

## ON THE PARAMETRIZATION OF THE THREE-DIMENSIONAL ROTATION GROUP\*

JOHN STUELPNAGEL†

**1. Introduction.** The problem of parametrizing the group of rotations of Euclidean 3-space has been of interest since 1776, when Euler first showed that this group is itself a 3-dimensional manifold. A primary application of such a parametrization occurs in the integration of the equations of motion of a rigid body. To describe the orientation of the body relative to its center of mass, we assume given two sets of mutually orthogonal unit vectors, or frames, one frame being attached to the body and moving with it, the other being constant and coinciding with the moving frame at time  $t = 0$ . The moving frame at time  $t$  is obtained by applying a rotation  $X(t)$  to the fixed frame, and  $X(t)$  satisfies the differential equation  $\dot{X}(t) = \Omega(t)X(t)$ , with  $X(0) = I$ , the identity matrix;  $\Omega(t)$  is defined by the relation  $\Omega(t)v = v \times \omega(t)$  for all 3-vectors  $v$ , where  $\omega(t)$  is the angular velocity vector. We assume  $\Omega(t)$  is known, so it is necessary to integrate the matrix differential equation, or equivalently, a system of nine scalar equations, to obtain  $X(t)$ . However, if it is possible to represent  $X(t)$  by a set of less than nine parameters, then the given system is equivalent to a system with fewer than nine scalar equations, and the problem may be simplified.

In this paper we show why it is topologically impossible to have a global 3-dimensional parametrization without singular points for the rotation group. This is a special case of a corollary to Brouwer's theorem on the invariance of domain. We also point out that, although Hopf showed in 1940 that five is the minimum number of parameters which suffices to represent the rotation group in a 1-1 global manner, the so-called "quaternion method" of parametrizing the group in a 1-2 way, using 4 parameters, is sufficient for practical purposes. In addition, three 3-dimensional parametrizations, as well as Hopf's method of using 5 parameters, are examined.

This paper is aimed primarily at those who have been led by their involvement with the practical applications of this problem to wonder if there were not a way to improve the present methods of parametrizing rotations without adding redundant parameters; while the answer is negative, it is possible, by adding only one redundant parameter, to obtain a method of representing unrestricted rotations, which leads to simpler differential equations than any of the other methods presented.

\* Received by the editors March 9, 1964.

This research was supported in part by the United States Air Force through the Air Force Office of Scientific Research, Office of Aerospace Research, under Contract No. AF 49(638)-1242, in part by the National Aeronautics and Space Administration under Contract No. NASw-845 and in part by the Office of Naval Research under Contract No. Nonr-3693(00).

† Research Institute for Advanced Studies (RIAS), 7212 Bellona Avenue, Baltimore, Maryland. Now at Air Arm Division, Westinghouse Electric Corporation, Box 746, Baltimore, Maryland.

The author is grateful to Dr. R. E. Kalman, of RIAS, for suggesting that this paper be written, as well as for many helpful conversations.

**2. Notation and preliminaries.** We shall use  $K$  to denote the set of complex  $2 \times 2$  matrices of the form  $\kappa = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ ; taking as basis

$$\kappa_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \kappa_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \kappa_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \kappa_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

we see that  $K$  is a 4-dimensional associative noncommutative division algebra over the real numbers, and the map sending  $\alpha_0\kappa_0 + \alpha_1\kappa_1 + \alpha_2\kappa_2 + \alpha_3\kappa_3$  onto  $\frac{1}{2}(\alpha_0 + \alpha_1i + \alpha_2j + \alpha_3k)$  is an isomorphism of  $K$  with the quaternions, where  $i, j, k$  here represent the usual basis for the quaternions. The determinant of  $\kappa$  is the square of the quaternion norm.  $U$  will stand for the subset of  $K$  of determinant 1, and  $K_0$  the subset of trace zero. The elements of  $U$  are just the  $2 \times 2$  complex unitary matrices of determinant 1, so  $U$  is a group, and  $U$  is topologically equivalent to the 3-sphere (the unit sphere in  $E^4$ ), since any element of  $U$  is of the form

$$u = \begin{pmatrix} u_1 + iu_2 & u_3 + iu_4 \\ -u_3 + iu_4 & u_1 - iu_2 \end{pmatrix}$$

with  $\sum u_j^2 = 1$ .  $K_0$  is spanned by the set  $Q = \{\kappa_1, \kappa_2, \kappa_3\}$ . For fixed  $u \in U, x \in K_0$ , define linear maps of  $K_0$  into itself by  $\Gamma_u(\kappa) = u\kappa u^{-1}$ , and  $\Delta_x(\kappa) = \kappa x - x\kappa$ , for any  $\kappa \in K_0$ . The matrices of  $\Gamma_u$  and  $\Delta_x$  with respect to  $Q$  will be denoted by  $\gamma(u)$  and  $\delta(x)$ , respectively.

