathbf{e}} + \mathrm{e}_0\mathbf{I} \end{bmatrix} \begin{bmatrix} -\dot{\mathbf{e}}^{\mathrm{T}} \\ \tilde{\dot{\mathbf{e}}} + \dot{\mathrm{e}}_0\mathbf{I} \end{bmatrix} \\ &= \mathbf{G}\dot{\mathbf{G}}^{\mathrm{T}} \end{aligned} \tag{9.3.3.6}$$

Recalling the definition of angular velocity, $\tilde{\boldsymbol{\omega}}' = \mathbf{A}^{\mathrm{T}}\dot{\mathbf{A}}$, and using Eq. (9.3.3.4),

$$\tilde{\boldsymbol{\omega}}' = \mathbf{A}^{\mathrm{T}}\dot{\mathbf{A}} = 2\mathbf{G}\mathbf{E}^{\mathrm{T}}\mathbf{E}\dot{\mathbf{G}}^{\mathrm{T}} = 2\mathbf{G}\left(\mathbf{I} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right)\dot{\mathbf{G}}^{\mathrm{T}} = 2\mathbf{G}\dot{\mathbf{G}}^{\mathrm{T}} \tag{9.3.3.7}$$

Equations (9.3.3.6) and(9.3.3.7) yield

$$\tilde{\boldsymbol{\omega}}' = \widetilde{2\mathbf{G}\dot{\mathbf{p}}} \tag{9.3.3.8}$$

or, equivalently, the desired relation

$$\boldsymbol{\omega}' = 2\mathbf{G}\dot{\mathbf{p}} \tag{9.3.3.9}$$

Multiplying both sides of Eq. (9.3.3.9) on the left by $\mathbf{G}^{\mathrm{T}}$ yields

$$\mathbf{G}^{\mathrm{T}}\boldsymbol{\omega}' = 2\mathbf{G}^{\mathrm{T}}\mathbf{G}\dot{\mathbf{p}} = 2\left(\mathbf{I} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right)\dot{\mathbf{p}} \tag{9.3.3.10}$$

Providing the Euler parameter normalization constraint is satisfied, in particular Eq. (9.3.3.3), this yields in inverse relationship

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^{\mathrm{T}}\boldsymbol{\omega}' \tag{9.3.3.11}$$

Thus, even though the matrix $\mathbf{G}$ is not square, it behaves much like and orthogonal matrix in Eqs. (9.3.3.9) and (9.3.3.11).

From Eq. (9.3.3.9),

$$\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega}' = 2\mathbf{E}\mathbf{G}^{\mathrm{T}}\mathbf{G}\dot{\mathbf{p}} = 2\mathbf{E}\left(\mathbf{I} - \mathbf{p}\mathbf{p}^{\mathrm{T}}\right)\dot{\mathbf{p}} \tag{9.3.3.12}$$

Providing Euler parameter constraints are satisfied, in particular Eq. (9.3.3.3), this reduces to

$$\boldsymbol{\omega} = 2\mathbf{E}\dot{\mathbf{p}} \tag{9.3.3.13}$$

and the inverse relation follows, just as in Eq. (9.3.3.11); i.e.,

$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^{\mathrm{T}}\boldsymbol{\omega} \tag{9.3.3.14}$$

Variations in Euler Parameters and Virtual Rotations

Repeating the manipulations of Eqs. (9.3.3.3) through (9.3.3.14), replacing $\frac{d}{dt}$ with δ, yields

$$\mathbf{p}^{\mathrm{T}}\delta\mathbf{p} = 0 \tag{9.3.3.15}$$

$$\delta\mathbf{A} = 2\mathbf{E}\delta\mathbf{G}^{\mathrm{T}} = 2\delta\mathbf{E}\mathbf{G}^{\mathrm{T}} \tag{9.3.3.16}$$

$$\widetilde{\mathbf{G}\delta\mathbf{p}} = \mathbf{G}\delta\mathbf{G}^{\mathrm{T}} \tag{9.3.3.17}$$

$$\delta\boldsymbol{\pi}' = 2\mathbf{G}\delta\mathbf{p} \tag{9.3.3.18}$$

$$\delta\mathbf{p} = \frac{1}{2}\mathbf{G}^{\mathrm{T}}\delta\tilde{\boldsymbol{\pi}}' \tag{9.3.3.19}$$

Each of Eqs. (9.3.3.15) through (9.3.3.19) is valid only if the Euler parameter normalization constraint is satisfied.

9.6 Kinematic Analysis

10-21-01

Constraint Equations

A kinematic model of a mechanism comprised of nb bodies is defined by a set of $7 \times \text{nb}$ generalized coordinates $\mathbf{q}$ and constraint equations of the form

$$\mathbf{\Phi}(\mathbf{q},t) \equiv \begin{bmatrix} \mathbf{\Phi}^{K}(\mathbf{q}) \\ \mathbf{\Phi}^{D}(\mathbf{q},t) \\ \mathbf{\Phi}^{\mathbf{p}}(\mathbf{q}) \end{bmatrix} = \mathbf{0} \tag{9.6.1}$$

where kinematic constraint equations comprised of combinations of basic kinematic constraint equations of Section 9.4 are

$$\mathbf{\Phi}^{K}(\mathbf{q}) = \mathbf{0} \tag{9.6.2}$$

Euler parameter normalization constraints are nb equations of the form

$$\mathbf{\Phi}^{\mathbf{p}}(\mathbf{q}) = \begin{bmatrix} \mathbf{p}_1^T \mathbf{p}_1 - 1 \\ \vdots \\ \mathbf{p}_{nb}^T \mathbf{p}_{nb} - 1 \end{bmatrix} = \mathbf{0} \tag{9.6.3}$$

and driving constraints comprised of combinations of basic driving constraint equations of Section 9.5 are

$$\mathbf{\Phi}^{D}(\mathbf{q},t) = \mathbf{0} \tag{9.6.4}$$

Let each of the constraint equations of Eq. (9.6.1) have k continuous derivatives with respect to both $\mathbf{q}$ and t. Define the Jacobian matrix of the constraint function as

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t) = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{q}}^{K}(\mathbf{q}) \\ \mathbf{\Phi}_{\mathbf{q}}^{D}(\mathbf{q},t) \\ \mathbf{\Phi}_{\mathbf{q}}^{\mathbf{p}}(\mathbf{q}) \end{bmatrix} \tag{9.6.5}$$

At an initial time t_0, let $\mathbf{q}_0$ denote an assembled configuration of the mechanism; i.e.,

$$\mathbf{\Phi}(\mathbf{q}_0,t_0) = \mathbf{0} \tag{9.6.6}$$

Under the above conditions, if the Jacobian is nonsingular at the initial time and assembled configuration; i.e.,

$$\left|\mathbf{\Phi}_{\mathbf{q}}\left(\mathbf{q}_0,t_0\right)\right| \neq 0 \qquad (9.6.7)$$

the implicit function theorem [**reference**] guarantees that Eq. (9.6.1) has a unique, k-times continuously differentiable solution $\mathbf{q}(t)$; i.e., $\mathbf{\Phi}(\mathbf{q}(t),t)=\mathbf{0}$, for all t in a neighborhood of t_0.

Differentiating Eq. (9.6.1) with respect to time yields the velocity equations

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t)\dot{\mathbf{q}} = -\mathbf{\Phi}_t(\mathbf{q},t) \equiv \boldsymbol{\nu}(\mathbf{q},t) \qquad (9.6.8)$$

Since the kinematic and Euler parameter constraint functions of Eq. (9.6.1) do not depend explicitly on time,

$$\boldsymbol{\nu}(\mathbf{q},t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{\Phi}_t^{D}(\mathbf{q},t) \\ \mathbf{0} \end{bmatrix} \qquad (9.6.9)$$

Under the assumption that the Jacobian is nonsingular for all time, Eq. (9.6.8) has a $(k-1)$-times continuously differentiable solution $\dot{\mathbf{q}}(t)$.

Differentiating Eq. (9.6.8) with respect to time yields the acceleration equation

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t)\ddot{\mathbf{q}} = -\left(\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q},t)\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{t\mathbf{q}}(\mathbf{q},t)\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}(\mathbf{q},t) \equiv \boldsymbol{\gamma}(\mathbf{q},\dot{\mathbf{q}},t) \qquad (9.6.10)$$

Under the assumption that the Jacobian is nonsingular for all time, Eq. (9.6.10) has a $(k-2)$-times continuously differentiable solution $\ddot{\mathbf{q}}(t)$.

Position Analysis

Since Eq. (9.6.1) is a system of $7\times nb$ highly nonlinear equations in a vector $\mathbf{q}$ containing $7\times nb$ variables, there is little hope of finding an analytical solution $\mathbf{q}(t)$, even though the implicit function theorem guarantees existence of a unique solution with k-continuous derivatives. It is necessary, therefore, to resort to construction of a numerical approximate solution at discrete points and time.

