CHAPTER TWO

Planar Vectors, Matrices, and Differential Calculus

Vector and matrix algebra form the mathematical foundation for kinematics and dynamics. Geometry of motion is at the heart of both the kinematics and dynamics of mechanical systems. Vector analysis is the time-honored tool for describing geometry. In its geometric form, however, vector algebra is not well suited to computer implementation. In this chapter, a systematic matrix formulation of planar vector algebra, referred to as the algebraic vector representation, is presented for use throughout Part One. This form of vector representation, in contrast to the more traditional geometric form, is easier to use for both formula manipulation and computer implementation. Multivariable differential calculus plays a key role in kinematic analysis for both writing and solving the equations of motion of mechanical systems. Basic ideas and notations of matrices and multivariable differential calculus are developed in this chapter for use throughout the text. Key formulas are summarized at the end of the chapter for easy reference.

2.1 GEOMETRIC VECTORS

The *geometric vector* $\vec{a}$ in Fig. 2.1.1, beginning at point A and ending at point B, is defined as the directed line segment from A to B. The *magnitude of a vector* $\vec{a}$ is its length and is denoted by a or $|\vec{a}|$.

Multiplication of a vector $\vec{a}$ *by a scalar* $\alpha > 0$ is defined as a vector in the same direction as $\vec{a}$, but having magnitude αa. Multiplication of a vector $\vec{a}$ by a scalar $\alpha < 0$ is the vector with magnitude $|\alpha| a$ and opposite direction of $\vec{a}$. The *negative of a vector* is obtained by multiplying the vector by -1. It is a vector with the same magnitude but opposite direction. A *unit vector,* that is, a vector having a magnitude of 1 unit, in the direction $\vec{a} \neq \vec{0}$ is $(1/a)\vec{a}$.

Two vectors $\vec{a}$ and $\vec{b}$ are added according to the *parallelogram rule,* as

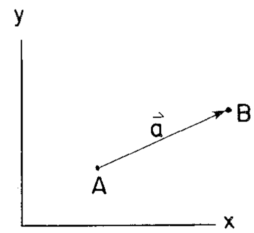

Figure 2.1.1 Vector from point A to point B.

shown in Fig. 2.1.2. In vector notation, the vector sum is written as

$$\vec{c} = \vec{a} + \vec{b} \tag{2.1.1}$$

Addition of vectors and multiplication of vectors by scalars obey the following rules [21]:

$$\begin{aligned} \vec{a} + \vec{b} &= \vec{b} + \vec{a} \\ (\alpha + \beta)\vec{a} &= \alpha\vec{a} + \beta\vec{a} \end{aligned} \tag{2.1.2}$$

where α and β are scalars.

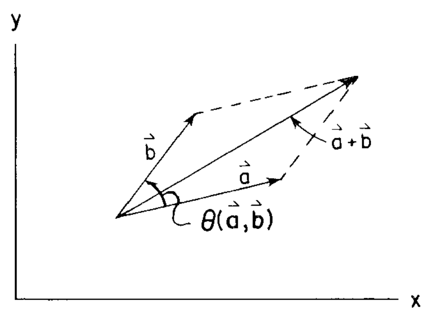

Figure 2.1.2 Addition of vectors.

Orthogonal reference frames are used extensively in representing vectors. Use in this text is limited to *right-hand orthogonal reference frames*; that is, the y axis is oriented $\pi/2$ rad counterclockwise from the x axis, as shown in Figs. 2.1.1 and 2.1.2. Such frames are called *Cartesian reference frames*.

A vector $\vec{a}$ can be resolved into components a_x and a_y along the x and y axes of a Cartesian reference frame, as shown in Fig. 2.1.3. These components are called the *Cartesian components of the vector*. Here, the *unit coordinate vectors* $\vec{i}$ and $\vec{j}$ are directed along the x and y coordinate axes, as shown in Fig. 2.1.3. In terms of the components of a vector and the unit coordinate vectors,

$$\vec{a} = a_x\vec{i} + a_y\vec{j} \tag{2.1.3}$$

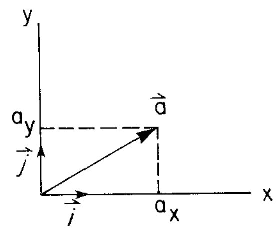

Figure 2.1.3 Components of a vector.

Addition of vectors $\vec{a}$ and $\vec{b}$ may now be expressed in terms of their components as

$$\begin{aligned}\vec{c} = \vec{a} + \vec{b} &= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} \\ &\equiv c_x\vec{i} + c_y\vec{j}\end{aligned} \tag{2.1.4}$$

where $c_x = a_x + b_x$ and $c_y = a_y + b_y$ are Cartesian components of vector $\vec{c}$. Thus, addition of vectors occurs component by component. Using this idea, three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ may be added, to verify (Prob. 2.1.1) that

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) = \vec{a} + \vec{b} + \vec{c} \tag{2.1.5}$$

Example 2.1.1: Given vectors $\vec{a} = \vec{i} + 2\vec{j}$, $\vec{b} = -2\vec{i} + \vec{j}$, and $\vec{c} = \vec{j}$, their vector sum is

$$\begin{aligned}\vec{a} + \vec{b} + \vec{c} &= (\vec{i} + 2\vec{j}) + (-2\vec{i} + \vec{j}) + (\vec{j}) \\ &= -\vec{i} + 4\vec{j}\end{aligned}$$

Denote the angle from vector $\vec{a}$ to vector $\vec{b}$ by $\theta(\vec{a}, \vec{b})$, with counterclockwise positive, as shown in Fig. 2.1.2. The *scalar product* (or *dot product*) of two vectors $\vec{a}$ and $\vec{b}$ is defined as the product of the magnitudes of the vectors and the cosine of the angle between them; that is,

$$\vec{a} \cdot \vec{b} = ab \cos \theta(\vec{a}, \vec{b}) \tag{2.1.6}$$

Note that if the vectors are nonzero, that is, if $a \neq 0$ and $b \neq 0$, then their scalar product is zero only if $\cos \theta(\vec{a}, \vec{b}) = 0$. Two nonzero vectors are said to be *orthogonal vectors* if their scalar product is zero. Since $\theta(\vec{b}, \vec{a}) = 2\pi - \theta(\vec{a}, \vec{b})$ and $\cos(2\pi - \alpha) = \cos \alpha$, the order of terms appearing on the right side of Eq. 2.1.6 is immaterial; so

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \tag{2.1.7}$$

Example 2.1.2: Let $\vec{u}$ be a unit vector, as shown in Fig. 2.1.4. Geometrically, $\vec{u} \cdot \vec{a} = a \cos \theta(\vec{u}, \vec{a})$ is the projection of $\vec{a}$ onto the directed line segment defined by $\vec{u}$, as shown in Fig. 2.1.4. Note that this is a geometric property of the scalar product, independent of the reference frame.

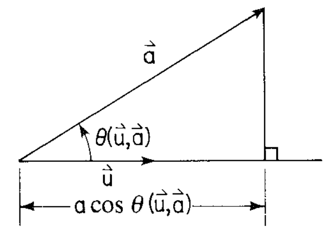

Figure 2.1.4 Projection of $\vec{a}$ onto $\vec{u}$.

Based on the definition of the scalar product, the following identities hold for the unit vectors $\vec{i}$ and $\vec{j}$:

$$\begin{aligned} \vec{i}\cdot\vec{j} &= \vec{j}\cdot\vec{i} = 0 \\ \vec{i}\cdot\vec{i} &= \vec{j}\cdot\vec{j} = 1 \end{aligned} \qquad \textbf{(2.1.8)}$$

For any vector $\vec{a}$, since $\theta(\vec{a}, \vec{a}) = 0$,

$$\vec{a}\cdot\vec{a} = aa \cos 0 = a^2 \qquad \textbf{(2.1.9)}$$

While not obvious on geometrical grounds, the scalar product satisfies the relation [21]

$$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} \qquad \textbf{(2.1.10)}$$

Using Eq. 2.1.10 and the identities of Eq. 2.1.8, a direct calculation yields

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y \qquad \textbf{(2.1.11)}$$

From Eqs. 2.1.9 and 2.1.11, note that

$$a = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_x^2 + a_y^2} \qquad \textbf{(2.1.12)}$$

It is often helpful to define a vector that is perpendicular to a given vector $\vec{a}$, has the same length as $\vec{a}$, and is oriented $\pi/2$ rad counterclockwise from $\vec{a}$. This vector is denoted as $\vec{a}^{\perp}$ and is written as

$$\vec{a}^{\perp} = -a_y\vec{i} + a_x\vec{j} \qquad \textbf{(2.1.13)}$$

as shown in Fig. 2.1.5. To verify that $\vec{a}^{\perp}$ is indeed orthogonal to $\vec{a}$, note that

$$\vec{a}^{\perp}\cdot\vec{a} = -a_y a_x + a_x a_y = 0$$

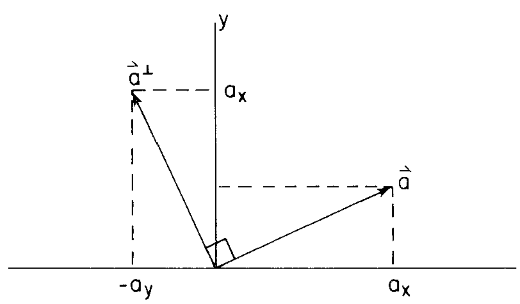

Figure 2.1.5 Vector perpendicular to $\vec{a}$.