The rotation group will be denoted by  $R$ ; it consists of those orthogonal  $3 \times 3$  matrices with determinant 1.  $I_n$  and  $O_n$  will designate the identity and zero matrices, respectively, of dimension  $n$ ; the subscript will be omitted except when confusion is possible. The transpose of a vector or matrix will be indicated by a prime, e.g.,  $x'$  is the row vector obtained by transposing the column vector  $x$ . For any matrix  $A$ ,  $\text{Tr } A$  denotes the trace of  $A$ .

The theorem of Brouwer on the invariance of domain, to which we will appeal in a later section, is stated as follows, and is proved in Hurewicz and Wallman [2]: If  $A$  and  $B$  are homeomorphic subsets of a Euclidean space  $E^n$  and  $A$  is open, then  $B$  is open.

**3. The topology of  $R$ .** The matrix  $\gamma(u)$  of  $\Gamma_u$  with respect to  $Q$  is easily seen to be

$$\begin{pmatrix} u_1^2 + u_2^2 - u_3^2 - u_4^2 & 2(-u_1u_4 + u_2u_3) & 2(u_1u_3 + u_2u_4) \\ 2(u_1u_4 + u_2u_3) & u_1^2 - u_2^2 + u_3^2 - u_4^2 & 2(-u_1u_2 + u_3u_4) \\ 2(-u_1u_3 + u_2u_4) & 2(u_1u_2 + u_3u_4) & u_1^2 - u_2^2 - u_3^2 + u_4^2 \end{pmatrix},$$

which is orthogonal for all  $u \in U$ , and has determinant 1. Also, for  $u, v \in U$ ,  $\gamma(u)\gamma(v) = \gamma(uv)$ , so  $\gamma$  is a group homomorphism. Since  $\gamma$  is continuous, and  $U$  is compact and connected,  $\gamma(U)$  is a compact connected subgroup of  $R$ . The

only compact connected subgroups of  $R$  are known to be  $I$ ,  $R$ , and the groups of rotations about a fixed axis. Since  $\gamma(U)$  leaves no axis fixed,  $\gamma(U) = R$ . Note that  $\gamma(u) = \gamma(v)$  if and only if  $u = -v$ , so  $\gamma$  is a two-to-one map of  $U$  onto  $R$ . Recalling that  $U$  is topologically a 3-sphere, we see that  $R$  is topologically equivalent to the sphere with antipodal points identified, that is, projective 3-space.

To find a 1-1 global parametrization of the rotation group using  $k$  parameters, it is necessary to embed the rotation group  $R$  in the Euclidean space  $E^k$ , that is, to find a differentiable 1-1 map with differentiable inverse which carries  $R$  into  $E^k$ , and use the image points as representatives of the rotation matrices.

Since  $R$  is a 3-dimensional manifold, each point  $r$  has a neighborhood  $U_r$  which is homeomorphic to an open subset of  $E^3$ . If there were a homeomorphism  $h$  of  $R$  into  $E^3$ , then  $h(U_r)$  would be open in  $E^3$  by Brouwer's theorem, so  $h(R)$ , being the union of all  $h(U_r)$  for  $r \in R$ , would be open in  $E^3$ . But  $R$  is compact, and  $h(R)$ , being the continuous image of a compact space, would be compact. It is a well-known fact that no Euclidean space contains an open compact subset, so there can exist no such homeomorphism.

The impossibility of embedding  $R$  topologically in  $E^4$  was first proved by H. Hopf in 1940 [1]. The proof is based on a knowledge of the homology ring of projective 3-space, and will not be included here. It is possible, however, to embed  $R$  in  $E^5$ , as Hopf showed, and we shall examine this embedding in the next section.

**4. Five- and six-dimensional parametrizations.** An element of  $R$  is determined when its first two columns are specified, since the third column is the cross-product of these two. Thus the six-vector obtained by vertical juxtaposition of these two columns serves to parametrize the group in a 1-1 global manner. So if  $X \in R$ , and  $X_0$  denotes the  $3 \times 2$  matrix obtained by deleting the last column of  $X$ , then the differential equation  $\dot{X}_0(t) = \Omega(t)X_0(t)$  is equivalent to the equation  $\dot{X}(t) = \Omega(t)X(t)$ , but contains only 6 scalar variables.