At $t=t_0$, let an estimate of the solution of Eq. (9.6.1) be $\mathbf{q}^0 \approx \mathbf{q}_0 = \mathbf{q}(t_0)$. Using Newton's method [**reference**], the following linear equations are solved:

$$\mathbf{\Phi_q}\left(\mathbf{q}^0,t_0\right)\Delta\mathbf{q}^0 = -\mathbf{\Phi}\left(\mathbf{q}^0,t_0\right) \tag{9.6.11}$$

A new estimate of the solution is defined as

$$\mathbf{q}^1 = \mathbf{q}^0 + \Delta\mathbf{q}^0 \tag{9.6.12}$$

This process continues, with the approximate solution of Eq. (9.6.1) at t_0 denoted $\mathbf{q}^i$. The subsequent approximation is defined by

$$\begin{aligned} &\mathbf{\Phi_q}\left(\mathbf{q}^i,t_0\right)\Delta\mathbf{q}^i = -\mathbf{\Phi}\left(\mathbf{q}^i,t_0\right) \\ &\mathbf{q}^{i+1} = \mathbf{q}^i + \Delta\mathbf{q}^i \end{aligned} \tag{9.6.13}$$

The process continues until the following convergence criteria are met:

$$\begin{aligned} &\left\|\mathbf{\Phi}\left(\mathbf{q}^{i+1},t_0\right)\right\| \le \varepsilon \\ &\left\|\Delta\mathbf{q}^{i+1}\right\| \le \delta \end{aligned} \tag{9.6.14}$$

where ε and δ are solution error tolerances.

If the Jacobian and right side of Eq.(9.6.13) are evaluated accurately and the Jacobian on the left side of Eq.(9.6.13) is nonsingular at all iterations, then Newton's method has attractive convergence properties. In particular, for sufficiently small error $\left\|\mathbf{q}^0 - \mathbf{q}\left(t_0\right)\right\|$ in the initial estimate, the method will converge quadratically. This is equivalent to the condition that there exists a constant c such that for iteration number i sufficiently large,

$$\left\|\mathbf{q}^{i+1} - \mathbf{q}\left(t_0\right)\right\| \le c\left\|\mathbf{q}^i - \mathbf{q}\left(t_0\right)\right\|^2 \tag{9.6.15}$$

Example

It is important to evaluate all terms in Eq.(9.6.13) accurately, if Newton's method is to perform according to theory. There is a great danger that identities involving Euler parameter's that hold only when **p** is normalized are used to simplify either the Jacobian or the right side of Eq. (9.6.13), leading to the inaccuracy when the estimate for **p** does not satisfy the normalization constraint.

To illustrate this problem, consider the simple example in which the unknown vector $\mathbf{q} = [q_1, q_2]^T$ is to satisfy the constraint

$$\mathbf{q}^T\mathbf{q} = q_1^2 + q_2^2 = 1$$

Consider the function

$$f(\mathbf{q}) = q_1^2$$

Define next the function

$$\bar{f}(\mathbf{q}) \equiv 1 - q_2^2$$

If $\mathbf{q}$ satisfies the normalization constraint, then $\bar{f}(\mathbf{q}) = f(\mathbf{q})$. However, the two functions have distinctly different Jacobians; i.e.,

$$f_{\mathbf{q}} = \begin{bmatrix} 2q_1 & 0 \end{bmatrix}$$

$$\bar{f}_{\mathbf{q}} = \begin{bmatrix} 0 & -2q_2 \end{bmatrix}$$

Consider next the problem of solving the nonlinear equations

$$\mathbf{\Phi}(q) \equiv \begin{bmatrix} f(\mathbf{q}) - \frac{1}{4} \\ q_1^2 + q_2^2 - 1 \end{bmatrix} = \begin{bmatrix} q_1^2 - \frac{1}{4} \\ q_1^2 + q_2^2 - 1 \end{bmatrix} = \mathbf{0}$$

The Newton equations are

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^i)\Delta\mathbf{q}^i = \begin{bmatrix} 2q_1^i & 0 \\ 2q_1^i & 2q_2^i \end{bmatrix}\Delta\mathbf{q}^i = -\begin{bmatrix} q_1^{i\,2} - \frac{1}{4} \\ q_1^{i\,2} + q_2^{i\,2} - 1 \end{bmatrix} = -\mathbf{\Phi}(\mathbf{q}^i)$$

If $\bar{f}(\mathbf{q}) \equiv 1 - q_2^2$ is used on the right side of the Newton equations; i.e.,

$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^i)\Delta\mathbf{q}^i = \begin{bmatrix} 2q_1^i & 0 \\ 2q_1^i & 2q_2^i \end{bmatrix}\Delta\mathbf{q}^i = -\begin{bmatrix} \frac{3}{4} - q_2^2 \\ q_1^{i\,2} + q_2^{i\,2} - 1 \end{bmatrix} = -\bar{\mathbf{\Phi}}(\mathbf{q}^i)$$

then there is no reason to believe that the Newton method would necessarily converge. To evaluate this possibility, numerical solutions are attempted with each of the above formulations and the Newton correction formula

$$\mathbf{q}^{i+1} = \mathbf{q}^i + \Delta\mathbf{q}^i$$

A time grid is defined by the analyst/engineer, with $t_{i+1} = t_i + h,\ i = 0,1,2,...$, where h is the time step, generally a small positive number. Given the numerical solution $\mathbf{q}^i$, along with $\dot{\mathbf{q}}^i$ and $\ddot{\mathbf{q}}^i$ as determined below, at t_i, an approximate solution at t_{i+1} is estimated as

$$\mathbf{q}^0 = \mathbf{q}^i + h\dot{\mathbf{q}}^i + \frac{1}{2}h^2\ddot{\mathbf{q}}^i \approx \mathbf{q}(t_{i+1}) \tag{9.6.16}$$

and Newton iteration is carried out, as follows:

$$\begin{aligned} &\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}^j, t_{i+1})\Delta\mathbf{q}^j = -\mathbf{\Phi}(\mathbf{q}^j, t_{i+1}) \\ &\mathbf{q}^{j+1} = \mathbf{q}^j + \Delta\mathbf{q}^j \end{aligned} \tag{9.6.17}$$

for $j = 0,1,\ldots$. In this way, approximate solutions for **q** are found at each specified time step.

Velocity Analysis

Beginning at t_0, velocities $\dot{\mathbf{q}}^i \approx \dot{\mathbf{q}}(t_i)$, i = 0, 1, 2,…; i.e., and the specified time steps, are determined by solving the velocity equations

$$\mathbf{\Phi_q}\left(\mathbf{q}^i, t_i\right)\dot{\mathbf{q}}^i = -\mathbf{\Phi}_t\left(\mathbf{q}^i, t_i\right) \equiv \boldsymbol{\nu}\left(\mathbf{q}^i, t_i\right) \tag{9.6.18}$$

Note that no iterative computation is required, since the equations are linear in velocities.

Acceleration Analysis

Much as in the case of velocity analysis, accelerations $\ddot{\mathbf{q}}^i \approx \ddot{\mathbf{q}}(t_i)$, i=0, 1,... are determined by solving the acceleration equations

$$\mathbf{\Phi_q}\left(\mathbf{q}^i, t_i\right)\ddot{\mathbf{q}}^i = -\left(\mathbf{\Phi_q}\left(\mathbf{q}^i, t_i\right)\dot{\mathbf{q}}^i\right)_{\mathbf{q}}\dot{\mathbf{q}}^i - 2\mathbf{\Phi}_{t\mathbf{q}}\left(\mathbf{q}^i, t_i\right)\dot{\mathbf{q}}^i - \mathbf{\Phi}_{tt}\left(\mathbf{q}^i, t_i\right) \equiv \boldsymbol{\gamma}\left(\mathbf{q}^i, \dot{\mathbf{q}}^i, t_i\right) \tag{9.6.19}$$

Note that no iterative computation is required, since the equations are linear in accelerations.

9.7 Derivatives of Basic Constraints

10-25-01

As shown in Section 9.6, kinematic analysis requires evaluation of the Jacobian of constraint equations, including combinations of the basic constraints comprising kinematic joint definition, Euler parameter normalization conditions, and driving constraints. Since each of the basic constraints involves only a pair of bodies, numbered i and j, all columns in each basic constraint Jacobian, except those involving bodies i and j, are 0. Thus, all basic constraint Jacobian information is contained in the matrix

$$\mathbf{\Phi_q} \equiv \begin{bmatrix} \mathbf{\Phi_{r_i}} & \mathbf{\Phi_{p_i}} & \mathbf{\Phi_{r_j}} & \mathbf{\Phi_{p_j}} \end{bmatrix} \tag{9.7.1}$$

where $\mathbf{q} \equiv \begin{bmatrix} \mathbf{r}_i^T & \mathbf{p}_i^T & \mathbf{r}_j^T & \mathbf{p}_j^T \end{bmatrix}_{14\times 1}^T$.