Example 2.1.3: Let $0 \leq \text{Arccos}\,\alpha < \pi$ be the principal value of the arccos function. Since, if $-\pi < \phi < 0$, $\text{Arccos}(\cos\phi) = -\phi$, knowing $\cos\phi$ does not determine ϕ; that is, $\text{Arccos}(\cos\phi) = \mp\phi$. Thus, from Eq. 2.1.6 and Fig. 2.1.2,

$$\begin{aligned}\text{Arccos}(\cos\theta(\vec{a}, \vec{b})) &= \text{Arccos}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right)\\ &= \mp\theta(\vec{a}, \vec{b})\end{aligned} \tag{2.1.14}$$

To resolve the sign ambiguity, note first that the plus sign is selected in Eq. 2.1.14 if $\vec{b}$ is directed to the left of $\vec{a}$ and the minus sign is selected if $\vec{b}$ is to the right of $\vec{a}$. Since $\vec{a}^{\perp}$ is directed to the left of $\vec{a}$, as shown in Fig. 2.1.5, then $\vec{b}$ is to the left of $\vec{a}$ if $\vec{a}^{\perp}\cdot\vec{b} \geq 0$ and to the right otherwise. Defining

$$\text{sgn}\,x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

then

$$\theta(\vec{a}, \vec{b}) = \text{sgn}(\vec{a}^{\perp}\cdot\vec{b}) \times \text{Arccos}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right) \tag{2.1.15}$$

Using Eqs. 2.1.11 through 2.1.13, this relation can be written in terms of the components of $\vec{a}$ and $\vec{b}$ (Prob. 2.1.2).

2.2 MATRIX ALGEBRA

Matrix notation [22] permits systematic representation of a system of equations. Matrix manipulation also allows for organized development, simplification, and solution of systems of equations.

A *matrix* is a rectangular array of numbers, taken here to be real. If it has m rows and n columns, the *dimension of the matrix* is said to be $m \times n$. A matrix is denoted by a boldface capital letter if m and n are greater than 1 and is written in the form

$$\mathbf{A} \equiv [a_{ij}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{2.2.1}$$

where element a_{ij} is located in the ith row and jth column. The *transpose of a matrix* is formed by interchanging rows and columns and is designated by the superscript T. Thus, if a_{ij} is the i-j element of matrix $\mathbf{A}$, a_{ji} is the i-j element of its transpose $\mathbf{A}^T$.

A matrix with only one column is called a *column matrix* and is denoted by a boldface lowercase letter; for example, $\mathbf{a}$. A matrix with only one row is called a *row matrix* and is denoted by a boldface lowercase letter. An $m \times n$ matrix can be considered as being constructed of n column matrices $\mathbf{a}_j = [a_{1j}, \ldots, a_{mj}]^T$, $j = 1, \ldots, n$, or m row matrices $\mathbf{b}_i = [a_{i1}, \ldots, a_{in}]$, $i = 1, \ldots, m$; that is,

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{bmatrix} \tag{2.2.2}$$

Example 2.2.1: The 2×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

may be thought of as made up of the three 2×1 column matrices

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or the two 1×3 row matrices

$$\mathbf{b}_1 = [1,\ 2,\ 0], \quad \mathbf{b}_2 = [2,\ 3,\ 1]$$

That is,

$$\mathbf{A} = [\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3] = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

A *square matrix* has an equal number of rows and columns. A *diagonal matrix* is a square matrix with $a_{ij} = 0$ for $i \neq j$ and at least one nonzero diagonal term. An $n \times n$ diagonal matrix $\mathbf{A}$ can be represented as

$$\mathbf{A} \equiv \text{diag}[a_{11},\ a_{22}, \ldots, a_{nn}] \tag{2.2.3}$$

The $n \times n$ *identity matrix*, denoted $\mathbf{I}$ or $\mathbf{I}_n$, is a diagonal matrix with $a_{ii} = 1$, $i = 1, \ldots, n$. A *zero matrix* of any dimension, designated as $\mathbf{0}$, has $a_{ij} = 0$, for all i and j.

If two matrices $\mathbf{A}$ and $\mathbf{B}$ have the same dimension, they are defined to be *equal matrices* if $a_{ij} = b_{ij}$, for all i and j; that is, entries in the same position are equal. The *sum of two matrices* $\mathbf{A}$ and $\mathbf{B}$ that have the same dimension is a matrix with the same dimension defined as

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{2.2.4}$$

where $c_{ij} = a_{ij} + b_{ij}$, for all i and j. That is, matrices with the same dimension add component by component. The *difference between two matrices* $\mathbf{A}$ and $\mathbf{B}$ of the same dimension is defined as

$$\mathbf{D} = \mathbf{A} - \mathbf{B}$$

where $d_{ij} = a_{ij} - b_{ij}$, for all i and j. If three matrices have the same dimension, then (Prob. 2.2.1)

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C} \tag{2.2.5}$$

Similarly, for matrices **A** and **B** with the same dimension (Prob. 2.2.2),

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{2.2.6}$$

Example 2.2.2: The sum of the 2×2 matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

is

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

and the difference is

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Let **A** be an $m \times p$ matrix and **B** be a $p \times n$ matrix, written in the form

$$\mathbf{A} \equiv [a_{ij}] = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_m \end{bmatrix}, \qquad \mathbf{B} \equiv [b_{ij}] = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] \tag{2.2.7}$$

where $\mathbf{d}_i$, $i = 1, \ldots, m$, are rows of **A** with p elements and $\mathbf{b}_i$, $i = 1, \ldots, n$, are columns of **B** with p elements. The *matrix product* of **A** and **B** is defined as the $m \times n$ matrix

$$\mathbf{C} = \mathbf{AB} \tag{2.2.8}$$

where

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \tag{2.2.9}$$

or, in terms of the rows of **A** and columns of **B**,

$$c_{ij} = \mathbf{d}_i \mathbf{b}_j \tag{2.2.10}$$

Example 2.2.3: The product of the 2×2 matrix $\mathbf{B}$ of Example 2.2.2 and the 2×3 matrix $\mathbf{A}$ of Example 2.2.1 is the 2×3 matrix

$$\mathbf{C}=\mathbf{BA}=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 3 & 1\end{bmatrix}=\begin{bmatrix}4 & 6 & 2\\ 3 & 5 & 1\end{bmatrix}$$

It is important to note that the product of two matrices is defined only if the number of columns in the first matrix equals the number of rows in the second matrix. From the definition of matrix multiplication, in general,

$$\mathbf{AB}\neq\mathbf{BA} \tag{2.2.11}$$

To see that this is the case, the matrices of Example 2.2.2 yield

$$\mathbf{AB}=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ 1 & 5\end{bmatrix}\neq\begin{bmatrix}4 & 2\\ 3 & 2\end{bmatrix}=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}=\mathbf{BA}$$

In fact, the products $\mathbf{AB}$ and $\mathbf{BA}$ are defined only if both $\mathbf{A}$ and $\mathbf{B}$ are square and of equal dimension.

If $\mathbf{A}$ and $\mathbf{B}$ are $m\times p$ matrices and $\mathbf{C}$ is a $p\times n$ matrix [22],

$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC} \tag{2.2.12}$$

Similarly, if $\mathbf{A}$ is an $m\times p$ matrix, $\mathbf{B}$ is a $p\times q$ matrix, and $\mathbf{C}$ is a $q\times n$ matrix [22],

$$(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})=\mathbf{ABC} \tag{2.2.13}$$

Multiplication of a matrix $\mathbf{A}$ *by a scalar* α is defined as

$$\mathbf{C}=\alpha\mathbf{A} \tag{2.2.14}$$

where $c_{ij}=\alpha a_{ij}$, for all i and j; that is, all terms in the matrix are multiplied by the same scalar.

If $a_{ij}=a_{ji}$, for all i and j, the square matrix $\mathbf{A}=[a_{ij}]$ is called a *symmetric matrix*; that is, $\mathbf{A}=\mathbf{A}^T$. If $a_{ij}=-a_{ji}$, for all i and j, $\mathbf{A}$ is called a *skew-symmetric matrix*; that is, $\mathbf{A}=-\mathbf{A}^T$. Note that in this case $a_{ii}=0$, for all i.

The transpose of the sum of two matrices is the sum of their transposes [22]; that is,

$$(\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T \tag{2.2.15}$$

Also, if $\mathbf{A}$ is an $m\times p$ matrix and $\mathbf{B}$ is a $p\times n$ matrix, then [22]

$$(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T \tag{2.2.16}$$

A set of column matrices $\mathbf{a}_j$, $j=1,\ldots,m$, is called *linearly dependent* if there are constants α_j, $j=1,\ldots,m$, that are not all zero such that

$$\sum_{j=1}^{m}\alpha_j\mathbf{a}_j=\mathbf{0}$$

If a set of column matrices is not linearly dependent, it is called *linearly independent*. Equivalently, column matrices $\mathbf{a}_j$, $j = 1, \ldots, m$, are linearly independent if and only if

$$\sum_{j=1}^{m} \alpha_j \mathbf{a}_j = \mathbf{0}$$

implies that $\alpha_j = 0$, $j = 1, \ldots, m$.

