Example : Chain Rule

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad f(y) = 3y_1^2 + \sin y_2 .$$

In turn, y depends on a variable $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as follows:

$$y = y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 2x_1 + \log x_2 + \sqrt{x_3} \\ (x_1 - x_2)^2 \end{bmatrix}$$

Therefore, f depends on x (because it depends on y which depends on x): $f = f(y(x)) = f(x)$.

According to the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$\frac{\partial f}{\partial y} = \begin{bmatrix} 6y_1 & \cos y_2 \end{bmatrix}$$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} 2 & \frac{1}{x_2} & \frac{1}{2\sqrt{x_3}} \\ 2(x_1 - x_2) & 2(x_2 - x_1) & 0 \end{bmatrix}$$

Therefore

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 6y_1 & \cos y_2 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{x_2} & \frac{1}{2\sqrt{x_3}} \\ 2(x_1 - x_2) & 2(x_2 - x_1) & 0 \end{bmatrix}$$

$$= \left[12y_1 + 2(x_1 - x_2)\cos y_2 \quad \frac{6y_1}{x_2} + 2\cos y_2 (x_2 - x_1) \quad \frac{3y_1}{\sqrt{x_3}} \right]$$

= (based on the expression of y_1 and y_2)

$$= \left[12(2x_1 + \log x_2 + \sqrt{x_3}) + 2(x_1 - x_2)\cos(x_1 - x_2)^2 \;\middle|\; \frac{6(2x_1 + \log x_2 + \sqrt{x_3})}{x_2} + 2(x_2 - x_1)\cos(x_1 - x_2)^2 \;\middle|\; \frac{3(2x_1 + \log x_2 + \sqrt{x_3})}{\sqrt{x_3}} \right].$$

To verify the above result, note that f can be expressed as a function of x as follows:

$$f(x) = 3(2x_1 + \log x_2 + \sqrt{x_3})^2 + \sin(x_1 - x_2)^2$$

Then:

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \;\middle|\; \frac{\partial f}{\partial x_2} \;\middle|\; \frac{\partial f}{\partial x_3} \right]$$

$$= \left[12(2x_1 + \log x_2 + \sqrt{x_3}) + 2(x_1 - x_2)\cdot\cos(x_1 - x_2)^2 \;\middle|\; \frac{6(2x_1 + \log x_2 + \sqrt{x_3})}{x_2} + 2(x_2 - x_1)\cos(x_1 - x_2)^2 \;\middle|\; \frac{3(2x_1 + \log x_2 + \sqrt{x_3})}{\sqrt{x_3}} \right]$$

This confirms that indeed,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial x}.$$