Jacobians

Since the kinematic and driving constraints are assembled using the basic constraints, their contribution to the Jacobian is assembled using derivatives of the associated basic constraint expressions. In preparation for basic constraint Jacobian calculation, note that for a body fixed vector $\mathbf{a}'$, $\mathbf{a} = \mathbf{A}\mathbf{a}'$, so

$$\mathbf{a_q} = \left(\mathbf{A}(\mathbf{q})\mathbf{a}'\right)_{\mathbf{q}} = \mathbf{B}(\mathbf{p},\mathbf{a}') \tag{9.7.2}$$

and, with $\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{s}_j - \mathbf{r}_i - \mathbf{s}_i$,

$$\begin{aligned}(\mathbf{d}_{ij})_{\mathbf{q}} &= \begin{bmatrix} -\mathbf{I} & -(\mathbf{s}_i)_{\mathbf{p}_i} & \mathbf{I} & (\mathbf{s}_j)_{\mathbf{p}_j} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{I} & -\mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P) & \mathbf{I} & \mathbf{B}(\mathbf{p}_j,\mathbf{s}_j'^P) \end{bmatrix}\end{aligned} \tag{9.7.3}$$

For the "dot-1" constraint,

$$\begin{aligned}\Phi_{\mathbf{q}}^{d1}(\mathbf{a}_i,\mathbf{a}_j) &= \left(\mathbf{a}_i^T\mathbf{a}_j\right)_{\mathbf{q}} \\ &= \mathbf{a}_i^T(\mathbf{a}_j)_{\mathbf{q}} + \mathbf{a}_j^T(\mathbf{a}_i)_{\mathbf{q}} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j') \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{a}_j^T\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i') & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{a}_j^T\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i') & \mathbf{0} & \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j') \end{bmatrix}\end{aligned} \tag{9.7.4}$$

For the "dot-2 constraint,

$$
\begin{aligned}
\Phi_{\mathbf{q}}^{d2}\left(\mathbf{a}_i,\mathbf{d}_{ij}\right) &= \mathbf{a}_i^T\left(\mathbf{d}_{ij}\right)_{\mathbf{q}} + \mathbf{d}_{ij}^T\left(\mathbf{a}_i\right)_{\mathbf{q}} \\
&= \mathbf{a}_i^T\left[-\mathbf{I} \quad -\mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \quad \mathbf{I} \quad \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\right] \\
&\quad + \mathbf{d}_{ij}^T\left[\mathbf{0} \quad \mathbf{B}\left(\mathbf{p}_i,\mathbf{a}_i'\right) \quad \mathbf{0} \quad \mathbf{0}\right] \\
&= \left[-\mathbf{a}_i^T \quad \mathbf{d}_{ij}^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{a}_i'\right) - \mathbf{a}_i^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \quad \mathbf{a}_i^T \quad \mathbf{a}_i^T\mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\right]
\end{aligned}
\tag{9.7.5}
$$

For the "spherical" constraint,

$$
\begin{aligned}
\mathbf{\Phi}_{\mathbf{q}}^{s}\left(P_i,P_j\right) &= \left(\mathbf{d}_{ij}\right)_{\mathbf{q}} \\
&= \left[-\mathbf{I} \quad -\mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \quad \mathbf{I} \quad \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\right]
\end{aligned}
\tag{9.7.6}
$$

For the "distance" constraint,

$$
\begin{aligned}
\mathbf{\Phi}_{\mathbf{q}}^{ss}\left(P_i,P_j,C\right) &= \left(\mathbf{d}_{ij}^T\mathbf{d}_{ij} - C^2\right)_{\mathbf{q}} = 2\mathbf{d}_{ij}^T\mathbf{d}_{ij\mathbf{q}} \\
&= 2\mathbf{d}_{ij}^T\left[-\mathbf{I} \quad -\mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \quad \mathbf{I} \quad \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\right]
\end{aligned}
\tag{9.7.7}
$$

The "screw" constraint is comprised of the cylindrical joint equations and Eq. 9.4.25 that involves the dot-2 constraint function and the rotation angle θ, so $\theta_{\mathbf{q}}$ must be determined. From Eqs. 9.2.26 and 27,

$$
\begin{aligned}
\cos\theta &= \mathbf{f}_i^T\mathbf{f}_j \\
\sin\theta &= \mathbf{g}_i^T\mathbf{f}_j
\end{aligned}
\tag{9.7.8}
$$

$$
\begin{aligned}
(-\sin\theta)\theta_{\mathbf{q}} &= \left[\mathbf{0} \quad \mathbf{f}_j^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right) \quad \mathbf{0} \quad \mathbf{f}_i^T\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right)\right] \\
(\cos\theta)\theta_{\mathbf{q}} &= \left[\mathbf{0} \quad \mathbf{f}_j^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right) \quad \mathbf{0} \quad \mathbf{g}_i^T\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right)\right]
\end{aligned}
\tag{9.7.9}
$$

Multiplying the first of Eqs. (9.7.9) by $-\sin\theta$ and the second by $\cos\theta$ and adding,

$$
\theta_{\mathbf{q}} = \left[\mathbf{0} \quad \mathbf{f}_j^T\left(\cos\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right)\right) \quad \mathbf{0} \quad \left(\cos\theta\mathbf{g}_i^T - \sin\theta\mathbf{f}_i^T\right)\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right)\right] \tag{9.7.10}
$$

For the "absolute" constraints, which involve only body i,

$$\boldsymbol{\Phi}_{\mathbf{q}_i}^{a}(\mathbf{q}) = \begin{bmatrix} \Phi^{a1}(\mathbf{q}) \\ \Phi^{a2}(\mathbf{q}) \\ \Phi^{a3}(\mathbf{q}) \end{bmatrix}_{\mathbf{q}_i} = \left(\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{C}\right)_{\mathbf{q}_i} \\ = \begin{bmatrix} \mathbf{I} & \mathbf{B}\left(\mathbf{p}_i, \mathbf{s}_i'^P\right) \end{bmatrix} \tag{9.7.11}$$

For "absolute drivers", the $\mathbf{C}$ in Eq.(9.7.11) is a function of time; i.e.,

$$\boldsymbol{\Phi}_{\mathbf{q}_i}^{ad}(\mathbf{q},t) = \begin{bmatrix} \Phi^{1ad}(\mathbf{q},t) \\ \Phi^{2ad}(\mathbf{q},t) \\ \Phi^{3ad}(\mathbf{q},t) \end{bmatrix}_{\mathbf{q}_i} = \left(\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{C}(t)\right)_{\mathbf{q}_i} \\ = \begin{bmatrix} \mathbf{I} & \mathbf{B}\left(\mathbf{p}_i, \mathbf{s}_i'^P\right) \end{bmatrix} \tag{9.7.12}$$

For the "distance driver",

$$\boldsymbol{\Phi}_{\mathbf{q}}^{ssd}(\mathbf{q},t) = \left(\boldsymbol{\Phi}^{ss}\left(P_i, P_j, 0\right) - C^2(t)\right)_{\mathbf{q}} = \boldsymbol{\Phi}_{\mathbf{q}}^{ss}\left(P_i, P_j, 0\right) \tag{9.7.13}$$

For the "relative translational driver",

$$\Phi_{\mathbf{q}}^{td}(\mathbf{q},t) = \left(\Phi^{d2}\left(\mathbf{a}_i, \mathbf{d}_{ij}\right) - C(t)\right)_{\mathbf{q}} = \Phi_{\mathbf{q}}^{d2}(\mathbf{q}) \tag{9.7.14}$$

Finally, for the "relative rotational driver",

$$\Phi_{\mathbf{q}}^{rotd}(\mathbf{q},t) = \left(\theta + 2n\pi - C(t)\right)_{\mathbf{q}} = \theta_{\mathbf{q}} \tag{9.7.15}$$

Velocity Equations

Differentiating a basic constraint equation $\boldsymbol{\Phi}(\mathbf{q},t) = \mathbf{0}$, whether it be a kinematic constraint or a driver,

$$\dot{\boldsymbol{\Phi}}(\mathbf{q},t) = \boldsymbol{\Phi}_{\mathbf{q}}(\mathbf{q},t)\dot{\mathbf{q}} - \boldsymbol{\Phi}_t(\mathbf{q},t) = \mathbf{0} \tag{9.7.16}$$

Equivalently, the velocity equation is

$$\boldsymbol{\Phi}_{\mathbf{q}}(\mathbf{q},t)\dot{\mathbf{q}} = -\boldsymbol{\Phi}_t(\mathbf{q},t) \equiv \boldsymbol{\nu}(\mathbf{q},t) \tag{9.7.17}$$

Since the basic kinematic constraints do not depend explicitly on time,

$$\begin{aligned} \nu^{d1}(\mathbf{q},t) = \nu^{d2}(\mathbf{q},t) = \nu^{ss}(\mathbf{q},t) = 0 \\ \boldsymbol{\nu}^{s}(\mathbf{q},t) = \boldsymbol{\nu}^{a}(\mathbf{q},t) = \mathbf{0} \end{aligned} \qquad (9.7.18)$$

For the "absolute drivers",

$$\boldsymbol{\nu}^{ad}(\mathbf{q},t) = \begin{bmatrix} \nu^{a1d}(\mathbf{q},t) \\ \nu^{a2d}(\mathbf{q},t) \\ \nu^{a3d}(\mathbf{q},t) \end{bmatrix} = -\begin{bmatrix} \Phi^{a1d}(\mathbf{q},t) \\ \Phi^{a2d}(\mathbf{q},t) \\ \Phi^{a3d}(\mathbf{q},t) \end{bmatrix}_t = -\left(\mathbf{r}_i + \mathbf{s}_i^P - \mathbf{C}(t)\right)_t = \dot{\mathbf{C}}(t) \qquad (9.7.19)$$

