Introduction to Dynamics

6.3.4, 6.6
November 25, 2014

Quote of the day: “Marge, don't discourage the boy! Weaseling out of things is important to learn. It's what separates us from the animals! Except the weasel.”

-- Homer Simpson
Before we get started...

- **Last time**
  - Lagrange Multiplier Theorem
  - EOM for a collection of rigid bodies connected through joints
  - Example of setting up the equations of motion, slider crank example

- **Today**
  - Setting up initial conditions for the dynamics analysis
  - Revisit the computation of reaction forces
  - Maybe start discussion of numerical integration

- **Assignment posted online due on December 2**
  - HW 9: 6.3.3, 6.4.2
  - MATLAB 8 – posted online
  - ADAMS 5 – posted online

- **Project 2 assigned on Tuesday, due 12/11 at 11:59 PM**

- **Exam on Th December 4, open everything**
  - Review session on Wd, December 3 at 7:15 PM (same idea like last time)
  - Room: 1163ME (next door room)
Most Important Slide of ME451

- Equations of Motion
  \[ M\ddot{q} + \Phi_q^T \lambda = Q^A \]

- Position Constraint Equations
  \[ \Phi(q, t) = 0 \]

- Velocity Constraint Equations
  \[ \Phi_q \dot{q} = -\Phi_t \triangleq \nu \]

- Acceleration Constraint Equations
  \[ \Phi_q \ddot{q} = - (\Phi_q \dot{q})_q \dot{q} - 2\Phi_q \dot{q} \dot{q} - \Phi_{tt} \triangleq \gamma \]
6.3.4

Initial Conditions
[making the simple complicated]

'Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creative.'
-- Charles Mingus
ICs for the EOM of Constrained Planar Systems

- We must provide ICs at the initial time $t_0$ to “seed” the numerical solution
  - How many can/should we specify?
  - How exactly do we specify them?

- Recall that the constraint and velocity equations must be satisfied at all times (including the initial time $t_0$)

- In other words, we have $nc$ generalized coordinates, but they are not independent, as they must satisfy

$$\Phi(q, t) = 0$$
$$\Phi_q q = \nu$$
Specifying Position ICs (1/2)

- We have $nc$ generalized coordinates that must satisfy $m$ equations, thus leaving $NDOF = nc - m$ degrees of freedom

$$\Phi(q_0, t_0) = \begin{bmatrix} \Phi^K(q_0) \\ \Phi^D(q_0, t_0) \end{bmatrix} = 0 \quad q_0 \in \mathbb{R}^{nc}, \Phi \in \mathbb{R}^m$$

- To completely specify the position configuration at $t_0$ we must therefore provide additional $NDOF$ conditions

- How can we do this?
  - Recall what we did in Kinematics to specify driver constraints (to “take care” of the excess DOFs): provide $NDOF$ additional conditions
  - In Dynamics, to specify IC, we provide $NDOF$ additional conditions of the form

$$\Phi^I(q_0, t_0) = 0 \quad \Phi^I \in \mathbb{R}^{NDOF}$$
Specifying Position ICs (2/2)

- The complete set of conditions that the generalized coordinates must satisfy at the initial time \( t_0 \) is therefore

\[
\Phi^*(q_0, t_0) = \begin{bmatrix}
\Phi^K(q_0) \\
\Phi^D(q_0, t_0) \\
\Phi^I(q_0, t_0)
\end{bmatrix} = 0
\]

- How do we know that the IC we imposed are properly specified?
- Implicit Function Theorem gives us the answer: the Jacobian must be nonsingular

\[
\det \left( \Phi^*_q(q_0, t_0) \right) \neq 0
\]

- If this is the case, we can solve the nonlinear system (using for example Newton’s method)

\[
\Phi^*(q_0, t_0) = 0
\]

to obtain the initial configuration \( q_0 \) at the initial time \( t_0 \)
Specifying Velocity ICs (1/2)

- Specifying a set of position ICs is not enough

- We are dealing with 2\textsuperscript{nd} order differential equations and we therefore also need ICs for the generalized velocities

- The generalized velocities must satisfy the velocity equation at all times, in particular at the initial time $t_0$

$$
\Phi_q(q_0, t_0)\dot{q}_0 = \begin{bmatrix} \Phi^K_q(q_0) \\ \Phi^D_q(q_0, t_0) \end{bmatrix} \dot{q}_0 = \begin{bmatrix} \nu^K \\ \nu^D \end{bmatrix} = \nu \quad \dot{q}_0 \in \mathbb{R}^{nc}, \Phi_q \in \mathbb{R}^{m \times nc}
$$

- We have two choices:
  - Specify velocity ICs for the same generalized coordinates for which we specified initial position ICs
    $$
    \Phi_q^I(q_0, t_0)\dot{q}_0 = \nu^I \quad \Phi_q^I \in \mathbb{R}^{NDOF \times nc}
    $$
  - Specify velocity ICs on a completely different set of generalized coordinates
    $$
    B^I(q_0, t_0)\dot{q}_0 = \nu^I \quad B^I \in \mathbb{R}^{NDOF \times nc}
    $$
Specifying Velocity ICs (2/2)