Let  $x = (x_1, x_2, x_3, x_4, x_5, x_6)'$  be the column vector representing the first two columns of the matrix  $(1/\sqrt{2})X$ , where  $X \in R$ . Then we have the identities  $x'x = 1, x'J_i x = 0, i = 1, 2$ , where

$$J_1 = \begin{pmatrix} I_3 & 0_3 \\ 0_3 & -I_3 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}.$$

Since  $x'x = 1$ , the set  $M$  of all points satisfying the above conditions is contained in the unit sphere  $S^5$  in  $E^6$ . If  $a$  is any point of  $S^5$  not in  $M$ , we can project  $S^5 - \{a\}$  stereographically onto the hyperplane orthogonal to  $a$ , and thus obtain an embedding of  $M$ , which is topologically equivalent to  $R$ , in  $E^5$ .

To do this explicitly, let  $V$  be a  $5 \times 6$  matrix with  $Va = 0, VV' = I_5$ . Then  $V'V$  is the projection along  $a$  onto the subspace of  $E^6$  orthogonal to  $a$ . For  $x \in M$ , let  $y = Vx/(1 - a'x)$ . This represents the point which is the intersection of the line joining  $x$  and  $a$  with the hyperplane orthogonal to  $a$ . It is defined for all  $x \in M$ , since the denominator vanishes only if  $x = a$ , but  $a \notin M$ . The cor-

respondence is 1-1, for if  $y$  is a 5-vector, then  $V'y$  is orthogonal to  $a$ , and the line joining  $V'y$  and  $a$  intersects the unit sphere in a single point  $x$ , where

$$x = \frac{a(y'y - 1) + 2V'y}{(y'y + 1)}.$$

If  $x \in M$ , then  $y$  satisfies the equations  $a'J_i a(y'y - 1)^2 + 4y'VJ_i a(y'y - 1) + 4y'VJ_i V'y = 0, i = 1, 2$ . We now have the 5-vector  $y$  satisfying the two equations above representing a point of  $R$  in a 1-1 manner, and we wish to find the differential equation satisfied by  $y$ , if  $\dot{x} = Ax$ , where

$$A = \begin{pmatrix} \Omega & 0_3 \\ 0_3 & \Omega \end{pmatrix}, \quad A' + A = 0.$$

Differentiating the equation for  $y$  above with respect to  $t$ , we obtain

$$\begin{aligned} \dot{y} &= \frac{(1 - a'x)V\dot{x} + (a'\dot{x})Vx}{(1 - a'x)^2} = \frac{1}{2}(y'y + 1)(V + ya')\dot{x} \\ &= \left(\frac{1}{2}(y'y - 1)I - yy'\right)VAa + VAV'y. \end{aligned}$$

The resulting equation for  $y$  is clearly not as simple as the original linear equation for  $x$ , and there is no apparent advantage in the reduction in the number of scalar variables by this method. It is possible that an embedding in  $E^5$  may be obtained which leads to a simpler equation for  $y$ . This parametrization is primarily of interest because it uses the smallest possible number of scalar variables for an everywhere defined, 1-1, continuous representation of  $R$ , and because the given embedding is the most obvious and probably the simplest which can be obtained with five parameters.

**5. The quaternion method.** As we saw in §3, there is a 2-1 correspondence  $\gamma$  between the quaternions of unit norm and the elements of  $R$ . Given the differential equation  $\dot{X}(t) = \Omega(t)X(t)$  in  $R$ , we can determine a differential equation  $\dot{u}(t) = \sigma(t)u(t)$  in  $U$  such that  $\gamma(u(t)) = X(t)$ , and we now indicate how this is done.