For the "distance driver",

$$\begin{aligned} \nu^{ssd}(\mathbf{q},t) &= -\left(\boldsymbol{\Phi}^{ss}(P_i, P_j, C) - C^2(t)\right)_t \\ &= 2C(t)\dot{C}(t) \end{aligned} \qquad (9.7.20)$$

For the "relative translational driver",

$$\nu^{td}(\mathbf{q},t) = -\left(\Phi^{d2}(\mathbf{a}_i, \mathbf{d}_{ij}) - C(t)\right)_t = \dot{C}(t) \qquad (9.7.21)$$

Finally, for the "relative rotational driver",

$$\nu^{rotd}(\mathbf{q},t) = -\left(\theta + 2n\pi - C(t)\right)_t = \dot{C}(t) \qquad (9.7.22)$$

Acceleration Equations

Taking time derivative of each of the basic kinematic constraint equations, suppressing arguments for notational convenience,

$$\dot{\Phi}^{d1} = \mathbf{a}_i^T\dot{\mathbf{a}}_j + \mathbf{a}_j^T\dot{\mathbf{a}}_i = 0 \qquad (9.7.23)$$

$$\dot{\Phi}^{d2} = \mathbf{a}_i^T\dot{\mathbf{d}}_{ij} + \mathbf{d}_{ij}^T\dot{\mathbf{a}}_i = 0 \qquad (9.7.24)$$

$$\dot{\boldsymbol{\Phi}}^{s} = \dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \dot{\mathbf{s}}_j^P - \dot{\mathbf{r}}_i - \dot{\mathbf{s}}_i^P = \mathbf{0} \qquad (9.7.25)$$

$$\dot{\Phi}^{ss} = 2\mathbf{d}_{ij}^T\dot{\mathbf{d}}_{ij} = 0 \qquad (9.7.26)$$

$$\dot{\boldsymbol{\Phi}}^{a} = \dot{\mathbf{r}}_i + \dot{\mathbf{s}}_i^P = \mathbf{0} \qquad (9.7.27)$$

Differentiating again,

$$\ddot{\Phi}^{d1} = \mathbf{a}_i^T \ddot{\mathbf{a}}_j + \mathbf{a}_j^T \ddot{\mathbf{a}}_i + 2\dot{\mathbf{a}}_j^T \dot{\mathbf{a}}_i = 0 \tag{9.7.28}$$

$$\ddot{\Phi}^{d2} = \mathbf{a}_i^T \ddot{\mathbf{d}}_{ij} + \mathbf{d}_{ij}^T \ddot{\mathbf{a}}_i + 2\dot{\mathbf{d}}_{ij}^T \dot{\mathbf{a}}_i = 0 \tag{9.7.29}$$

$$\ddot{\mathbf{\Phi}}^{s} = \ddot{\mathbf{d}}_{ij} = \mathbf{0} \tag{9.7.30}$$

$$\ddot{\Phi}^{ss} = 2\mathbf{d}_{ij}^T \ddot{\mathbf{d}}_{ij} + 2\dot{\mathbf{d}}_{ij}^T \dot{\mathbf{d}}_{ij} = 0 \tag{9.7.31}$$

$$\ddot{\mathbf{\Phi}}^{a} = \ddot{\mathbf{r}}_i + \ddot{\mathbf{s}}_i^P = \mathbf{0} \tag{9.7.32}$$

For $\mathbf{a} = \mathbf{A}(\mathbf{p})\mathbf{a}'$ and $\mathbf{d}_{ij} = \mathbf{r}_j + \mathbf{s}_j^P - \mathbf{r}_i - \mathbf{s}_i^P$,

$$\dot{\mathbf{a}} = \left(\mathbf{A}(\mathbf{p})\mathbf{a}'\right)_{\mathbf{p}} \dot{\mathbf{p}} = \mathbf{B}(\mathbf{p},\mathbf{a}')\dot{\mathbf{p}} \tag{9.7.33}$$

$$\dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\dot{\mathbf{p}}_j - \dot{\mathbf{r}}_i - \mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right)\dot{\mathbf{p}}_i \tag{9.7.34}$$

Taking the time derivative of Eqs.(9.7.33) and (9.7.34),

$$\begin{aligned}
\ddot{\mathbf{a}} &= \left(\mathbf{B}(\mathbf{p},\mathbf{a}')\dot{\mathbf{p}}\right)_{\dot{\mathbf{p}}} \ddot{\mathbf{p}} + \left(\mathbf{B}(\mathbf{p},\mathbf{a}')\dot{\mathbf{p}}\right)_{\mathbf{p}} \dot{\mathbf{p}} \\
&= \mathbf{B}(\mathbf{p},\mathbf{a}')\ddot{\mathbf{p}} + \left(\mathbf{B}(\dot{\mathbf{p}},\mathbf{a}')\mathbf{p}\right)_{\mathbf{p}} \dot{\mathbf{p}} \\
&= \mathbf{B}(\mathbf{p},\mathbf{a}')\ddot{\mathbf{p}} + \mathbf{B}(\dot{\mathbf{p}},\mathbf{a}')\dot{\mathbf{p}}
\end{aligned} \tag{9.7.35}$$

$$\begin{aligned}
\ddot{\mathbf{d}}_{ij} &= \ddot{\mathbf{r}}_j + \ddot{\mathbf{s}}_j^P - \ddot{\mathbf{r}}_i - \ddot{\mathbf{s}}_i^P \\
&= \ddot{\mathbf{r}}_j + \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right)\ddot{\mathbf{p}}_i + \mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\dot{\mathbf{p}}_j - \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right)\dot{\mathbf{p}}_i
\end{aligned} \tag{9.7.36}$$

An alternate derivation of Eq. (9.7.35) uses linearity of $\mathbf{B}(\mathbf{p},\mathbf{a}')$ in its first argument. For arbitrary 4-vectors $\mathbf{p} = \left[e_0 \quad \mathbf{e}^T\right]^T$ and $\mathbf{z} = \left[w_0 \quad \mathbf{w}^T\right]^T$ and constants α and β,

$$
\begin{aligned}
\mathbf{B}(\alpha\mathbf{p}+\beta\mathbf{z},\mathbf{a}') = 2\Big[&\big((\alpha e_0+\beta w_0)\mathbf{I}+(\alpha\tilde{\mathbf{e}}+\beta\tilde{\mathbf{w}})\big)\mathbf{a}' \\
&(\alpha\mathbf{e}+\beta\mathbf{w})\mathbf{a}'^{T} - \big((\alpha e_0+\beta w_0)\mathbf{I}+(\alpha\tilde{\mathbf{e}}+\beta\tilde{\mathbf{w}})\big)\tilde{\mathbf{a}}'\Big] \\
&= 2\Big[\alpha\left(e_0\mathbf{a}'+\tilde{\mathbf{e}}\mathbf{a}'\right)+\beta\left(w_0\mathbf{a}'+\tilde{\mathbf{w}}\mathbf{a}'\right) \\
&\quad \alpha\left(\mathbf{e}\mathbf{a}'^{T}-e_0\tilde{\mathbf{a}}'-\tilde{\mathbf{e}}\tilde{\mathbf{a}}'\right)+\beta\left(\mathbf{w}\mathbf{a}'^{T}-w_0\tilde{\mathbf{a}}'-\tilde{\mathbf{w}}\tilde{\mathbf{a}}'\right)\Big] \\
&= \alpha 2\left[e_0\mathbf{a}'+\tilde{\mathbf{e}}\mathbf{a}' \quad \mathbf{e}\mathbf{a}'^{T}-e_0\tilde{\mathbf{a}}'-\tilde{\mathbf{e}}\tilde{\mathbf{a}}'\right]+\beta 2\left[w_0\mathbf{a}'+\tilde{\mathbf{w}}\mathbf{a}' \quad \mathbf{w}\mathbf{a}'^{T}-w_0\tilde{\mathbf{a}}'-\tilde{\mathbf{w}}\tilde{\mathbf{a}}'\right] \\
&= \alpha 2\left[(e_0\mathbf{I}+\tilde{\mathbf{e}})\mathbf{a}' \quad \mathbf{e}\mathbf{a}'^{T}-(e_0\mathbf{I}+\tilde{\mathbf{e}})\tilde{\mathbf{a}}'\right]+\beta 2\left[(w_0\mathbf{I}+\tilde{\mathbf{w}})\mathbf{a}' \quad \mathbf{w}\mathbf{a}'^{T}-(w_0\mathbf{I}+\tilde{\mathbf{w}})\tilde{\mathbf{a}}'\right] \\
&= \alpha\mathbf{B}(\mathbf{p},\mathbf{a}')+\beta\mathbf{B}(\mathbf{z},\mathbf{a}')
\end{aligned}
\tag{9.7.37}
$$

which demonstrates the linearity property. Since derivative with respect to time is a linear operator,

$$
\frac{d}{dt}\mathbf{B}(\mathbf{p},\mathbf{a}') = \mathbf{B}\left(\frac{d}{dt}\mathbf{p},\mathbf{a}'\right) = \mathbf{B}(\dot{\mathbf{p}},\mathbf{a}') \tag{9.7.38}
$$

Using this relation in differentiating Eq. (9.7.33),

$$
\begin{aligned}
\ddot{\mathbf{a}} &= \mathbf{B}(\mathbf{p},\mathbf{a}')\frac{d}{dt}\dot{\mathbf{p}}+\left(\frac{d}{dt}\mathbf{B}(\mathbf{p},\mathbf{a}')\right)\dot{\mathbf{p}} \\
&= \mathbf{B}(\mathbf{p},\mathbf{a}')\ddot{\mathbf{p}}+\mathbf{B}(\dot{\mathbf{p}},\mathbf{a}')\dot{\mathbf{p}}
\end{aligned}
\tag{9.7.39}
$$

which is the desired result of Eq. (9.7.35).