- In either case, we must be able to find a unique solution for the initial generalized velocities $\dot{q}_0$ at the initial time $t_0$.

\[
\begin{bmatrix}
\Phi_q^K(q_0) \\
\Phi_q^D(q_0, t_0) \\
\Phi_q^I(q_0, t_0)
\end{bmatrix} \begin{bmatrix}
\dot{q}_0 \\
\nu^K \\
\nu^I
\end{bmatrix} = \begin{bmatrix}
\nu^K \\
\nu^D \\
\nu^I
\end{bmatrix}
\]

- In both cases, we solve the linear system

\[
\Phi_q^*(q_0, t_0) \dot{q}_0 = \nu
\]

for the initial generalized velocities and therefore we must ensure that

\[
\det (\Phi_q^*(q_0, t_0)) \neq 0
\]
Initial Conditions: Conclusions

- The IC problem is actually simple if we remember what we did in Kinematics regarding driver constraints.

- We only do this at the initial time $t_0$ to provide a starting configuration for the mechanism. Otherwise, the dynamics problem is underdefined.

- Initial conditions can be provided either by
  - Specifying a consistent initial configuration (that is a set of $nc$ generalized coordinates and $nc$ generalized velocities that satisfy the constraint and velocity equations at $t_0$).
    - This is what you should do for simEngine2D.
  - Specifying additional $NDOF$ conditions (that are independent of the existing kinematic and driver constraints) and relying on the Kinematic solver to compute the consistent initial configuration.
    - This is what a general purpose solver might do, such as ADAMS.
Specifying ICs in simEngine2D

- Recall a typical body definition in an ADM file (JSON format)

```json
{
    "name": "slider",
    "id": 1,
    "mass": 2,
    "jbar": 0.3,
    "q0": [2, 0, 0],
    "qd0": [0, 0, 0]
}
```

- In other words, we include in the definition of a body its initial position and velocity (values for the generalized coordinates and velocities at the initial time, which we will always assume to be $t_0 = 0$)

- Make sure that the values $q_0$ and $q_d0$ are such that $\Phi(q_0, 0) = 0$ and $\Phi_q \dot{q}_0 = v$, and then you have a well defined Dynamic Analysis
ICs for a Simple Pendulum
[handout]

- Specify ICs for the simple pendulum such that
  - it starts from a vertical configuration (hanging down), and
  - it has an initial angular velocity $2\pi \frac{rad}{s}$.

- Assume that $l = 0.2 \, m$
6.6

Constraint Reaction Forces
[somewhat hard to grasp]
Reaction Forces

- Remember that we jumped through some hoops to get rid of the reaction forces that develop in joints

- Now, we want to go back and recover them, since they are important:
  - Durability analysis
  - Stress/Strain analysis
  - Selecting bearings in a mechanism
  - Etc.

- The key ingredient needed to compute the reaction forces in all joints is the set of Lagrange multipliers \( \lambda \)
Reaction Forces: The Basic Idea

• Recall the partitioning of the total force acting on the mechanical system

\[
Q = \begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_{nb}
\end{bmatrix} = \begin{bmatrix}
Q_1^A + Q_1^C \\
Q_2^A + Q_2^C \\
\vdots \\
Q_{nb}^A + Q_{nb}^C
\end{bmatrix} = \begin{bmatrix}
Q_1^A \\
Q_2^A \\
\vdots \\
Q_{nb}^A
\end{bmatrix} + \begin{bmatrix}
Q_1^C \\
Q_2^C \\
\vdots \\
Q_{nb}^C
\end{bmatrix} = Q^A + Q^C
\]

• Applying a variational approach (principle of virtual work) we ended up with this equation of motion

\[
\delta q^T (M\ddot{q} - Q) = 0 \quad \Leftrightarrow \quad \delta q^T (M\ddot{q} - Q^A - Q^C) = 0 \quad \Rightarrow \quad M\ddot{q} - Q^A - Q^C = 0
\]

• After jumping through hoops, we ended up with this:

\[
M\ddot{q} + \Phi_q^T \lambda = Q^A \quad \Leftrightarrow \quad M\ddot{q} - Q^A + \Phi_q^T \lambda = 0
\]

• It’s easy to see that

\[
Q^C = -\Phi_q^T \lambda
\]
What we obtain by multiplying the transposed Jacobian of a constraint, $\Phi_q^T$, with the computed corresponding Lagrange multiplier(s), $\lambda$, is the constraint reaction force expressed as a generalized force:

$$Q^C = -\Phi_q^T \lambda$$

Important Observation: One might want a physical representation of this generalized force
- We would like to find $F_x$, $F_y$, and a torque $T$ due to the constraint
- We would like to report these quantities as acting at some point $P$ on a body

In other words: Look for a fictitious force which, when acting on the body at the point $P$, would lead to a generalized force equal to $Q^C$
Reaction Forces: Framework [2/6]

- Assume that the $k$-th joint in the system constrains points $P_i$ on body $i$ and $P_j$ on body $j$.