Example 2.2.4: To see if the column matrices

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly dependent, form

$$\sum_{j=1}^{3} \beta_j \mathbf{b}_j = \mathbf{0}$$

This can be viewed as a system of equations for β_1, β_2, and β_3:

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= 0 \\ \beta_1 + 2\beta_3 &= 0 \\ \beta_2 - \beta_3 &= 0 \end{aligned}$$

Thus, $\beta_1 = -2\beta_3$ and $\beta_2 = \beta_3$ satisfy all three equations for any value of β_3; for example, $\beta_3 = 1$. Thus, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are linearly dependent.

Consider the $p \times m$ matrix $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_m]$. If a linear combination of the columns $\mathbf{a}_i$ of $\mathbf{A}$ is zero (see Prob. 2.2.5),

$$\mathbf{A}\boldsymbol{\alpha} = \sum_{j=1}^{m} \alpha_j \mathbf{a}_j = \mathbf{0} \tag{2.2.17}$$

for some $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T \neq \mathbf{0}$, then the columns of $\mathbf{A}$ are linearly dependent. Otherwise, they are linearly independent. Rows $\mathbf{d}_i$ of $\mathbf{A}$ (see Eq. 2.2.7) are linearly dependent if (Prob. 2.2.5)

$$\boldsymbol{\beta}^T \mathbf{A} = \sum_{i=1}^{n} \beta_i \mathbf{d}_i = \mathbf{0} \tag{2.2.18}$$

for some $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_n]^T \neq \mathbf{0}$. Otherwise, they are linearly independent.

Example 2.2.5: The matrix $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$, with columns given in Example 2.2.4, has linearly dependent columns, as shown in Example 2.2.4. To check for linear dependence of its rows, the product of Eq. 2.2.18 is

$$\boldsymbol{\beta}^T\mathbf{B} = [\beta_1,\ \beta_2,\ \beta_3]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = [0,\ 0,\ 0] = \mathbf{0}$$

From the first two equations, $\beta_2 = -\beta_1 = \beta_3$. The third equation is then $\beta_1 - 2\beta_1 + \beta_1 = 0$. Thus, if $\beta_1 = 1$ and $\beta_2 = \beta_3 = -1$, the equation is satisfied; so the rows of $\mathbf{B}$ are linearly dependent.

The *row rank* (*column rank*) of a matrix is defined as the largest number of linearly independent rows (columns) in the matrix. The row and column ranks of any matrix are equal [22], hence defining the *rank* of the matrix. The rank of a matrix is also equal to the dimension of the largest square submatrix (obtained by deleting rows and columns) with nonzero determinant [22] (Prob. 2.2.6). A square matrix with linearly independent rows (columns) is said to have *full rank.*

When a square matrix does not have full rank, it is called a *singular matrix.* A square matrix $\mathbf{A}$ of full rank is called a *nonsingular matrix.* For such a matrix, there is an *inverse* [22] denoted by $\mathbf{A}^{-1}$, such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad \textbf{(2.2.19)}$$

where $\mathbf{I}$ is the identity matrix. Using the definition of Eq. 2.2.19 and Eq. 2.2.16, it may be shown that [22] (Prob. 2.2.7)

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \qquad \textbf{(2.2.20)}$$

and that (Prob. 2.2.8)

$$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \textbf{(2.2.21)}$$

A special nonsingular matrix that arises often in kinematics is called an *orthogonal matrix,* with the property that

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I} \qquad \textbf{(2.2.22)}$$

That is, from the definition of Eq. 2.2.19,

$$\mathbf{A}^{-1} = \mathbf{A}^T \qquad \textbf{(2.2.23)}$$

Since constructing the inverse of a nonsingular matrix is time consuming, it is important to know when a matrix is orthogonal. In case a matrix is orthogonal, its inverse is easily constructed using Eq. 2.2.23.

Example 2.2.6: The 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

is orthogonal, for any value of ϕ, since

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \cos^2\phi + \sin^2\phi & 0 \\ 0 & \cos^2\phi + \sin^2\phi \end{bmatrix} = \mathbf{I}$$

2.3 ALGEBRAIC VECTORS

Recall, from Eq. 2.1.3, that a vector $\vec{a}$ can be written in component form as

$$\vec{a} = a_x\vec{i} + a_y\vec{j} \tag{2.3.1}$$

The vector $\vec{a}$ is thus uniquely defined by its Cartesian components, which may be written using matrix notation as

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x,\ a_y]^T \tag{2.3.2}$$

This is the *algebraic representation of a vector* in the x-y Cartesian reference frame. An *algebraic vector* is defined as a column matrix. When an algebraic vector represents a geometric vector in the plane (i.e., it contains the x-y Cartesian components of the vector), the algebraic vector has two components.

If two vectors $\vec{a}$ and $\vec{b}$ are represented in algebraic form as $\mathbf{a} = [a_x, a_y]^T$ and $\mathbf{b} = [b_x, b_y]^T$, then their vector sum $\vec{c} = \vec{a} + \vec{b}$ is represented in algebraic form by (Prob. 2.3.1)

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{2.3.3}$$

Similarly, $\vec{a} = \vec{b}$ if and only if the components of the vectors are equal; that is, $\mathbf{a} = \mathbf{b}$. Multiplication of a vector $\vec{a}$ by a scalar α occurs component by component, so the geometric vector $\alpha\vec{a}$ is represented by the algebraic vector $\alpha\mathbf{a}$. A *zero algebraic vector,* denoted by $\mathbf{0}$, has all its components equal to zero.

Algebraic representation of vectors allows algebraic vectors with more than three components to be defined (i.e., algebraic vectors with higher dimension than 2). An algebraic vector with n components is called an *n vector*. For example, the algebraic vectors $\mathbf{a} = [a_x, a_y]^T$, $\mathbf{b} = [b_x, b_y]^T$, and $\mathbf{c} = [c_x, c_y]^T$ may be combined to form the 6-vector

$$\begin{aligned} \mathbf{d} &= [a_x, a_y, b_x, b_y, c_x, c_y]^T \\ &= [\mathbf{a}^T, \mathbf{b}^T, \mathbf{c}^T]^T \end{aligned}$$

Since there is a one-to-one correspondence between geometric vectors in a plane and 2×1 algebraic vectors formed from their components, no distinction other than notation will be made between them in the remainder of Part I.

The scalar product of two geometric vectors may be expressed in algebraic form, using the result of Eq. 2.1.11, as

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y = \mathbf{a}^T\mathbf{b} \tag{2.3.4}$$

The vector $\vec{a}^{\perp}$ that is orthogonal to $\vec{a}$, given by Eq. 2.1.13, is represented in algebraic form as

$$\mathbf{a}^{\perp} = \begin{bmatrix} -a_y \\ a_x \end{bmatrix} = \mathbf{R}\mathbf{a} \tag{2.3.5}$$

where $\mathbf{R}$ is the *orthogonal rotation matrix* (Prob. 2.3.2):

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.3.6}$$

The matrix $\mathbf{R}$ is called orthogonal since, consistent with Eq. 2.2.22,

$$\mathbf{R}^T\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Finally, by direct computation,

$$\mathbf{RR} = -\mathbf{I} \tag{2.3.7}$$

Example 2.3.1: Using the result of Example 2.1.3, specifically Eq. 2.1.15, and Prob. 2.3.2,

$$\begin{aligned} \theta(\mathbf{a}, \mathbf{a}^{\perp}) &= \operatorname{sgn}(\mathbf{a}^{\perp T}\mathbf{a}^{\perp})\operatorname{Arccos}\frac{\mathbf{a}^T\mathbf{a}^{\perp}}{aa} \\ &= \operatorname{Arccos}\frac{-a_x a_y + a_y a_x}{a^2} \\ &= \operatorname{Arccos} 0 = \frac{\pi}{2} \end{aligned}$$

which shows that the operation $\mathbf{Ra} = \mathbf{a}^{\perp}$ indeed rotates a vector $\mathbf{a}$ by an angle $\pi/2$ counterclockwise. Using this result and the result of Prob. 2.3.3, the result of successive applications of $\mathbf{R}$ to a vector $\mathbf{a}$ is shown in Fig. 2.3.1.

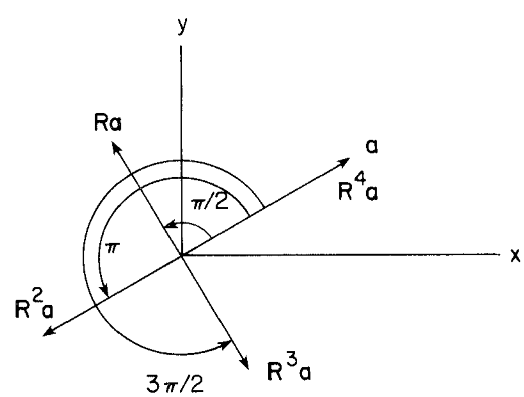

Figure 2.3.1 Rotation of a vector **a**.

2.4 TRANSFORMATION OF COORDINATES

It is shown in Section 2.3 that, in a fixed x-y Cartesian reference frame, geometric vectors are represented by algebraic vectors that contain their components. The components of a vector, however, are defined in a specific Cartesian reference frame. Consider a second Cartesian x'-y' frame, with the same origin as the x-y frame, with angle ϕ between the x and x' axes, counterclockwise positive, as shown in Fig. 2.4.1.