Manipulating Eqs. (9.7.28) through (9.7.32), using Eqs. (9.7.35) and (9.7.36), yields the basic constraint acceleration equations.

For the "dot-1" constraint,

$$
\begin{aligned}
\ddot{\Phi}^{d1} &= \mathbf{a}_i^T\ddot{\mathbf{a}}_j+\mathbf{a}_j^T\ddot{\mathbf{a}}_i+2\dot{\mathbf{a}}_i^T\dot{\mathbf{a}}_j \\
&= \mathbf{a}_i^T\left(\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j')\ddot{\mathbf{p}}_j+\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{a}_j')\dot{\mathbf{p}}_j\right)+\mathbf{a}_j^T\left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i')\ddot{\mathbf{p}}_i+\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i\right)+2\dot{\mathbf{a}}_i^T\dot{\mathbf{a}}_j \\
&= 0
\end{aligned}
\tag{9.7.40}
$$

Manipulating and using Eq. (9.7.4), this is the "dot-1" acceleration equation,

$$\begin{aligned}\Phi^{d1}_{\mathbf{q}}\ddot{\mathbf{q}} &= \mathbf{a}_j^T\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i')\ddot{\mathbf{p}}_i + \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j')\ddot{\mathbf{p}}_j \\ &= -\left\{\mathbf{a}_i^T\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{a}_j')\dot{\mathbf{p}}_j + \mathbf{a}_j^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{a}}_j\right\} \\ &\equiv \gamma^{d1}(\mathbf{q},\dot{\mathbf{q}})\end{aligned} \tag{9.7.41}$$

For the "dot-2" constraint,

$$\begin{aligned}\ddot{\Phi}^{d2} &= \mathbf{a}_i^T\ddot{\mathbf{d}}_{ij} + \mathbf{d}_{ij}^T\ddot{\mathbf{a}}_i + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{d}}_{ij} \\ &= \mathbf{a}_i^T\left(\ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\mathbf{s}_j'^P)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P)\ddot{\mathbf{p}}_i + \mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\dot{\mathbf{p}}_j - \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\dot{\mathbf{p}}_i\right) \\ &\quad + \mathbf{d}_{ij}^T\left(\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i')\ddot{\mathbf{p}}_i + \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i\right) + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{d}}_{ij} \\ &= 0\end{aligned} \tag{9.7.42}$$

Manipulating and using Eq. (9.7.5), this is the "dot-2" acceleration equation,

$$\begin{aligned}\Phi^{d2}_{\mathbf{q}}\ddot{\mathbf{q}} &= \mathbf{a}_i^T\left(\ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\mathbf{s}_j'^P)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P)\ddot{\mathbf{p}}_i\right) + \mathbf{d}_{ij}^T\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i')\ddot{\mathbf{p}}_i \\ &= -\left\{\mathbf{a}_i^T\left(\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\dot{\mathbf{p}}_j - \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\dot{\mathbf{p}}_i\right) + \mathbf{d}_{ij}^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{d}}_{ij}\right\} \\ &\equiv \gamma^{d2}(\mathbf{q},\dot{\mathbf{q}})\end{aligned} \tag{9.7.43}$$

For the "spherical" constraint,

$$\ddot{\boldsymbol{\Phi}}^{s} = \ddot{\mathbf{d}}_{ij} = \mathbf{0} \tag{9.7.44}$$

Manipulating and using Eq. (9.7.6), this is the "spherical" acceleration equation,

$$\begin{aligned}\boldsymbol{\Phi}^{s}_{\mathbf{q}}\ddot{\mathbf{q}} &= \ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\mathbf{s}_j'^P)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P)\ddot{\mathbf{p}}_i \\ &= -\left\{\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\dot{\mathbf{p}}_j - \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\dot{\mathbf{p}}_i\right\} \\ &\equiv \boldsymbol{\gamma}^{s}(\mathbf{q},\dot{\mathbf{q}})\end{aligned} \tag{9.7.45}$$

For the "distance" constraint,

$$\begin{aligned}\ddot{\Phi}^{ss} &= 2\mathbf{d}_{ij}^T\ddot{\mathbf{d}}_{ij} + 2\dot{\mathbf{d}}_{ij}^T\dot{\mathbf{d}}_{ij} \\ &= 2\mathbf{d}_{ij}^T\left(\ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j,\mathbf{s}_j'^P)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P)\ddot{\mathbf{p}}_i\right) \\ &\quad + 2\mathbf{d}_{ij}^T\left(\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\dot{\mathbf{p}}_j - \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\dot{\mathbf{p}}_i\right) + 2\dot{\mathbf{d}}_{ij}^T\dot{\mathbf{d}}_{ij} \\ &= 0\end{aligned} \tag{9.7.46}$$

Manipulating and using Eq. (9.7.7), this is the "distance" acceleration equation,

$$
\begin{aligned}
\Phi_{\mathbf{q}}^{ss}\ddot{\mathbf{q}} &= 2\mathbf{d}_{ij}^{T}\left(\ddot{\mathbf{r}}_j + \mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^{P}\right)\ddot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^{P}\right)\ddot{\mathbf{p}}_i\right) \\
&= -\left\{2\mathbf{d}_{ij}^{T}\left(\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^{P}\right)\dot{\mathbf{p}}_j - \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^{P}\right)\dot{\mathbf{p}}_i\right) + 2\dot{\mathbf{d}}_{ij}^{T}\dot{\mathbf{d}}_{ij}\right\} \\
&\equiv \gamma^{ss}\left(\mathbf{q},\dot{\mathbf{q}}\right)
\end{aligned}
\tag{9.7.47}
$$

Since the "screw constraint" includes an equation with θ, its acceleration equation includes terms involving $\ddot{\theta}$. Differentiating both sides of Eqs. (9.7.8) with respect to time yields

$$
\begin{aligned}
-\dot{\theta}\sin\theta &= \mathbf{f}_j^{T}\dot{\mathbf{f}}_i + \mathbf{f}_i^{T}\dot{\mathbf{f}}_j \\
\dot{\theta}\cos\theta &= \mathbf{f}_j^{T}\dot{\mathbf{g}}_i + \mathbf{g}_i^{T}\dot{\mathbf{f}}_j
\end{aligned}
\tag{9.7.48}
$$

Differentiating again,

$$
\begin{aligned}
-\ddot{\theta}\sin\theta &= \mathbf{f}_i^{T}\ddot{\mathbf{f}}_j + \mathbf{f}_j^{T}\ddot{\mathbf{f}}_i + \dot{\theta}^2\cos\theta + 2\dot{\mathbf{f}}_i^{T}\dot{\mathbf{f}}_j \\
\ddot{\theta}\cos\theta &= \mathbf{g}_i^{T}\ddot{\mathbf{f}}_j + \mathbf{f}_j^{T}\ddot{\mathbf{g}}_i + \dot{\theta}^2\sin\theta + 2\dot{\mathbf{g}}_i^{T}\dot{\mathbf{f}}_j
\end{aligned}
\tag{9.7.49}
$$

Multiplying the first equation by $-\sin\theta$ and the second by $\cos\theta$ and adding yields

$$
\begin{aligned}
\ddot{\theta} &= \left(\cos\theta\mathbf{g}_i^{T} - \sin\theta\mathbf{f}_i^{T}\right)\left(\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right)\ddot{\mathbf{p}}_j + \mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\dot{\mathbf{p}}_j\right) \\
&\quad + \mathbf{f}_j^{T}\left[\cos\theta\left(\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right)\ddot{\mathbf{p}}_i + \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right)\dot{\mathbf{p}}_i\right) - \sin\theta\left(\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right)\ddot{\mathbf{p}}_i + \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\dot{\mathbf{p}}_i\right)\right] \\
&\quad + 2\left(\cos\theta\dot{\mathbf{g}}_i^{T} - \sin\theta\dot{\mathbf{f}}_i^{T}\right)\dot{\mathbf{f}}_j
\end{aligned}
\tag{9.7.50}
$$