- We are interested in finding the reaction forces and torques $F_i^{(k)}$ and $T_i^{(k)}$ acting on body $i$ at point $P_i$, as well as $F_j^{(k)}$ and $T_j^{(k)}$ acting on body $j$ at point $P_j$.

- The book complicates the formulation for no good reason by expressing these reaction forces with respect to some arbitrary body-fixed RFs attached at the points $P_i$ and $P_j$, respectively.

- It is much easier to derive the reaction forces and torques in the GRF and, if desired, re-express them in any other frame by using the appropriate rotation matrices.
Reaction Forces: Setup [3/6]

- Let the $m_k$ constraint equations defining the $k$-th joint be

$$\Phi^{(k)}(q_i, q_j, t) = 0, \quad \Phi^{(k)} \in \mathbb{R}^{m_k}$$

- Let the $m_k$ Lagrange multipliers associated with this joint be

$$\lambda^{(k)} \in \mathbb{R}^{m_k}$$
For the sake of this discussion assume that $m_k = 1$

- In other words, look at a constraint and not a joint such as revolute, translational, etc. – which has $m_k = 2$
- Discussion is simpler this way – carries over to $m_k = 2$ as well

Start by taking a closer look at the expression of the constraint reaction force induced by the presence of the kinematic constraint $k$

$$ [Q_{q_i}^{(k)}]^C = -[\Phi_q^{(k)}]^T \lambda^{(k)} = \begin{bmatrix} -[\Phi_{r_i}^{(k)}]^T \\ -[\Phi_{\phi_i}^{(k)}]^T \end{bmatrix} \lambda^{(k)} = \begin{bmatrix} -[\Phi_{r_i}^{(k)}]^T \cdot \lambda^{(k)} \\ -[\Phi_{\phi_i}^{(k)}]^T \cdot \lambda^{(k)} \end{bmatrix} $$
The presence of the $k$-th joint leads to the following reaction force and torque at point $P_i$ on body $i$

$$F_i^{(k)} = - \left( \Phi_{r_i}^{(k)} \right)^T \lambda^{(k)}$$

$$T_i^{(k)} = \left( \Phi_{r_i}^{(k)} B_i s_{i}^{P} - \Phi_{\phi_{i}}^{(k)} \right)^T \lambda^{(k)}$$
For constraint equations that act between two bodies $i$ and $j$, there will also be a $F_j$, $T_j$ pair associated with such constraints, representing the constraint reaction forces on body $j$

- To get $F_i$ and $T_i$, respectively

\[
F_{j}^{(k)} = -\left(\Phi_{r_{j}}^{(k)}\right)^{T}\lambda^{(k)}
\]
\[
T_{j}^{(k)} = \left(\Phi_{r_{j}}^{(k)}B_{j}^{P} - \Phi_{\phi_{j}}^{(k)}\right)^{T}\lambda^{(k)}
\]

- Note that the only thing that we had to do was to replace $i$ with $j$.
  - There is nothing special about $i$ relative to $j$. 
**Reaction Forces: Comments**

- Note that there is one Lagrange multiplier associated with each constraint equation
  - Number of Lagrange multipliers in mechanism is equal to number of constraints
  - Example: the revolute joint brings along a set of two kinematic constraints and therefore there will be two Lagrange multipliers associated with this joint

- Each Lagrange multiplier produces (leads to) a reaction force/torque combo

- Therefore, to each constraint equation corresponds a reaction force/torque pair that “enforces” the satisfaction of the constraint, throughout the time evolution of the mechanism

- If the system is kinematically driven (meaning there are driver constraints), the same approach is applied to obtain reaction forces associated with such constraints
  - In this case, we obtain the force/torque required to impose that driving constraint
Reaction Forces: Summary

- Each constraint in the system has a Lagrange multiplier associated with it.
- The Lagrange multiplier results in the following reaction force and torque:

\[
F_{i}^{(k)} = - \left( \Phi_{r_i}^{(k)} \right)^T \lambda^{(k)} \\
T_{i}^{(k)} = \left( \Phi_{r_i}^{(k)} B_i s_{i}^{P} - \Phi_{\phi_i}^{(k)} \right)^T \lambda^{(k)}
\]

Note: The expression of \( \Phi \) for all the usual joints is known, so a boiler plate approach can be used to obtain the reaction force in all these joints.

- An alternative expression for the reaction torque is:

\[
T_{i}^{(k)} = - \left( B_i s_{i}^{P} \right)^T F_{i}^{(k)} - \left( \Phi_{\phi_i}^{(k)} \right)^T \lambda^{(k)}
\]
Exam Question

- What is the expression of the reaction force/torque on body $i$ induced by the constraint $k$ if the point $P$ is chosen to be the COM for body $i$?
Consider the following kinematically driven simple pendulum, with

\[ \phi_1 = 2\pi t + \frac{3\pi}{2} \]

1. Find the reaction force in the revolute joint that connects the pendulum to ground.

2. Express the reaction force in the \( O''x''y'' \) reference frame