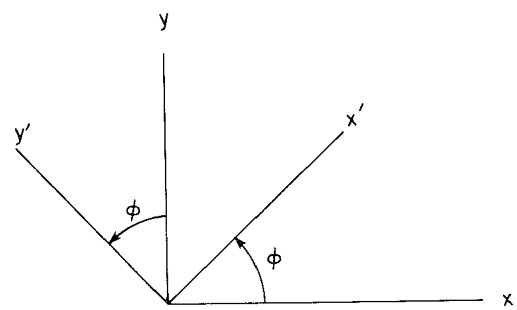

Figure 2.4.1 Two Cartesian reference frames.

A vector $\vec{s}$ in Fig. 2.4.2 can be represented by algebraic vectors

$$\begin{aligned} \mathbf{s} &= [s_x, s_y]^T \\ \mathbf{s}' &= [s_{x'}, s_{y'}]^T \end{aligned} \tag{2.4.1}$$

in the x-y and x'-y' frames, respectively.

By elementary trigonometry,

$$\begin{aligned} s_x &= s_{x'} \cos \phi - s_{y'} \sin \phi \\ s_y &= s_{x'} \sin \phi + s_{y'} \cos \phi \end{aligned} \tag{2.4.2}$$

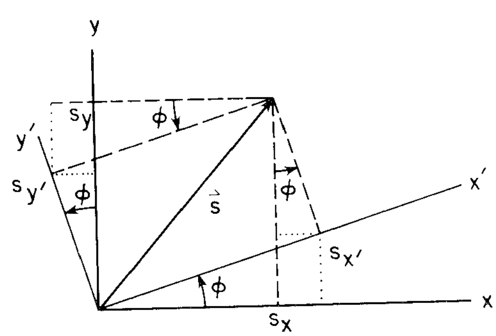

Figure 2.4.2 Vector $\vec{s}$ in two frames.

Thus, $\mathbf{s}$ and $\mathbf{s}'$ are related by the matrix transformation

$$\mathbf{s} = \mathbf{A}\mathbf{s}' \tag{2.4.3}$$

where $\mathbf{A}$ is the planar *rotation transformation matrix*

$$\mathbf{A} = \mathbf{A}(\phi) \equiv \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \tag{2.4.4}$$

By direct expansion, note that

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \cos^2\phi + \sin^2\phi & -\cos\phi\sin\phi + \sin\phi\cos\phi \\ -\cos\phi\sin\phi + \sin\phi\cos\phi & \sin^2\phi + \cos^2\phi \end{bmatrix} = \mathbf{I} \tag{2.4.5}$$

Thus, $\mathbf{A}$ is orthogonal and, by Eq. 2.2.23,

$$\mathbf{A}^T = \mathbf{A}^{-1} \tag{2.4.6}$$

The inverse of the transformation of Eq. 2.4.3 is thus

$$\mathbf{s}' = \mathbf{A}^T\mathbf{s} \tag{2.4.7}$$

When the origins of the x-y and x'-y' frames do not coincide, as shown in Fig. 2.4.3, the foregoing analysis is applied between the x'-y' and translated x-y frames, as shown in Fig. 2.4.3. If the vector $\mathbf{s}'^P$ locates point P in the x'-y' frame, then in the translated x-y frame this vector is just $\mathbf{A}\mathbf{s}'^P$. Thus,

$$\begin{aligned} \mathbf{r}^P &= \mathbf{r} + \mathbf{s}^P \\ &= \mathbf{r} + \mathbf{A}\mathbf{s}'^P \end{aligned} \tag{2.4.8}$$

where $\mathbf{r}$ is the vector from the origin of the x-y frame to the origin of the x'-y' frame, as shown in Fig. 2.4.3.

Consider the pair of x_i'-y_i' and x_j'-y_j' frames shown in Fig. 2.4.4. An arbitrary vector $\mathbf{s}$ in the x-y frame has representations $\mathbf{s}_i'$ and $\mathbf{s}_j'$ in the x_i'-y_i' and x_j'-y_j' frames, respectively; that is,

$$\mathbf{s} = \mathbf{A}_i\mathbf{s}_i' = \mathbf{A}_j\mathbf{s}_j' \tag{2.4.9}$$

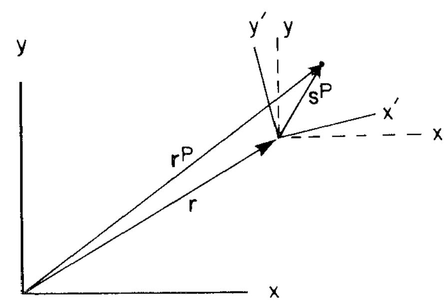

Figure 2.4.3 Translation and rotation of reference frames.

where $\mathbf{A}_i$ and $\mathbf{A}_j$ are transformation matrices from the x_i'-y_i' and x_j'-y_j' frames to the x-y frame, respectively. Since $\mathbf{A}_i$ and $\mathbf{A}_j$ are orthogonal, Eq. 2.4.9 yields

$$\mathbf{s}_i' = \mathbf{A}_i^T\mathbf{A}_j\mathbf{s}_j' \equiv \mathbf{A}_{ij}\mathbf{s}_j' \tag{2.4.10}$$

Since $\mathbf{s}_j'$ is an arbitrary vector,

$$\mathbf{A}_{ij} = \mathbf{A}_i^T\mathbf{A}_j \tag{2.4.11}$$

Thus, $\mathbf{A}_{ij}$ is the transformation matrix from the x_j'-y_j' frame to the x_i'-y_i' frame. A direct calculation shows that $\mathbf{A}_{ij}$ is an orthogonal matrix; that is,

$$\mathbf{A}_{ij}^T\mathbf{A}_{ij} = \mathbf{A}_j^T\mathbf{A}_i\mathbf{A}_i^T\mathbf{A}_j = \mathbf{A}_j^T\mathbf{A}_j = \mathbf{I}$$

Using the definition of $\mathbf{A}_i$ and $\mathbf{A}_j$ in Eq. 2.4.4 and standard trigonometric identities, Eq. 2.4.11 may be expanded to obtain

$$\mathbf{A}_{ij} = \begin{bmatrix} \cos(\phi_j - \phi_i) & -\sin(\phi_j - \phi_i) \\ \sin(\phi_j - \phi_i) & \cos(\phi_j - \phi_i) \end{bmatrix} = \mathbf{A}(\phi_j - \phi_i) \tag{2.4.12}$$

which is just the orthogonal rotation transformation matrix due to a rotation of the x_j'-y_j' frame by angle $\phi_j - \phi_i$, relative to the x_i'-y_i' frame. Since $\mathbf{A}_{ij}$ transforms vectors from the x_j'-y_j' frame to the x_i'-y_i' frame, this result could have been anticipated from Fig. 2.4.4.

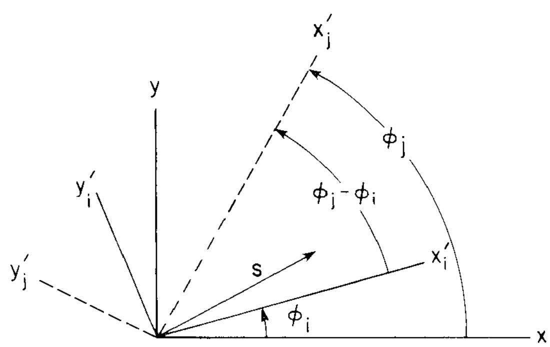

Figure 2.4.4 Three reference frames with coincident origins.

Example 2.4.1: If actuators control angles θ_1 and θ_2 in the positioning mechanism shown in Fig. 2.4.5, then $\phi_1 = \theta_1$ and $\phi_2 = \theta_1 + \theta_2$. Thus,

$$\mathbf{r}^Q = \mathbf{A}(\theta_1)\mathbf{s}_1'^Q$$

$$\mathbf{r}^P = \mathbf{r}^Q + \mathbf{A}(\theta_1 + \theta_2)\mathbf{s}_2'^{QP}$$

where $\mathbf{s}_2'^{QP}$ is the vector fixed in the x_2'-y_2' frame that locates point P relative to

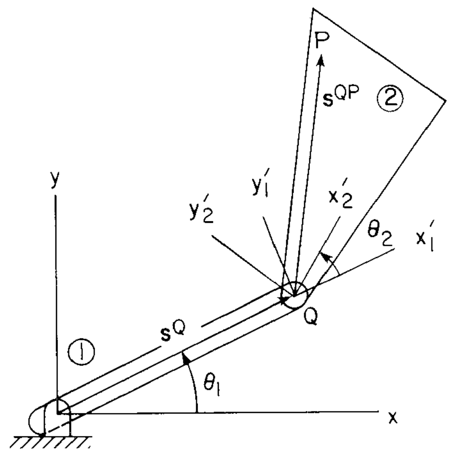

Figure 2.4.5 Two-body positioning mechanism.

point Q. Thus,

$$\mathbf{r}^P = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \mathbf{s}_1'^Q + \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \mathbf{s}_2'^{QP} \quad \textbf{(2.4.13)}$$