Let  $\alpha = \alpha_1\kappa_1 + \alpha_2\kappa_2 + \alpha_3\kappa_3 \in K_0$ , and consider  $\Delta_\alpha$  acting on  $K_0$ . The matrix of  $\Delta_\alpha$  with respect to the basis  $Q$  is

$$\begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}.$$

Now if

$$\Omega(t) = \begin{pmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{pmatrix},$$

and

$$\sigma(t) = -\sum_{i=1}^3 \omega_i(t) \kappa_i,$$

then  $\Delta_\sigma = -\Omega$ . If  $u(t)$  is the solution of  $\dot{u}(t) = \sigma(t)u(t)$  such that  $u(0) = I$ , it is easily seen that  $u(t) \in U$ , since  $\sigma(t) \in K_0$ . Also, for any fixed  $\kappa \in K_0$ ,  $\dot{\Gamma}_u(\kappa) = (u\dot{\kappa}u^{-1}) = \dot{u}\kappa u^{-1} - u\kappa u^{-1}\dot{u}u^{-1} = \sigma(u\kappa u^{-1}) - (u\kappa u^{-1})\sigma = -\Delta_\sigma(\Gamma_u(\kappa))$ . It follows that if  $X = \gamma(u)$ , the matrix of  $\Gamma_u$  with respect to  $Q$ , then  $\dot{X} = \Omega X$ , and  $\gamma$  thus maps solutions of  $\dot{u} = \sigma u$  onto solutions of  $\dot{X} = \Omega X$ . If  $u(0) = I$ , then  $X(0) = \gamma(u(0)) = I$ , so the desired particular solution is obtained.

In terms of the real parameters  $u_1, u_2, u_3, u_4$  appearing in  $u$ , the differential equation  $\dot{u} = \sigma u$  becomes

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ -\omega_1 & 0 & \omega_3 & -\omega_2 \\ -\omega_2 & -\omega_3 & 0 & \omega_1 \\ -\omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

It should be noted that the original linear equation is transformed into a linear equation; this was not the case with the 5-dimensional method, so this method is obviously far superior to the previous one. Although the parametrization is not 1-1, no difficulties arise, since  $\gamma$  is a local homeomorphism.

It would be reasonable to ask whether it might be possible to obtain a representation of this form, that is, one-to-many, using only three parameters, but still possessing the property of being a local homeomorphism, and having no singular points. The answer is no, for this would force the parameter set to be a "covering space" of  $R$ , and it is known that the 3-sphere, which cannot be represented topologically by less than 4 parameters, is the only covering space of  $R$ , except for  $R$  itself.

**6. Three-dimensional parametrizations.** As we showed earlier, no 3-dimensional parametrization can be both global and nonsingular; however, there are at least three such parametrizations in common use, each of which has certain advantages, and we present them here.

The Euler angles are defined in many different ways, depending on the problem to be solved. The definition adopted here is convenient for problems involving orientation of aircraft, etc., since the Euler angles  $\phi, \theta, \psi$  correspond to the commonly used parameters of roll, pitch, and yaw, respectively.

If

$$X = \begin{pmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_3 \\ x_3 & x_6 & x_9 \end{pmatrix}$$

is in  $R$ , we define the Euler angles for  $X$  as follows: Let  $\xi^2 = x_1^2 + x_4^2$ ,  $\xi \geq 0$ . If  $\xi \neq 0$ , define  $\phi, \theta, \psi$  by

$$\begin{aligned} \cos \phi &= x_9/\xi, & \sin \phi &= x_3/\xi; \\ \cos \theta &= \xi, & \sin \theta &= -x_7; \\ \cos \psi &= x_1/\xi, & \sin \psi &= x_4/\xi. \end{aligned}$$

If  $\xi = 0$ , so  $x_7^2 = 1$ , then  $\theta = -x_7\pi/2$ , but  $\phi$  and  $\psi$  are not uniquely determined, being subject only to the conditions  $\cos(x_7\phi + \psi) = x_5$ ,  $\sin(x_7\phi + \psi) = -x_2$ . In particular, we may, if we wish, always choose  $\psi = 0$  if  $\theta = \pm\pi/2$ . This determines  $\phi$  uniquely, but the resulting parameters are not continuous functions of  $X$  at  $\theta = \pm\pi/2$ . The Euler angles enable us to factor  $X$  into a product of rotations about the vertical, transverse, and longitudinal axes of the moving rigid body; in fact,

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that the Euler angles give a parametrization of the rotation group except at the points  $\xi = 0$ , or  $\theta = \pm\pi/2$ .

If  $\dot{X} = \Omega X$ , where  $\Omega$  is as in §5, then it is seen by direct computation that

$$\begin{pmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Since the determinant of the matrix on the left is  $\cos \theta$ , it is clearly singular if  $\theta = \pm\pi/2$ , so  $\dot{\phi}, \dot{\theta}, \dot{\psi}$  are determined only if  $\theta$  does not take on these values. If it is known in advance that certain orientations of the rigid body cannot be assumed, then we may be able to choose the original coordinate system in such a way that these orientations correspond to the singular points. In this case, the Euler angles furnish a satisfactory method for representing the necessary subset of  $R$ .

