Example: Partial derivative with chain Rule

Let $f(y) = \begin{bmatrix} 2y_1 + y_2^2 \\ y_1 \cdot y_2 \end{bmatrix}$

$$y = y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_1^2 - x_2 \end{bmatrix}, \quad \text{here } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

f depends on y which depends on x. Therefore f depends on x.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x} = \begin{bmatrix} 2 & 2y_2 \\ y_2 & y_1 \end{bmatrix} \begin{bmatrix} x_2 & x_1 \\ 2x_1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_2 + 4x_1 y_2 & 2x_1 - 2y_2 \\ x_2 y_2 + 2x_1 y_1 & x_1 y_2 - y_1 \end{bmatrix} = \begin{bmatrix} 2x_2 + 4x_1(x_1^2 - x_2) & 2x_1 - 2(x_1^2 - x_2) \\ x_2(x_1^2 - x_2) + 2x_1^2 x_2 & x_1(x_1^2 - x_2) - x_1 x_2 \end{bmatrix}$$

The same result can be computed without the chain rule:

$$f(x) = \begin{bmatrix} 2x_1 x_2 + (x_1^2 - x_2)^2 \\ x_1 x_2 (x_1^2 - x_2) \end{bmatrix}$$

Then,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2x_2 + 4x_1(x_1^2 - x_2) & 2x_1 + 2(x_2 - x_1^2) \\ x_2(x_1^2 - x_2) + 2x_1^2 x_2 & x_1(x_1^2 - x_2) - x_1 x_2 \end{bmatrix}$$

The same, as expected.