Example 2.4.2: Using Eq. 2.4.12, where $\phi_2 - \phi_1 = \theta_2$ in Fig. 2.4.5,

$$\mathbf{A}_{12}(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

Thus, the vector $\mathbf{s}^{QP}$ from point Q to point P in Fig. 2.4.5, represented in the x_1'-y_1' frame, is

$$\mathbf{s}_1'^{QP} = \mathbf{A}_{12}\mathbf{s}_2'^{QP}$$

In the x-y frame, this is

$$\mathbf{s}^{QP} = \mathbf{A}_1\mathbf{A}_{12}\mathbf{s}_2'^{QP}$$

where

$$\mathbf{A}_1(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Thus, since $\mathbf{s}_1'^Q$ and $\mathbf{s}_2'^{QP}$ are constant, $\mathbf{r}^P$ as a function of θ_1 and θ_2 is

$$\mathbf{r}^P = \mathbf{A}_1(\theta_1)\mathbf{s}_1'^Q + \mathbf{A}_1(\theta_1)\mathbf{A}_{12}(\theta_2)\mathbf{s}_2'^{QP} \quad \textbf{(2.4.14)}$$

Example 2.4.3: The slider–crank mechanism shown in Fig. 2.4.6(a) may be modeled using the angles θ_1 and θ_2 shown in Fig. 2.4.6(b). Since the geometry of this mechanism is the same as that of Fig. 2.4.5, with $\mathbf{s}_1'^Q = [1, 0]^T$ and

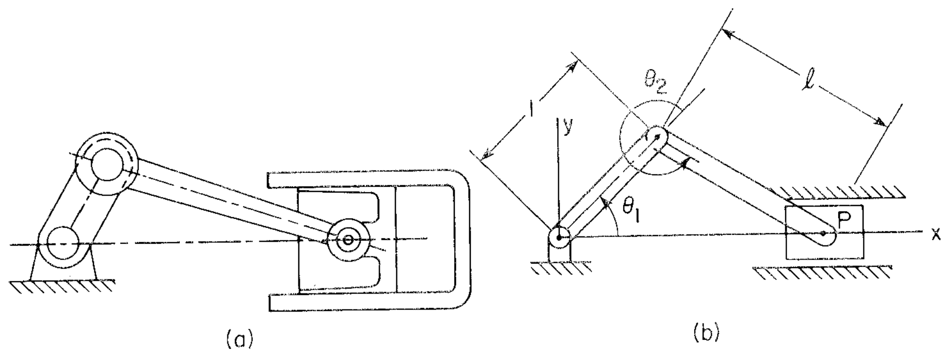

Figure 2.4.6 Slider–crank mechanism. (a) Physical system. (b) Kinematic model.

$\mathbf{s}_2'^{QP} = [\ell, 0]^T$, from Eq. 2.4.13,

$$\mathbf{r}^P = \begin{bmatrix} \cos\theta_1 + \ell\cos(\theta_1 + \theta_2) \\ \sin\theta_1 + \ell\sin(\theta_1 + \theta_2) \end{bmatrix}$$

Since the piston wrist pin at point P is constrained to slide along the x axis,

$$r_y^P \equiv \Phi(\theta_1, \theta_2) = \sin\theta_1 + \ell\sin(\theta_1 + \theta_2) = 0 \tag{2.4.15}$$

Thus, angles θ_1 and θ_2 are not independent; that is, they must satisfy the constraint equation of Eq. 2.4.15. These variables are typical of *generalized coordinates* that are used to define the geometry of a mechanism.

2.5 VECTOR AND MATRIX DIFFERENTIATION

In kinematics and dynamics of mechanical systems, vectors that represent the positions of points on bodies or equations that describe the geometry of the system are functions of time or some other variables. In analyzing these equations, time derivatives and partial derivatives are needed. In this section these derivatives are defined, and the *matrix calculus notation* that is used throughout the text is introduced.

In analyzing velocities and accelerations, time derivatives of vectors that locate points must be calculated. Consider a vector $\vec{a}(t)$ with components $\mathbf{a} \equiv \mathbf{a}(t) = [a_x(t), a_y(t)]^T$ in a *stationary Cartesian reference frame,* that is, a frame with $\vec{i}$ and $\vec{j}$ constant. The *time derivative of a vector* $\vec{a}$ is

$$\begin{aligned} \dot{\vec{a}}(t) &\equiv \frac{d}{dt}\vec{a}(t) = \frac{d}{dt}[a_x(t)\vec{i} + a_y(t)\vec{j}] \\ &= \left[\frac{d}{dt}a_x(t)\right]\vec{i} + \left[\frac{d}{dt}a_y(t)\right]\vec{j} \end{aligned}$$

Note that this is valid only if $\vec{i}$ and $\vec{j}$ are not time dependent. In matrix notation, this is

$$\begin{aligned}\dot{\mathbf{a}} &\equiv \frac{d}{dt}\mathbf{a} \\ &= \left[\frac{d}{dt}a_x, \frac{d}{dt}a_y\right]^T = [\dot{a}_x, \dot{a}_y]^T\end{aligned} \tag{2.5.1}$$

where an overdot denotes derivative with respect to time. Thus, for vectors that are written in terms of their components in a stationary Cartesian reference frame, the derivative of a vector is obtained by differentiating its components.

The derivative of the sum of two vectors $\mathbf{a} = \mathbf{a}(t)$ and $\mathbf{b} = \mathbf{b}(t)$ is

$$\frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \dot{\mathbf{a}} + \dot{\mathbf{b}} \tag{2.5.2}$$

which is analogous to the ordinary differentiation rule that the derivative of a sum is the sum of the derivatives. The following vector forms of the *product rule of differentiation* are also valid (Prob. 2.5.1):

$$\frac{d}{dt}(\alpha\mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}} \tag{2.5.3}$$

$$\frac{d}{dt}(\mathbf{a}^T\mathbf{b}) = \dot{\mathbf{a}}^T\mathbf{b} + \mathbf{a}^T\dot{\mathbf{b}} \tag{2.5.4}$$

Many uses may be made of these derivative formulas. For example, if the length of a vector $\mathbf{a}(t)$ is fixed; that is, $\mathbf{a}(t)^T\mathbf{a}(t) = c$, then Eq. 2.5.4 yields

$$\dot{\mathbf{a}}^T\mathbf{a} = 0 \tag{2.5.5}$$

If $\mathbf{a}$ is a position vector that locates a point in a stationary Cartesian reference frame, then $\dot{\mathbf{a}}$ is the *velocity* of that point. Hence, Eq. 2.5.5 indicates that the velocity of a point whose distance from the origin is constant is orthogonal to the position vector of the point. The second time derivative of $\mathbf{a}(t)$ is the *acceleration* of the point, denoted as

$$\ddot{\mathbf{a}} = \frac{d}{dt}\left(\frac{d}{dt}\mathbf{a}(t)\right) = [\ddot{a}_x, \ddot{a}_y]^T \tag{2.5.6}$$

Thus, for vectors that are written in terms of their components in a stationary Cartesian reference frame, the second time derivative of the vector may be calculated in terms of second time derivatives of components of the vector.

Example 2.5.1: If $\mathbf{s}_1'^Q = [1, 0]^T$ and $\mathbf{s}_2'^{QP} = [1, 0]^T$ in the positioning mechanism of Example 2.4.1, from Eq. 2.4.13,

$$\mathbf{r}^P = \begin{bmatrix} \cos\theta_1 + \cos(\theta_1 + \theta_2) \\ \sin\theta_1 + \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Let the actuators be driven with constant angular velocities ω_1 and ω_2, so that $\theta_1 = \omega_1 t$ and $\theta_2 = \omega_2 t$. Then

$$\mathbf{r}^P = \begin{bmatrix} \cos \omega_1 t + \cos(\omega_1 + \omega_2)t \\ \sin \omega_1 t + \sin(\omega_1 + \omega_2)t \end{bmatrix}$$

From Eqs. 2.5.1 and 2.5.6, the velocity and acceleration of point P are obtained by direct differentiation as

$$\dot{\mathbf{r}}^P = \begin{bmatrix} -\omega_1 \sin \omega_1 t - (\omega_1 + \omega_2)\sin(\omega_1 + \omega_2)t \\ \omega_1 \cos \omega_1 t + (\omega_1 + \omega_2)\cos(\omega_1 + \omega_2)t \end{bmatrix}$$

$$\ddot{\mathbf{r}}^P = \begin{bmatrix} -\omega_1^2 \cos \omega_1 t - (\omega_1 + \omega_2)^2 \cos(\omega_1 + \omega_2)t \\ -\omega_1^2 \sin \omega_1 t - (\omega_1 + \omega_2)^2 \sin(\omega_1 + \omega_2)t \end{bmatrix}$$

Just as in differentiation of a vector whose components are functions of t, the *derivative of a matrix* whose components depend on t may be defined. Consider a matrix $\mathbf{B}(t) = [b_{ij}(t)]$. The derivative of $\mathbf{B}(t)$ is defined as

$$\dot{\mathbf{B}} \equiv \frac{d}{dt}\mathbf{B} = \left[\frac{d}{dt} b_{ij}\right] \tag{2.5.7}$$

With this definition and elementary rules of differentiation (Prob. 2.5.3),

$$\frac{d}{dt}(\mathbf{B} + \mathbf{C}) = \dot{\mathbf{B}} + \dot{\mathbf{C}} \tag{2.5.8}$$

$$\frac{d}{dt}(\mathbf{BC}) = \dot{\mathbf{B}}\mathbf{C} + \mathbf{B}\dot{\mathbf{C}} \tag{2.5.9}$$

$$\frac{d}{dt}(\alpha\mathbf{B}) = \dot{\alpha}\mathbf{B} + \alpha\dot{\mathbf{B}} \tag{2.5.10}$$

where $\alpha = \alpha(t)$ is a scalar function of time.