Manipulating and using Eq. (9.7.10), this is

$$
\begin{aligned}
\theta_{\mathbf{q}}\ddot{\mathbf{q}} &= \left[\mathbf{0} \quad \mathbf{f}_j^{T}\left(\cos\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right)\right) \quad \mathbf{0} \quad \left(\cos\theta\mathbf{g}_i^{T} - \sin\theta\mathbf{f}_i^{T}\right)\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right)\right]\ddot{\mathbf{q}} \\
&= -\left\{\left(\cos\theta\mathbf{g}_i^{T} - \sin\theta\mathbf{f}_i^{T}\right)\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\dot{\mathbf{p}}_j + \mathbf{f}_j^{T}\left(\cos\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right)\dot{\mathbf{p}}_i \right. \\
&\qquad \left. + 2\left(\cos\theta\dot{\mathbf{g}}_i^{T} - \sin\theta\dot{\mathbf{f}}_i^{T}\right)\dot{\mathbf{f}}_j\right\} \\
&\equiv \gamma^{\theta}\left(\mathbf{q},\dot{\mathbf{q}}\right)
\end{aligned}
\tag{9.7.51}
$$

For the "absolute" constraint,

$$
\ddot{\mathbf{\Phi}}^{a} = \begin{bmatrix} \ddot{\Phi}^{a1} \\ \ddot{\Phi}^{a2} \\ \ddot{\Phi}^{a3} \end{bmatrix} = \ddot{\mathbf{r}}_i + \mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^{P}\right)\ddot{\mathbf{p}}_i + \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^{P}\right)\dot{\mathbf{p}}_i = \mathbf{0}
\tag{9.7.52}
$$

Manipulating and using Eq. (9.7.11), this is the "absolute" acceleration equation,

$$\mathbf{\Phi}^{a}_{\mathbf{q}_i}\ddot{\mathbf{q}} = \begin{bmatrix} \Phi^{a1} \\ \Phi^{a2} \\ \Phi^{a3} \end{bmatrix}_{\mathbf{q}_i} \ddot{\mathbf{q}}_i = -\mathbf{B}\left(\dot{\mathbf{p}}_i, \mathbf{s}_i'^{P}\right)\dot{\mathbf{p}}_i \equiv \begin{bmatrix} \gamma^{a1} \\ \gamma^{a2} \\ \gamma^{a3} \end{bmatrix} = \boldsymbol{\gamma}^{a}\left(\mathbf{q},\dot{\mathbf{q}}\right) \qquad (9.7.53)$$

For the driving constraints, each constraint function is a sum of terms, one of which depends only on generalized coordinates and the other depends only on time. Therefore, the derivative $\Phi_{t\mathbf{q}}\left(\mathbf{q},t\right)$ for each of the drivers is zero. In particular, for the "absolute drivers",

$$\begin{aligned} \boldsymbol{\gamma}^{ad}\left(\mathbf{q},\dot{\mathbf{q}},t\right) &= \begin{bmatrix} \gamma^{a1d}\left(\mathbf{q},\dot{\mathbf{q}},t\right) \\ \gamma^{a2d}\left(\mathbf{q},\dot{\mathbf{q}},t\right) \\ \gamma^{a3d}\left(\mathbf{q},\dot{\mathbf{q}},t\right) \end{bmatrix} = \boldsymbol{\gamma}^{a}\left(\mathbf{q},\dot{\mathbf{q}}\right) - \mathbf{\Phi}^{ad}_{tt}\left(\mathbf{q},t\right) \\ &= \boldsymbol{\gamma}^{a}\left(\mathbf{q},\dot{\mathbf{q}}\right) - \left(\mathbf{r}_i + \mathbf{s}_i^{P} - \mathbf{C}(t)\right)_{tt} \\ &= -\left(\mathbf{B}\left(\dot{\mathbf{p}}_i, \mathbf{s}_i'^{P}\right)\right)\dot{\mathbf{p}}_i + \ddot{\mathbf{C}}(t) \end{aligned} \qquad (9.7.54)$$

For the "distance driver",

$$\begin{aligned} \gamma^{ssd}\left(\mathbf{q},\dot{\mathbf{q}},t\right) &= \gamma^{ss}\left(\mathbf{q},\dot{\mathbf{q}}\right) - \left(\Phi^{ss}\left(\mathbf{q}\right) - C^{2}(t)\right)_{tt} \\ &= \gamma^{ss}\left(\mathbf{q},\dot{\mathbf{q}}\right) + 2C(t)\ddot{C}(t) + 2\dot{C}^{2}(t) \end{aligned} \qquad (9.7.55)$$

For the "relative translational driver",

$$\begin{aligned} \gamma^{td}\left(\mathbf{q},\dot{\mathbf{q}},t\right) &= \gamma^{d2}\left(\mathbf{q},\dot{\mathbf{q}}\right) - \left(\Phi^{d2}\left(\mathbf{q}\right) - C(t)\right)_{tt} \\ &= \gamma^{d2}\left(\mathbf{q},\dot{\mathbf{q}}\right) + \ddot{C}(t) \end{aligned} \qquad (9.7.56)$$

For the "relative rotational driver",

$$\begin{aligned} \gamma^{rotd}\left(\mathbf{q},\dot{\mathbf{q}},t\right) &= \gamma^{\theta}\left(\mathbf{q},\dot{\mathbf{q}}\right) - \left(\theta\left(\mathbf{q}\right) + 2n\pi - C(t)\right)_{tt} \\ &= \gamma^{\theta}\left(\mathbf{q},\dot{\mathbf{q}}\right) + \ddot{C}(t) \end{aligned} \qquad (9.7.57)$$

9.8 Derivatives for Implicit Integration and Sensitivity Analysis

10-25-01

In addition to the derivatives obtained above, one additional derivative of the equations of motion with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$ is required for implicit numerical integration and sensitivity analysis. In particular, a derivative of the right side of the acceleration equation is required, involving a third derivative of constraint expressions.

Derivatives of Right Side of Acceleration Equation

Using the relation $\dot{\mathbf{a}} = \mathbf{B}(\mathbf{p},\mathbf{a}')\dot{\mathbf{p}} = \mathbf{B}(\dot{\mathbf{p}},\mathbf{a}')\mathbf{p}$ and equations of Section 9.7 defining right sides of acceleration equations, the derivatives $\boldsymbol{\gamma}_{\mathbf{q}}(\mathbf{q},\dot{\mathbf{q}},t)$ and $\boldsymbol{\gamma}_{\dot{\mathbf{q}}}(\mathbf{q},\dot{\mathbf{q}},t)$ can be obtained.

For the "dot-1" constraint,

$$\gamma^{d1}(\mathbf{q},\dot{\mathbf{q}}) = -\left\{\mathbf{a}_i^T\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{a}_j')\dot{\mathbf{p}}_j + \mathbf{a}_j^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{a}}_j\right\} \quad (9.8.1)$$

$$\left(\gamma^{d1}\right)_{\mathbf{q}} = -\left[\mathbf{0} \quad \dot{\mathbf{p}}_j^T\mathbf{B}^T(\dot{\mathbf{p}}_j,\mathbf{a}_j')\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i') + 2\dot{\mathbf{a}}_j^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i') \quad \mathbf{0} \quad \dot{\mathbf{p}}_i^T\mathbf{B}^T(\dot{\mathbf{p}}_i,\mathbf{a}_i')\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j') + 2\dot{\mathbf{a}}_i^T\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{a}_j')\right] \quad (9.8.2)$$

Similarly,

$$\left(\gamma^{d1}\right)_{\dot{\mathbf{q}}} = -\left[\mathbf{0} \quad 2\mathbf{a}_j^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i') + 2\dot{\mathbf{a}}_j^T\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i') \quad \mathbf{0} \quad 2\mathbf{a}_i^T\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{a}_j') + 2\dot{\mathbf{a}}_i^T\mathbf{B}(\mathbf{p}_j,\mathbf{a}_j')\right] \quad (9.8.3)$$

For the "dot-2" constraint,

$$\gamma^{d2}(\mathbf{q},\dot{\mathbf{q}}) = -\left\{\mathbf{a}_i^T\left(\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\dot{\mathbf{p}}_j - \mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\dot{\mathbf{p}}_i\right) + \mathbf{d}_{ij}^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i')\dot{\mathbf{p}}_i + 2\dot{\mathbf{a}}_i^T\dot{\mathbf{d}}_{ij}\right\} \quad (9.8.4)$$

$$\begin{aligned}\left(\gamma^{d2}\right)_{\mathbf{q}} = &-\dot{\mathbf{p}}_i^T\mathbf{B}^T(\dot{\mathbf{p}}_i,\mathbf{a}_i')\left[-\mathbf{I} \quad -\mathbf{B}(\mathbf{p}_i,\mathbf{s}_i'^P) \quad \mathbf{I} \quad \mathbf{B}(\mathbf{p}_j,\mathbf{s}_i'^P)\right]\\ &-\Big[\mathbf{0} \quad \left(\dot{\mathbf{p}}_j^T\mathbf{B}^T(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P) - \dot{\mathbf{p}}_i^T\mathbf{B}^T(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\right)\mathbf{B}(\mathbf{p}_i,\mathbf{a}_i') + 2\dot{\mathbf{d}}_{ij}^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{a}_i') - 2\dot{\mathbf{a}}_i^T\mathbf{B}(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P)\\ &\quad \mathbf{0} \quad 2\dot{\mathbf{a}}_i^T\mathbf{B}(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P)\Big]\end{aligned} \quad (9.8.5)$$

$$\left(\gamma^{d2}\right)_{\dot{\mathbf{q}}} = -\Big[-2\dot{\mathbf{a}}_i^T \quad 2\mathbf{d}_{ij}^T\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{a}_i'\right) - 2\mathbf{a}_i^T\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right) + 2\dot{\mathbf{d}}_{ij}^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{a}_i'\right) - 2\dot{\mathbf{a}}_i^T\mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \\ 2\dot{\mathbf{a}}_i^T \quad 2\mathbf{a}_i^T\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right) + 2\dot{\mathbf{a}}_i^T\mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\Big] \tag{9.8.6}$$