A second method of obtaining a 3-dimensional parametrization of the rotation group is based on the facts that (1) for any  $3 \times 3$  skew-symmetric matrix  $S$ ,  $\exp S$  is orthogonal, and (2) any rotation matrix is the exponential of some skew-symmetric matrix.

Let  $S$  be a  $3 \times 3$  skew-symmetric matrix and  $\sigma^2 = -\frac{1}{2} \text{Tr } S^2$ ,  $\sigma \geq 0$ . Then the characteristic polynomial of  $S$  is  $\lambda^3 + \sigma^2\lambda$ , so  $S^3 = -\sigma^2 S$ . The power series for  $X = \exp S$  may consequently be simplified, using the relations  $S^{2n} = (-1)^{n-1} \cdot \sigma^{2n-2} S^2$ ,  $S^{2n+1} = (-1)^n \sigma^{2n} S$ , and collecting terms, to

$$X = I + \frac{\sin \sigma}{\sigma} S + \frac{1 - \cos \sigma}{\sigma^2} S^2.$$

The characteristic roots of  $X$  are  $1, \cos \sigma \pm i \sin \sigma$ . It is not hard to see that  $\exp S_1 = \exp S_2$  if and only if  $S_2 = 0$  and  $\sigma_1^2 = 2k\pi$ , or  $S_1 = S_2 + (2k\pi/\sigma_2)S_2$ , for some integer  $k$ , where  $\sigma_i = -\frac{1}{2} \text{Tr } S_i^2$ . In particular, if we restrict our attention to those skew-symmetric matrices  $S$  for which  $\sigma \leq \pi$ , then  $\exp S_1 = \exp S_2$  if and only if  $S_1 = \pm S_2$  and  $\sigma_1 = \sigma_2 = \pi$ .

Conversely, let  $X \in R$ , and let  $\alpha = \text{Tr } X$ , so  $X^3 - \alpha X^2 + \alpha X - I = 0$ , and  $-1 \leq \alpha \leq 3$ . Then  $-1 \leq (\alpha - 1)/2 \leq 1$ ; hence there is a unique angle  $\sigma$ ,  $0 \leq \sigma < \pi$ , with  $\cos \sigma = (\alpha - 1)/2$ . If  $\alpha \neq -1, 3$ , let

$$S = \frac{-\sigma(1 + 2 \cos \sigma)}{\sin \sigma} I + \frac{\sigma(1 + \cos \sigma)}{2 \sin \sigma} X - \frac{\sigma}{2 \sin \sigma} X^2;$$

if  $\alpha = 3$ , let  $S = 0$ . Then  $\exp S = X$ , and  $S$  is skew-symmetric. If  $\alpha = -1$ , then  $S^2 = (\pi^2/2)(X - I)$  has two skew-symmetric solutions  $\pm S$ , and  $\exp S = \exp(-S) = X$ .

Using the correspondence above, we can parametrize the rotation group by the set of skew-symmetric matrices  $S$  with  $\sigma \leq \pi$ ; every rotation matrix corresponds to at least one skew-symmetric matrix, and those rotations which are involutions ( $X$  is an involution if  $X^2 = I$ ) correspond to two skew-symmetric matrices. If we identify

$$S = \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix}$$

with the vector  $s = (s_1, s_2, s_3)'$ , then  $\sigma = |s|$ , and  $R$  is seen to be topologically equivalent to the ball  $|s| \leq \pi$  with boundary points identified.

The original differential equation  $\dot{X} = \Omega X$  is transformed by this substitution into the equation

$$\dot{S} = \Omega - \frac{1}{2}(\Omega S - S\Omega) + \left( \frac{2 - \sigma \cot(\sigma/2)}{2\sigma^2} \right) (S^2\Omega + \Omega S^2 - 2S\Omega S).$$

The derivation of this equation requires some lengthy computations, which are omitted. Although the number of parameters has been reduced to three, it is clear that the form of the transformed differential equation is considerably more complex than that of the original. Also, the transformed equation has a pole at  $\sigma = 2\pi$ , just as we would expect from the nature of the map  $X \rightarrow S$ , since the set of  $S$  for which  $-\frac{1}{2} \text{Tr } S^2 = 4\pi^2$  is collapsed by the exponential map into the identity.