In dealing with systems of nonlinear differential and algebraic equations in several variables, which govern the kinematics and dynamics of mechanical systems, it is essential that a *matrix calculus notation* be employed. To introduce the notation used here, let $\mathbf{q} = [q_1, \ldots, q_k]^T$ be a k vector of real variables, $a(\mathbf{q})$ be a scalar differentiable function of $\mathbf{q}$, and $\mathbf{\Phi}(\mathbf{q}) = [\Phi_1(\mathbf{q}), \ldots, \Phi_n(\mathbf{q})]^T$ be an n vector of differentiable functions of $\mathbf{q}$. Using i as row index and j as column index, the following matrix calculus notations are defined:

$$a_{\mathbf{q}} \equiv \frac{\partial a}{\partial \mathbf{q}} \equiv \left[\frac{\partial a}{\partial q_j}\right]_{1\times k} \tag{2.5.11}$$

$$\mathbf{\Phi}_{\mathbf{q}} \equiv \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \equiv \left[\frac{\partial \Phi_i}{\partial q_j}\right]_{n\times k} \tag{2.5.12}$$

Note that the derivative of a scalar function with respect to a vector variable in

Eq. 2.5.11 is a row matrix. This is one of the few matrix symbols in the text that is a row matrix, rather than the more common column matrix. Note also that, as defined by Eq. 2.5.12, the derivative of a vector function $\mathbf{\Phi}$, whose elements are functions of the vector variable $\mathbf{q}$, is a matrix. The subscript notation used here to denote differentiation is helpful in deriving needed relations without becoming entangled in cumbersome index and partial differentiation notation. To take advantage of this notation, however, it is critically important that the correct matrix definition of derivatives be used.

Example 2.5.2: If $\mathbf{q} = [\theta_1, \theta_2]^T$, then the derivative of the slider–crank constraint function Φ of Eq. 2.4.15 in Example 2.4.3 is

$$\Phi_{\mathbf{q}} = [\cos\theta_1 + \ell\cos(\theta_1+\theta_2),\ \ell\cos(\theta_1+\theta_2)]$$

Similarly, the matrix of derivatives of $\mathbf{r}^P$ in the same example is

$$\mathbf{r}^P_{\mathbf{q}} = \begin{bmatrix} -\sin\theta_1 - \ell\sin(\theta_1+\theta_2) & -\ell\sin(\theta_1+\theta_2) \\ \cos\theta_1 + \ell\cos(\theta_1+\theta_2) & \ell\cos(\theta_1+\theta_2) \end{bmatrix}$$

The partial derivative of the scalar product of two n vector functions $\mathbf{g}(\mathbf{q}) = [g_1(\mathbf{q}), \ldots, g_n(\mathbf{q})]^T$ and $\mathbf{h}(\mathbf{q}) = [h_1(\mathbf{q}), \ldots, h_n(\mathbf{q})]^T$, by careful manipulation, is the *product rule of differentiation*

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{q}}(\mathbf{g}^T\mathbf{h}) &= (\mathbf{g}^T\mathbf{h})_{\mathbf{q}} \\
&= \frac{\partial}{\partial \mathbf{q}}\left(\sum_{k=1}^{n} g_k h_k\right) \\
&= \left[\frac{\partial}{\partial q_j}\left(\sum_{k=1}^{n} g_k h_k\right)\right] \\
&= \left[\sum_{k=1}^{n}\left(\frac{\partial g_k}{\partial q_j} h_k + g_k \frac{\partial h_k}{\partial q_j}\right)\right] \\
&= \left[\sum_{k=1}^{n}\left(h_k \frac{\partial g_k}{\partial q_j}\right) + \sum_{k=1}^{n}\left(g_k \frac{\partial h_k}{\partial q_j}\right)\right] \\
&= \mathbf{h}^T \frac{\partial \mathbf{g}}{\partial \mathbf{q}} + \mathbf{g}^T \frac{\partial \mathbf{h}}{\partial \mathbf{q}} \\
&= \mathbf{h}^T \mathbf{g}_{\mathbf{q}} + \mathbf{g}^T \mathbf{h}_{\mathbf{q}}
\end{aligned} \tag{2.5.13}$$

Note that what might have intuitively appeared to be the appropriate product rule of differentiation is not even defined, much less valid; that is,

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{g}^T\mathbf{h}) \neq \mathbf{g}_{\mathbf{q}}^T\mathbf{h} + \mathbf{g}^T\mathbf{h}_{\mathbf{q}}$$

If $\mathbf{\Phi}(\mathbf{g}) = [\Phi_1(\mathbf{g}), \ldots, \Phi_m(\mathbf{g})]^T$ and $\mathbf{g} = \mathbf{g}(\mathbf{q}) = [g_1(\mathbf{q}), \ldots, g_n(\mathbf{q})]^T$ are vector functions of vector variables, the *chain rule of differentiation,* in matrix calculus form, is obtained as

$$
\begin{aligned}
\mathbf{\Phi_q} &\equiv \left[\frac{\partial \Phi_i(\mathbf{g}(\mathbf{q}))}{\partial q_j}\right]_{m\times k} \\
&= \left[\sum_{\ell=1}^{n}\left(\frac{\partial \Phi_i}{\partial g_\ell}\frac{\partial g_\ell}{\partial q_j}\right)\right]_{m\times k} \\
&= \frac{\partial \mathbf{\Phi}}{\partial \mathbf{g}}\frac{\partial \mathbf{g}}{\partial \mathbf{q}} \\
&= \mathbf{\Phi_g g_q}
\end{aligned}
\tag{2.5.14}
$$

If $\mathbf{B}$ is a constant $m \times n$ matrix and $\mathbf{p}$ and $\mathbf{q}$ are m and n vectors of variables, respectively, the following useful relations may be verified (Prob. 2.5.7):

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{Bq}) = \mathbf{B} \tag{2.5.15}$$

$$\frac{\partial}{\partial \mathbf{p}}(\mathbf{p}^T\mathbf{Bq}) = \mathbf{q}^T\mathbf{B}^T \tag{2.5.16}$$

$$\frac{d}{dt}(\mathbf{p}^T\mathbf{Bq}) = \mathbf{q}^T\mathbf{B}^T\dot{\mathbf{p}} + \mathbf{p}^T\mathbf{B}\dot{\mathbf{q}} \tag{2.5.17}$$

Example 2.5.3: If $\ell > 1$ for the slider–crank mechanism of Example 2.4.3, then $|\sin(\theta_1 + \theta_2)| = |\sin\theta_1|/\ell < 1$ and Eq. 2.4.15 can be solved for θ_2 as

$$\theta_2 = 2\pi - \text{Arcsin}\left(\frac{\sin\theta_1}{\ell}\right) - \theta_1 \tag{2.5.18}$$

where $-\pi/2 \leq \text{Arcsin}\,\alpha \leq \pi/2$. Thus, θ_2 may be treated as a function of θ_1, and the motion of the slider–crank mechanism may be controlled by specifying the time history $\theta_1(t)$ of the crank angle. By direct differentiation,

$$\dot{\theta}_2 = -\left[\frac{\cos\theta_1}{\sqrt{\ell^2 - \sin^2\theta_1}} + 1\right]\dot{\theta}_1 \tag{2.5.19}$$

An alternative approach to calculating $\dot{\theta}_2$ in terms of $\dot{\theta}_1$ that is useful in kinematic analysis, even when complicated nonlinear constraint equations cannot be solved in closed form, is to differentiate both sides of Eq. 2.4.15, to obtain

$$\dot{\theta}_1\cos\theta_1 + \ell(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2) = 0 \tag{2.5.20}$$

This can be solved for $\dot{\theta}_2$ as

$$\dot{\theta}_2 = -\left[\frac{\cos\theta_1}{\ell\cos(\theta_1 + \theta_2)} + 1\right]\dot{\theta}_1 \tag{2.5.21}$$

Similarly, Eq. 2.5.20 can be differentiated to obtain

$$\ddot{\theta}_1 \cos \theta_1 - \dot{\theta}_1^2 \sin \theta_1 + \ell(\ddot{\theta}_1 + \ddot{\theta}_2) \cos(\theta_1 + \theta_2) - \ell(\dot{\theta}_1 + \dot{\theta}_2)^2 \sin(\theta_1 + \theta_2) = 0 \quad \textbf{(2.5.22)}$$

which can be solved for $\ddot{\theta}_2$ as a function of $\dot{\theta}_1$ and $\ddot{\theta}_1$, using Eq. 2.5.21 to eliminate $\dot{\theta}_2$ (Prob. 2.5.9).

2.6 VELOCITY AND ACCELERATION OF A POINT FIXED IN A MOVING FRAME

Quite often in applications, an x'-y' Cartesian reference frame is fixed in a moving body to define its position and orientation, relative to a stationary global x-y reference frame. Consider a point P that is fixed in an x'-y' frame, as shown in Fig. 2.6.1. The vector that locates P in the x-y frame is given by Eq. 2.4.8 as

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \quad \textbf{(2.6.1)}$$

where $\mathbf{s}'^P$ is the constant vector of coordinates of P in the x'-y' frame and $\mathbf{A}$ is the transformation matrix from the x'-y' frame to the x-y frame.