For the "spherical" constraint,

$$\boldsymbol{\gamma}^s\left(\mathbf{q},\dot{\mathbf{q}}\right) = -\left\{\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\dot{\mathbf{p}}_j - \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right)\dot{\mathbf{p}}_i\right\} \tag{9.8.7}$$

$$\left(\gamma^s\right)_{\mathbf{q}} = \mathbf{0} \tag{9.8.8}$$

$$\left(\gamma^s\right)_{\dot{\mathbf{q}}} = \Big[\mathbf{0} \quad 2\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right) \quad \mathbf{0} \quad -2\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\Big] \tag{9.8.9}$$

For the "distance" constraint,

$$\gamma^{ss}\left(\mathbf{q},\dot{\mathbf{q}}\right) = -\left\{2\mathbf{d}_{ij}^T\left(\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\dot{\mathbf{p}}_j - \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right)\dot{\mathbf{p}}_i\right) + 2\dot{\mathbf{d}}_{ij}^T\dot{\mathbf{d}}_{ij}\right\} \tag{9.8.10}$$

$$\left(\gamma^{ss}\right)_{\mathbf{q}} = 2\left(\dot{\mathbf{p}}_j^T\mathbf{B}^T\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right) - \dot{\mathbf{p}}_i^T\mathbf{B}^T\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right)\right)\Big[\mathbf{I} \quad \mathbf{B}\left(\mathbf{p}_i,\mathbf{s}_i'^P\right) \quad -\mathbf{I} \quad -\mathbf{B}\left(\mathbf{p}_j,\mathbf{s}_j'^P\right)\Big] \\ + 4\dot{\mathbf{d}}_{ij}^T\Big[\mathbf{0} \quad \mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right) \quad \mathbf{0} \quad -\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\Big] \tag{9.8.11}$$

$$\left(\gamma^{ss}\right)_{\dot{\mathbf{q}}} = 4\Big[\dot{\mathbf{d}}_{ij}^T \quad \mathbf{d}_{ij}^T\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right) + \dot{\mathbf{d}}_{ij}^T\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^P\right) \quad -\dot{\mathbf{d}}_{ij}^T \quad -\mathbf{d}_{ij}^T\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\dot{\mathbf{p}}_j - \dot{\mathbf{d}}_{ij}^T\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{s}_j'^P\right)\Big] \tag{9.8.12}$$

For terms involving θ in the "screw constraint",

$$\gamma^\theta\left(\mathbf{q},\dot{\mathbf{q}}\right) = -\Big\{\left(\cos\theta\mathbf{g}_i^T - \sin\theta\mathbf{f}_i^T\right)\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\dot{\mathbf{p}}_j + \mathbf{f}_j^T\left(\cos\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right)\dot{\mathbf{p}}_i \\ +2\left(\cos\theta\dot{\mathbf{g}}_i^T - \sin\theta\dot{\mathbf{f}}_i^T\right)\dot{\mathbf{f}}_j\Big\} \tag{9.8.13}$$

$$\left(\gamma^\theta\right)_{\mathbf{q}} = -\Big[\mathbf{0} \quad \dot{\mathbf{p}}_j^T\mathbf{B}^T\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\left(\cos\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right)\right) + 2\dot{\mathbf{f}}_j^T\left(\cos\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right) \\ \mathbf{0} \quad \dot{\mathbf{p}}_i^T\left(\cos\theta\mathbf{B}^T\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}^T\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right)\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right) + 2\left(\cos\theta\dot{\mathbf{g}}_i^T - \sin\theta\dot{\mathbf{f}}_i^T\right)\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\Big] \\ + \Big\{\left(\sin\theta\mathbf{g}_i^T + \cos\theta\mathbf{f}_i^T\right)\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right)\dot{\mathbf{p}}_j + \mathbf{f}_j^T\left(\sin\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) + \cos\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right)\dot{\mathbf{p}}_i \\ + 2\left(\sin\theta\dot{\mathbf{g}}_i^T + \cos\theta\dot{\mathbf{f}}_i^T\right)\dot{\mathbf{f}}\Big\}\theta_{\mathbf{q}}$$

(9.8.14)

$$\left(\gamma^{\theta}\right)_{\dot{\mathbf{q}}} = -\begin{bmatrix} \mathbf{0} & 2\mathbf{f}_j^T\left(\cos\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{f}_i'\right)\right) + 2\dot{\mathbf{f}}_j^T\left(\cos\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{g}_i'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_i,\mathbf{f}_i'\right)\right) \\ \mathbf{0} & 2\left(\cos\theta\mathbf{g}_i^T - \sin\theta\mathbf{f}_i^T\right)\mathbf{B}\left(\dot{\mathbf{p}}_j,\mathbf{f}_j'\right) + 2\left(\cos\theta\dot{\mathbf{g}}_i^T - \sin\theta\dot{\mathbf{f}}_i^T\right)\mathbf{B}\left(\mathbf{p}_j,\mathbf{f}_j'\right) \end{bmatrix} \tag{9.8.15}$$

For the "absolute" constraint,

$$\boldsymbol{\gamma}^{a}\left(\mathbf{q},\dot{\mathbf{q}}\right) = -\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^{P}\right)\dot{\mathbf{p}}_i \tag{9.8.16}$$

$$\boldsymbol{\gamma}^{a}_{\mathbf{q}_i} = \mathbf{0} \tag{9.8.17}$$

$$\boldsymbol{\gamma}^{a}_{\dot{\mathbf{q}}_i} = -\begin{bmatrix} \mathbf{0} & 2\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^{P}\right) \end{bmatrix} \tag{9.8.18}$$

For the "absolute driver" constraint,

$$\boldsymbol{\gamma}^{ad}\left(\mathbf{q},\dot{\mathbf{q}},t\right) = -\left(\mathbf{B}\left(\dot{\mathbf{p}}_i,\mathbf{s}_i'^{P}\right)\right)\dot{\mathbf{p}}_i + \ddot{\mathbf{C}}\left(t\right) \tag{9.8.19}$$

$$\boldsymbol{\gamma}^{a}_{\mathbf{q}_i} = \mathbf{0} \tag{9.8.20}$$

$$\boldsymbol{\gamma}^{ad}_{\dot{\mathbf{q}}_i} = \boldsymbol{\gamma}^{a}_{\dot{\mathbf{q}}_i} \tag{9.8.21}$$

For the "distance driver",

(9.8.22)

$$\gamma^{ssd}\left(\mathbf{q},\dot{\mathbf{q}},t\right) = \gamma^{ss}\left(\mathbf{q},\dot{\mathbf{q}}\right) + 2C\left(t\right)\ddot{C}\left(t\right) + 2\dot{C}^2\left(t\right) \tag{9.8.23}$$

$$\gamma^{ssd}_{\mathbf{q}} = \gamma^{ss}_{\mathbf{q}} \tag{9.8.24}$$

$$\gamma^{ssd}_{\dot{\mathbf{q}}} = \gamma^{ss}_{\dot{\mathbf{q}}} \tag{9.8.25}$$

For the "relative translation driver",

$$\gamma^{td}\left(\mathbf{q},\dot{\mathbf{q}},t\right) = \gamma^{d2}\left(\mathbf{q},\dot{\mathbf{q}}\right) + \ddot{C}\left(t\right) \tag{9.8.26}$$

$$\gamma^{td}_{\mathbf{q}} = \gamma^{d2}_{\mathbf{q}} \tag{9.8.27}$$

$$\gamma^{td}_{\dot{\mathbf{q}}} = \gamma^{d2}_{\dot{\mathbf{q}}} \tag{9.8.28}$$

For the "relative rotation driver",

$$\gamma^{rotd}(\mathbf{q},\dot{\mathbf{q}},t) = \gamma^{\theta}(\mathbf{q},\dot{\mathbf{q}}) + \ddot{C}(t) \tag{9.8.29}$$

$$\gamma^{rotd}_{\mathbf{q}} = \gamma^{\theta}_{\mathbf{q}} \tag{9.8.30}$$

$$\gamma^{rotd}_{\dot{\mathbf{q}}} = \gamma^{\theta}_{\dot{\mathbf{q}}} \tag{9.8.31}$$

Jacobian Transposed Times Lagrange Multiplier

The term $\left(\mathbf{\Phi}_{\mathbf{q}}^{T}\hat{\boldsymbol{\lambda}}\right)_{\mathbf{q}}$, where $\hat{}$ denotes a quantity that is held constant for purposes of partial differentiation, is required in sensitivity analysis. It can be computed for each of the basic constraints, using Jacobian is with respect to generalized coordinates of bodies i and j that are connected by the constraint. For these calculations, $\mathbf{q} = \left[\mathbf{q}_i^T, \mathbf{q}_j^T\right]^T$.