The final 3-dimensional parametrization we shall consider is known as the Cayley parametrization (not to be confused with the Cayley-Klein parameters), and also uses  $3 \times 3$  skew-symmetric matrices to represent rotations. If  $S$  is skew-symmetric, we again let  $\sigma^2 = -\frac{1}{2} \text{Tr } S^2$ , and now set

$$X = (I - S)(I + S)^{-1} = I - \frac{2}{1 + \sigma^2} S + \frac{2}{1 + \sigma^2} S^2.$$

Then  $X$  is orthogonal, and the characteristic equation of  $X$  is  $\lambda^3 - \alpha\lambda^2 + \alpha\lambda - 1 = 0$ , with  $\alpha = [(3 - \sigma^2)/(1 + \sigma^2)]$ , so the characteristic roots of  $X$  are  $1, [1/(1 + \sigma^2)](1 - \sigma^2 \pm 2i\sigma)$ . These roots are real only if  $\sigma = 0$ , in which case all roots are  $+1$ . Thus no rotation matrix having  $-1$  as an eigenvalue may be obtained from a skew-symmetric matrix in this manner.

Conversely, if  $X \in R$ , and  $\alpha = \text{Tr } X$ , then, for  $\alpha \neq -1$ , set

$$S = (I - X)(I + X)^{-1} = \frac{1}{1 + \alpha} (\alpha I - (1 + \alpha)X + X^2).$$

$S$  is then skew-symmetric, and this is the inverse of the above correspondence. Differentiating this last equation, substituting  $\dot{X} = \Omega X$ , and simplifying, we obtain  $\dot{S} = \frac{1}{2}(S\Omega S - S\Omega + \Omega S - \Omega)$ , a Riccati matrix equation for  $S$ .

In this case, if it is known beforehand that  $\text{Tr } X$  is never  $-1$ , this parametrization serves to represent all allowed orientations.

**7. Conclusion.** In evaluating the usefulness of a parametrization of  $R$ , several factors must be considered. Among these are (1) the number of parameters needed, (2) the form of the transformed differential equations, (3) the susceptibility to error of the new equations in machine integration of these equations, and (4) the ease with which a desired output can be obtained when these equations are integrated.

As we have seen, the 6-dimensional parametrization, using the first two columns of a rotation matrix to describe it, leads to linear equations, and the output is in a readily usable form, since  $X$  is very simply obtained from the given six parameters.

The 5-dimensional parametrization leads to nonlinear equations, and an undesirable amount of computation is necessary to obtain  $X$  as an output. This method, while using one less parameter than the previous method, does not appear to have anything in particular to recommend it, and it is included only because it uses the smallest possible number of parameters in a 1-1 global parametrization.

The 4-dimensional or quaternion method has the advantages of leading to linear equations while using only one redundant parameter, and representing the most general possible motion of the body. At the same time, the coefficients of  $X$  are obtained as quadratic functions of the coefficients of  $u$ .

As we showed in §3, no 3-dimensional parametrization can be both global and nonsingular. If the parametrization is global, i.e., every rotation matrix determines some finite values of the parameters, then there must be points where the parameter values are not uniquely defined, and in this case the derivatives of the parameters are obviously not defined, so the transformed differential equations become singular at these points, that is, the derivatives become infinite. This occurs, for example, with the Euler angles and the exponential parametrization. On the other hand, the Cayley parametrization leads to a well-defined differential equation, being nonsingular, but it does not represent any rotation matrices of trace  $-1$ , which is a distinct disadvantage, since this will not even allow  $180^\circ$  rotations about a fixed axis.

The only commonly used methods among those presented here are the 6- and 4-parameter methods, and the Euler angles. A comparison of the advantages and disadvantages of these methods is made by Robinson in [3]; he concludes that the quaternion method is the best, at any rate from the standpoint of analog computation, for handling unrestricted rotations, although the Euler angles are useful because of their simple interpretations as roll, pitch and yaw. That is, the Euler angles themselves provide a usable output, whereas with the quaternion method, it is necessary to transform the solution to the rotation group after integrating.

## REFERENCES

- [1] H. HOPF, *Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume*, Vierteljschr. Naturforsch. Ges. Zürich, 1940.
- [2] W. HUREWICZ AND H. WALLMAN, *Dimension Theory*, Princeton University Press, Princeton, 1948.
- [3] A. C. ROBINSON, *On the use of quaternions in simulation of rigid-body motion*, Wright Air Development Center Technical Report 58-17, Dayton, Ohio.