Since the x'-y' frame is moving and changing its orientation with time, the vector $\mathbf{r}$ and the transformation matrix $\mathbf{A}$ are functions of time. Differentiation results of Section 2.5 can be used to obtain the time derivative of $\mathbf{r}^P$ as

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\mathbf{s}'^P \quad \textbf{(2.6.2)}$$

From Eq. 2.4.4,

$$\dot{\mathbf{A}} = \dot{\phi}\frac{d}{d\phi}\mathbf{A} = \dot{\phi}\begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} \equiv \dot{\phi}\mathbf{B} \quad \textbf{(2.6.3)}$$

Thus, Eq. 2.6.2 becomes

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P \quad \textbf{(2.6.4)}$$

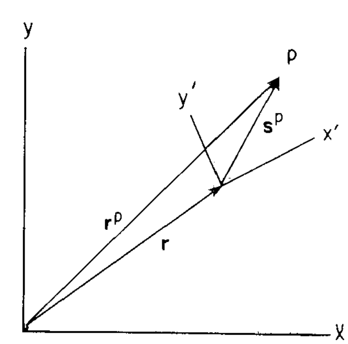

Figure 2.6.1 Point P fixed in an x'-y' frame.

Note that, with the orthogonal rotation matrix $\mathbf{R}$ of Eq. 2.3.6,

$$\mathbf{AR} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} = \mathbf{B} \tag{2.6.5}$$

Also,

$$\mathbf{AR} = \mathbf{RA} \tag{2.6.6}$$

Thus, Eq. 2.6.4 may be written as

$$\begin{aligned} \dot{\mathbf{r}}^P &= \dot{\mathbf{r}} + \dot{\phi}\mathbf{ARs}'^P \\ &= \dot{\mathbf{r}} + \dot{\phi}\mathbf{As}'^{P\perp} \\ &= \dot{\mathbf{r}} + \dot{\phi}\mathbf{s}^{P\perp} \end{aligned} \tag{2.6.7}$$

Finally, note that

$$\dot{\mathbf{B}} = \dot{\phi}\frac{d}{d\phi}\mathbf{B} = \dot{\phi}\begin{bmatrix} -\cos\phi & \sin\phi \\ -\sin\phi & -\cos\phi \end{bmatrix} = -\dot{\phi}\mathbf{A} \tag{2.6.8}$$

Taking the time derivative of Eq. 2.6.4 yields the acceleration of point P as

$$\begin{aligned} \ddot{\mathbf{r}}^P &= \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{Bs}'^P + \dot{\phi}\dot{\mathbf{B}}\mathbf{s}'^P \\ &= \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{Bs}'^P - \dot{\phi}^2\mathbf{As}'^P \end{aligned} \tag{2.6.9}$$

Alternative forms of this relation are obtained, using Eqs. 2.6.5 and 2.6.6, as

$$\begin{aligned} \ddot{\mathbf{r}}^P &= \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{As}'^{P\perp} - \dot{\phi}^2\mathbf{As}'^P \\ &= \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{s}^{P\perp} - \dot{\phi}^2\mathbf{s}^P \end{aligned} \tag{2.6.10}$$

Example 2.6.1 The position vector of point P on body 2 of the positioning mechanism of Example 2.4.1 was derived in Example 2.4.2 in the form (Eq. 2.4.14)

$$\mathbf{r}^P = \mathbf{A}_1\mathbf{s}_1'^Q + \mathbf{A}_1\mathbf{A}_{12}\mathbf{s}_2'^{QP}$$

where $\mathbf{A}_1 = \mathbf{A}(\theta_1)$, $\mathbf{A}_{12} = \mathbf{A}(\theta_2)$, and $\mathbf{s}_1'^Q$ and $\mathbf{s}_2'^{QP}$ are constant vectors. Using the chain and product rules of differentiation and Eq. 2.6.3,

$$\dot{\mathbf{r}}^P = \dot{\theta}_1(\mathbf{B}_1\mathbf{s}_1'^Q + \mathbf{B}_1\mathbf{A}_{12}\mathbf{s}_2'^{QP}) + \dot{\theta}_2\mathbf{A}_1\mathbf{B}_{12}\mathbf{s}_2'^{QP} \tag{2.6.11}$$

and, using Eq. 2.6.8 (Prob. 2.6.1),

$$\begin{aligned} \ddot{\mathbf{r}}^P = {} & \ddot{\theta}_1(\mathbf{B}_1\mathbf{s}_1'^Q + \mathbf{B}_1\mathbf{A}_{12}\mathbf{s}_2'^{QP}) + \ddot{\theta}_2\mathbf{A}_1\mathbf{B}_{12}\mathbf{s}_2'^{QP} \\ & - \dot{\theta}_1^2\mathbf{r}^P - \dot{\theta}_2^2\mathbf{A}_1\mathbf{A}_{12}\mathbf{s}_2'^{QP} + 2\dot{\theta}_1\dot{\theta}_2\mathbf{B}_1\mathbf{B}_{12}\mathbf{s}_2'^{QP} \end{aligned} \tag{2.6.12}$$

where $\mathbf{B}_1 = \mathbf{B}(\theta_1)$ and $\mathbf{B}_{12} = \mathbf{B}(\theta_2)$.

PROBLEMS

Section 2.1

2.1.1. Using the component forms of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, show that Eq. 2.1.5 holds.

2.1.2. Show that the component form of Eq. 2.1.15 is

$$\theta(\vec{a}, \vec{b}) = \operatorname{sgn}(-a_y b_x + a_x b_y)\ \text{Arccos}\left(\frac{a_x b_x + a_y b_y}{\sqrt{a_x^2 + a_y^2}\sqrt{b_x^2 + b_y^2}}\right)$$

2.1.3. Find $\theta(\vec{a}, \vec{b}_k)$, $k = 1, 2$, for $\vec{a} = \vec{i} + \vec{j}$, $\vec{b}_1 = \vec{j}$, and $\vec{b}_2 = \vec{i}$. Draw a figure to display and interpret the result.

Section 2.2

2.2.1. Use the definitions of equality and addition of matrices of equal dimension to show that Eq. 2.2.5 is valid.

2.2.2. Show that Eq. 2.2.6 is valid.

2.2.3. Calculate the sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$, difference $\mathbf{D} = \mathbf{A} - \mathbf{B}$, and product $\mathbf{E} = \mathbf{AB}$ of the 3×3 matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

2.2.4. Carry out the operations on both sides of Eqs. 2.2.12 and 2.2.13 with the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

2.2.5. Show that the expansion of $\mathbf{A}\boldsymbol{\alpha}$ as a linear combination of columns of $\mathbf{A}$ in Eq. 2.2.17 is valid. Similarly verify that the expansion in Eq. 2.2.18 is valid.

2.2.6. Show that the rank of matrix $\mathbf{B}$ of Example 2.2.5 is 2 by finding a 2×2 submatrix with nonzero determinant and showing that $|\mathbf{B}| = 0$.

2.2.7. Use Eqs. 2.2.16 and 2.2.19 to show that Eq. 2.2.20 is valid.

2.2.8. Show that $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$, to verify that Eq. 2.2.21 is valid.

2.2.9. Show that the following 3×3 matrix is orthogonal for any value of ϕ:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

2.2.10. Show that if $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are orthogonal then $\mathbf{A}^T\mathbf{B}$ is orthogonal (*Hint*: Use Eq. 2.2.16).

Section 2.3

2.3.1. Show that $\mathbf{c}$ of Eq. 2.3.3 is the algebraic representation of $\vec{c} = \vec{a} + \vec{b}$.

2.3.2. Show that $\mathbf{a}^T\mathbf{a}^{\perp} = \mathbf{a}^T\mathbf{R}\mathbf{a} = 0$.

2.3.3. Show that $\mathbf{RRa} = \mathbf{R}^2\mathbf{a} = -\mathbf{a}$, $\mathbf{RRRa} = \mathbf{R}^3\mathbf{a} = -\mathbf{a}^{\perp}$, and $\mathbf{RRRRa} = \mathbf{R}^4\mathbf{a} = \mathbf{a}$.

2.3.4. Show that $(\mathbf{a}+\mathbf{b})^{\perp} = \mathbf{a}^{\perp} + \mathbf{b}^{\perp}$ (*Hint*: Use the matrix property of Eq. 2.2.12 and $\mathbf{a}^{\perp} = \mathbf{Ra}$).

Section 2.4

2.4.1. Show that $\mathbf{R} = \mathbf{A}(\pi/2)$ and, more generally, $\mathbf{R}^n = \mathbf{A}(n\pi/2)$, $n = 2, 3, 4$.

2.4.2. Show that the alternative representations of $\mathbf{r}^P$ in Examples 2.4.1 and 2.4.2 yield identical expressions in θ_1 and θ_2.

2.4.3. Write the vector $\mathbf{r}^P$ from the origin of the x-y frame to point P, in terms of θ, for the parallelogram mechanism shown in Fig. P2.4.3, where $\mathbf{s}'^P$ is a known constant vector in the x'-y' frame that is fixed in the coupler.

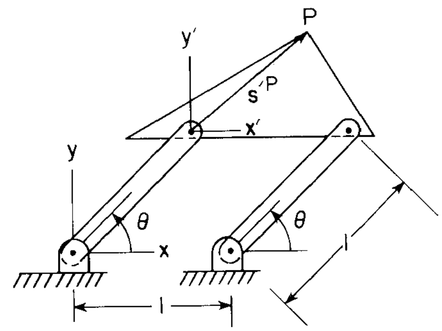

Figure P2.4.3

2.4.4. Pins in a body of unit length are constrained to slide in horizontal and vertical slots as shown in Fig. P2.4.4. Write the vector $\mathbf{r}^P$ from the origin of the x-y frame to point P in terms of generalized coordinates d and θ. Write the constraint equation in d and θ require that point P to slide in the vertical slot.