For the "dot-1 constraint", from Section 9.7,

$$\begin{aligned}
\left(\Phi_{\mathbf{q}}^{d1^T}\hat{\lambda}^{d1}\right)_{\mathbf{q}} &= \hat{\lambda}^{d1}\begin{bmatrix} \mathbf{0} \\ \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i)\mathbf{a}_i \\ \mathbf{0} \\ \mathbf{B}^T(\mathbf{p}_j,\mathbf{a}'_j)\mathbf{a}_j \end{bmatrix}_{\mathbf{q}} \\
&= \hat{\lambda}^{d1}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}(\mathbf{a}'_i,\mathbf{a}_i) + \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i)\mathbf{B}(\mathbf{p}_i,\mathbf{a}'_i) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}(\mathbf{a}'_j,\mathbf{a}_j) + \mathbf{B}^T(\mathbf{p}_j,\mathbf{a}'_j)\mathbf{B}(\mathbf{p}_j,\mathbf{a}'_j) \end{bmatrix}
\end{aligned} \tag{9.8.32}$$

For the "dot-2 constraint", from Section 9.7,

$$\left(\Phi_{\mathbf{q}}^{d2^T}\hat{\lambda}^{d2}\right)_{\mathbf{q}} = \hat{\lambda}^{d2}\begin{bmatrix} -\mathbf{a}_i \\ \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i)\mathbf{d}_{ij} - \mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\mathbf{a}_i \\ \mathbf{a}_i \\ \mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j)\mathbf{a}_i \end{bmatrix}_{\mathbf{q}}$$

$$= \hat{\lambda}^{d2}\begin{bmatrix} \mathbf{0} & -\mathbf{B}(\mathbf{p}_i,\mathbf{a}'_i) & \mathbf{0} & \mathbf{0} \\ -\mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i) & \left\{\begin{matrix} \mathbf{K}(\mathbf{a}'_i,\mathbf{d}_{ij}) - \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i)\mathbf{B}(\mathbf{p}_i,\mathbf{a}'_i) \\ -\mathbf{K}(\mathbf{s}'^P_i,\mathbf{a}_i) - \mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\mathbf{B}(\mathbf{p}_i,\mathbf{s}'^P_i) \end{matrix}\right\} & \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i) & \mathbf{B}^T(\mathbf{p}_i,\mathbf{a}'_i)\mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j) \\ \mathbf{0} & \mathbf{B}(\mathbf{p}_i,\mathbf{a}'_i) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j)\mathbf{B}(\mathbf{p}_i,\mathbf{a}'_i) & \mathbf{0} & \mathbf{K}(\mathbf{s}'^P_j,\mathbf{a}_i) \end{bmatrix}$$

(9.8.33)

For the "spherical constraint", from Section 9.7,

$$\left(\mathbf{\Phi}_{\mathbf{q}}^{s\,T}\hat{\boldsymbol{\lambda}}^{s}\right)_{\mathbf{q}} = \begin{bmatrix} -\hat{\boldsymbol{\lambda}}^s \\ -\mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\hat{\boldsymbol{\lambda}}^s \\ \hat{\boldsymbol{\lambda}}^s \\ \mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j)\hat{\boldsymbol{\lambda}}^s \end{bmatrix}_{\mathbf{q}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}(\mathbf{s}'^P_i,\hat{\boldsymbol{\lambda}}^s) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}(\mathbf{s}'^P_j,\hat{\boldsymbol{\lambda}}^s) \end{bmatrix} \qquad (9.8.34)$$

For the "distance constraint", from Section 9.7,

$$\left(\mathbf{\Phi}_{\mathbf{q}}^{ss^T}\hat{\lambda}^{ss}\right)_{\mathbf{q}} = 2\hat{\lambda}^{ss}\begin{bmatrix} -\mathbf{d}_{ij} \\ -\mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\mathbf{d}_{ij} \\ \mathbf{d}_{ij} \\ \mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j)\mathbf{d}_{ij} \end{bmatrix}_{\mathbf{q}}$$

$$= 2\hat{\lambda}^{ss}\begin{bmatrix} \mathbf{I} & \mathbf{B}(\mathbf{p}_i,\mathbf{s}'^P_i) & -\mathbf{I} & -\mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j) \\ \mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i) & \mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\mathbf{B}(\mathbf{p}_i,\mathbf{s}'^P_i) & -\mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i) & -\mathbf{B}^T(\mathbf{p}_i,\mathbf{s}'^P_i)\mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j) \\ \mathbf{I} & \mathbf{B}(\mathbf{p}_i,\mathbf{s}'^P_i) & -\mathbf{I} & -\mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j) \\ -\mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j) & -\mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j)\mathbf{B}(\mathbf{p}_i,\mathbf{s}'^P_i) & \mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j) & \mathbf{B}^T(\mathbf{p}_j,\mathbf{s}'^P_j)\mathbf{B}(\mathbf{p}_j,\mathbf{s}'^P_j) \end{bmatrix}$$

(9.8.35)

For the "screw constraint", which is comprised of the cylindrical joint equations and Eq. 9.4.25 that involves the dot-2 constraint function and the rotation angle θ, $\left(\theta_{\mathbf{q}}^{T}\lambda^{scr}\right)_{\mathbf{q}}$ must be determined. From Section 9.7,

$$\left(\theta_{\mathbf{q}}^{T}\lambda^{scr}\right)_{\mathbf{q}} = \begin{bmatrix} \mathbf{0} \\ \left(\cos\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) - \sin\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right)\mathbf{f}_{j} \\ \mathbf{0} \\ \mathbf{B}^{T}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right)\left(\cos\theta\mathbf{g}_{i} - \sin\theta\mathbf{f}_{i}\right) \end{bmatrix}_{\mathbf{q}}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left\{ \begin{array}{l} \left(\cos\theta\mathbf{K}\left(\mathbf{g}_{i}',\mathbf{f}_{j}\right) - \sin\theta\mathbf{K}\left(\mathbf{f}_{i}',\mathbf{f}_{j}\right)\right) \\ -\left(\sin\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) + \cos\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right)\mathbf{f}_{j} \\ \times\mathbf{f}_{j}^{T}\left(\cos\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right) \end{array} \right\} & \mathbf{0} & \left\{ \begin{array}{l} \left(\cos\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) - \sin\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right)\mathbf{B}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right) \\ -\left(\sin\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) + \cos\theta\mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right)\mathbf{f}_{j} \\ \times\left(\cos\theta\mathbf{g}_{i}^{T} - \sin\theta\mathbf{f}_{i}^{T}\right)\mathbf{B}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right) \end{array} \right\} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left\{ \begin{array}{l} \mathbf{B}^{T}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right)\left(\cos\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{f}'\right)\right) \\ -\mathbf{B}^{T}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right)\left(\sin\theta\mathbf{g}_{i} + \cos\theta\mathbf{f}_{i}\right) \\ \times\mathbf{f}_{j}^{T}\left(\cos\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{g}_{i}'\right) - \sin\theta\mathbf{B}\left(\mathbf{p}_{i},\mathbf{f}_{i}'\right)\right) \end{array} \right\} & \mathbf{0} & \left\{ \begin{array}{l} \left(\cos\theta\mathbf{K}\left(\mathbf{f}_{j}',\mathbf{g}_{i}\right) - \sin\theta\mathbf{K}\left(\mathbf{f}_{j}',\mathbf{f}_{i}\right)\right) \\ -\mathbf{B}^{T}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right)\left(\sin\theta\mathbf{g}_{i} + \cos\theta\mathbf{f}_{i}\right) \\ \times\left(\cos\theta\mathbf{g}_{i}^{T} - \sin\theta\mathbf{f}_{i}^{T}\right)\mathbf{B}\left(\mathbf{p}_{j},\mathbf{f}_{j}'\right) \end{array} \right\} \end{bmatrix}$$

(9.8.36)

For the "absolute" constraints, which involve only body i, from Section 9.7,

$$\begin{aligned} \left(\mathbf{\Phi}_{\mathbf{q}_{i}}^{a\,T}\hat{\boldsymbol{\lambda}}^{a}\right)_{\mathbf{q}} &= \begin{bmatrix} \mathbf{B}^{T}\left(\mathbf{p}_{i},\mathbf{s}_{i}'^{P}\right)\hat{\boldsymbol{\lambda}}^{a} \\ \mathbf{I} \end{bmatrix}_{\mathbf{q}_{i}} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{K}\left(\mathbf{s}_{i}'^{P},\hat{\boldsymbol{\lambda}}^{a}\right) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \qquad (9.8.37)$$