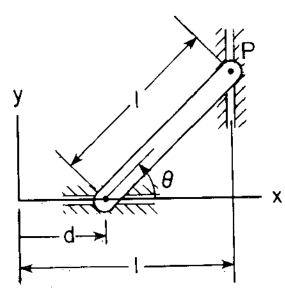

Figure P2.4.4

Section 2.5

2.5.1. Show that Eqs. 2.5.3 and 2.5.4 are valid.

2.5.2. Show that if $\omega_2 = 0$ in Example 2.5.1, then

$$\dot{\mathbf{r}}^P = \omega_1 \mathbf{r}^{P\perp}$$

$$\ddot{\mathbf{r}}^P = -\omega_1^2 \mathbf{r}^P$$

Interpret these results (*Hint*: If $\omega_2 = 0$, then there is no rotation of body 2 relative to body 1).

2.5.3. Show that Eqs. 2.5.8, 2.5.9, and 2.5.10 are valid.

2.5.4. Use the definition of Eq. 2.5.7 and Eq. 2.5.9 to evaluate $\dot{\mathbf{r}}^P$ in Eq. 2.4.13 in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$.

2.5.5. Use Eq. 2.5.9 and the definition of $\mathbf{A}_1$ and $\mathbf{A}_{12}$, as functions of θ_1 and θ_2, to evaluate $\dot{\mathbf{r}}^P$ from Eq. 2.4.14 in terms of $\dot{\theta}_1$ and $\dot{\theta}_2$. Verify that the result is the same as obtained in Prob. 2.5.4.

2.5.6. Use the chain rule of differentiation and the results of Example 2.5.2 to evaluate $\dot{\mathbf{\Phi}}$ and $\dot{\mathbf{r}}^P$ (*Hint*: $\dot{\mathbf{\Phi}} = \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}$ and $\dot{\mathbf{r}}^P = \mathbf{r}^P_{\mathbf{q}}\dot{\mathbf{q}}$).

2.5.7. Verify that Eqs. 2.5.15, 2.5.16, and 2.5.17 are valid.

2.5.8. Verify that Eqs. 2.5.19 and 2.5.21 give equivalent results.

2.5.9. Solve Eq. 2.5.22, using Eq. 2.5.21, to obtain $\ddot{\theta}_2$ as a function of $\dot{\theta}_1$ and $\ddot{\theta}_1$.

2.5.10. Let $\mathbf{q}$ be an n vector of real variables. Show that

$$\frac{\partial \mathbf{q}}{\partial \mathbf{q}} = \mathbf{I}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

2.5.11. Let $\mathbf{q}$ be an n vector of real variables and $\mathbf{A}$ be a real $n \times n$ constant matrix. Show that

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{q}^T\mathbf{A}\mathbf{q}) = \mathbf{q}^T(\mathbf{A}^T + \mathbf{A})$$

2.5.12. If the matrix $\mathbf{A}$ in Prob. 2.5.11 is symmetric, show that

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{q}^T\mathbf{A}\mathbf{q}) = 2\mathbf{q}^T\mathbf{A}$$

Section 2.6

2.6.1. Verify that Eqs. 2.6.11 and 2.6.12 are valid.

2.6.2 With $\mathbf{s}_1'^Q$ and $\mathbf{s}_2'^{QP}$ of Example 2.5.1 and with $\theta_1 = \omega_1 t$ and $\theta_2 = \omega_2 t$, expand Eqs. 2.6.11 and 2.6.12 and verify that $\dot{\mathbf{r}}^P$ and $\ddot{\mathbf{r}}^P$ derived by direct differentiation in Example 2.5.1 are correct.

SUMMARY OF KEY FORMULAS

Matrix Algebra

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}] \qquad \textbf{(2.2.4)}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}), \qquad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \textbf{(2.2.5, 6)}$$

$$\mathbf{AB} = \left[\sum_{k=1}^{p} a_{ik}b_{kj}\right], \qquad \mathbf{AB} \neq \mathbf{BA} \qquad \textbf{(2.2.8, 11)}$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}, \qquad (\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC}), \quad \alpha\mathbf{A}=[\alpha a_{ij}] \tag{2.2.12, 13, 14}$$

$$(\mathbf{A}+\mathbf{B})^T=\mathbf{A}^T+\mathbf{B}^T, \qquad (\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T \tag{2.2.15, 16}$$

$$\mathbf{AA}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}, \quad (\mathbf{A}^{-1})^T=(\mathbf{A}^T)^{-1}, \quad (\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1} \tag{2.2.19, 20, 21}$$

Algebraic Vectors

$$\vec{a}=a_x\vec{i}+a_y\vec{j}, \qquad \mathbf{a}=[a_x, a_y]^T \tag{2.3.1, 2}$$

$$\mathbf{a}^T\mathbf{b}=\vec{a}\cdot\vec{b}, \qquad \mathbf{a}^\perp=\mathbf{Ra}=[-a_y, a_x]^T \tag{2.3.4, 5}$$

$$\mathbf{R}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \qquad \mathbf{RR}=-\mathbf{I} \tag{2.3.6, 7}$$

Transformation of Coordinates

$$\mathbf{s}=[s_x, s_y]^T, \qquad \mathbf{s}'=[\mathbf{s}_{x'}, \mathbf{s}_{y'}]^T \tag{2.4.1}$$

$$\mathbf{s}=\mathbf{As}', \qquad \mathbf{s}'=\mathbf{A}^T\mathbf{s} \tag{2.4.3, 7}$$

$$\mathbf{A}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}, \qquad \mathbf{A}^T\mathbf{A}=\mathbf{I}, \qquad \mathbf{A}^{-1}=\mathbf{A}^T \tag{2.4.4, 5, 6}$$

$$\mathbf{r}^P=\mathbf{r}+\mathbf{s}^P=\mathbf{r}+\mathbf{As}'^P \tag{2.4.8}$$

$$\mathbf{s}_i'=\mathbf{A}_{ij}\mathbf{s}_j', \qquad \mathbf{A}_{ij}=\mathbf{A}_i^T\mathbf{A}_j=\begin{bmatrix}\cos(\phi_j-\phi_i) & -\sin(\phi_j-\phi_i)\\ \sin(\phi_j-\phi_i) & \cos(\phi_j-\phi_i)\end{bmatrix} \tag{2.4.10, 11, 12}$$

Vector and Matrix Differentiation

$$\frac{d}{dt}\mathbf{a}=\dot{\mathbf{a}}=[\dot{a}_x, \dot{a}_y]^T, \qquad \ddot{\mathbf{a}}=[\ddot{a}_x, \ddot{a}_y]^T \tag{2.5.1, 6}$$

$$\frac{d}{dt}(\mathbf{a}+\mathbf{b})=\dot{\mathbf{a}}+\dot{\mathbf{b}}, \qquad \frac{d}{dt}(\alpha\mathbf{a})=\dot{\alpha}\mathbf{a}+\alpha\dot{\mathbf{a}}, \qquad \frac{d}{dt}(\mathbf{a}^T\mathbf{b})=\dot{\mathbf{a}}^T\mathbf{b}+\mathbf{a}^T\dot{\mathbf{b}} \tag{2.5.2, 3, 4}$$

$$\dot{\mathbf{B}}=[\dot{b}_{ij}], \qquad \frac{d}{dt}(\mathbf{B}+\mathbf{C})=\dot{\mathbf{B}}+\dot{\mathbf{C}}, \qquad \frac{d}{dt}(\mathbf{BC})=\dot{\mathbf{B}}\mathbf{C}+\mathbf{B}\dot{\mathbf{C}} \tag{2.5.7, 8, 9}$$

$$\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{q})=[\Phi_1(\mathbf{q}), \ldots, \Phi_n(\mathbf{q})], \qquad \mathbf{\Phi}_{\mathbf{q}}=\left[\frac{\partial\Phi_i}{\partial q_j}\right] \tag{2.5.12}$$

$$(\mathbf{g}^T\mathbf{h})_{\mathbf{q}}=\mathbf{h}^T\mathbf{g}_{\mathbf{q}}+\mathbf{g}^T\mathbf{h}_{\mathbf{q}}, \qquad \mathbf{\Phi}_{\mathbf{q}}=\mathbf{\Phi}_{\mathbf{g}}\mathbf{g}_{\mathbf{q}} \tag{2.5.13, 14}$$

Velocity and Acceleration of a Point

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\phi}\mathbf{B}\mathbf{s}'^P, \qquad \dot{\mathbf{A}} = \dot{\phi}\mathbf{B}, \qquad \frac{d}{d\phi}\mathbf{A} = \mathbf{B} \tag{2.6.4, 3}$$

$$\mathbf{B} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix} = \mathbf{AR} = \mathbf{RA}, \qquad \dot{\mathbf{B}} = -\dot{\phi}\mathbf{A}, \qquad \frac{d}{d\phi}\mathbf{B} = -\mathbf{A} \tag{2.6.5, 6, 8}$$

$$\ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{B}\mathbf{s}'^P - \dot{\phi}^2\mathbf{A}\mathbf{s}'^P = \ddot{\mathbf{r}} + \ddot{\phi}\mathbf{s}^{P\perp} - \dot{\phi}^2\mathbf{s}^P \tag{2.6.9, 10}$$